Chapter 7

Lattice of Weakly Induced Principal $L$-topologies

7.1 Introduction

The concept of induced fuzzy topological space was introduced by Weiss [75]. Lowen called these spaces a topologically generated spaces. Martin [38] introduced a generalized concept, weakly induced spaces, which was called semi induced space by Mashhour et al. [40]. The notion of lower semi continuous functions plays an important tool in defining the above concepts. In ( [24],[5]), Aygun et al. introduced a new class of functions from a topological space $(X, \tau)$ to a fuzzy lattice($F$-lattice) $L$ with its

* Some results of this chapter are included in the following paper. 
scott topology called (completely) scott continuous functions, as a generalization of (completely) lower semi continuous functions from \((X, \tau)\) to \([0, 1]\).

It is known that \([30]\) lattice of \(L\)-topologies is complete, atomic and not complemented. In \([31]\), Jose and Johnson generalized weakly induced spaces introduced by Martin \([38]\) using the tool (completely) scott continuous functions and studied the lattice structure of the set \(W(X)\) of all weakly induced \(L\)-topologies on a given set \(X\). A related problem is to find subfamilies in \(W(X)\) having certain properties. The collection of all weakly induced principal \(L\) topologies \(W_P(X)\) form a lattice with the natural order of set inclusion. The concept of principal topologies in the crisp case was studied by Steiner \([58]\). The lattice of principal topologies is both atomic and dually atomic. Analogously we study the lattice structure of the set of all weakly induced principal \(L\)-topologies on a given set \(X\). Here we study properties of the lattice \(W_P\) of all weakly induced principal \(L\) topologies defined by families of (completely) scott continuous functions with reference to \(\tau\) on \(X\). From the lattice \(W_P\), we deduce the lattice \(W_P(X)\) of all weakly induced principal \(L\)-topologies on \(X\). It is join complemented. Also we prove that if \(L\) is a finite pseudocomplemented chain or a complemented \(F\)-lattice, then \(W_P(X)\) has dual atoms and if \(L\) has neither dual atoms nor atoms, then \(W_P(X)\) has no dual atoms.

7.2 Preliminaries

Let \(X\) be a nonempty ordinary set and \(L = (\leq, \lor, \land, ')\) be a completely distributive lattice with smallest element 0 and largest element 1, \(0 \neq 1,\)
and with an order reversing invalution $a \rightarrow a'$ ($a \in L$) called a $F$-lattice. We identify the constant function from $X$ to $L$ with value $\alpha$ by $\alpha$. The fundamental definition of $L$-fuzzy set theory and $L$-topology are assumed to be familiar to the reader in the sense of Chang [13].

A topological space is called principal if it is discrete or if it can be written as the meet of principal ultra topologies. Steiner [58] proved that this is equivalent to requiring that the arbitrary intersection of open sets is open. Analogously we define principal $L$-topology as

**Definition 7.2.1.** An $L$-topology is called principal $L$-topology if arbitrary intersection of open $L$ subsets is an open $L$ subset.

**Definition 7.2.2.** [34] An element of a lattice $L$ is called an atom if it is the minimal element of $L \setminus \{0\}$.

**Definition 7.2.3.** [34] An element of a lattice $L$ is called a dual atom if it is the maximal element of $L \setminus \{1\}$.

**Definition 7.2.4.** [15] A lattice is said to be bounded if it possess smallest element 0 and largest element 1.

**Definition 7.2.5.** [34] A bounded lattice $L$ is said to be join complemented if for all $x$ in $L$, there exists $y$ in $L$ such that $x \vee y = 1$.

**Definition 7.2.6.** [34] A bounded lattice $L$ is said to be meet complemented if for all $x$ in $L$, there exist $y$ in $L$ such that $x \wedge y = 0$.

**Definition 7.2.7.** [34] A bounded lattice is said to be complemented if it is both join complemented and meet complemented.

**Definition 7.2.8.** [34] A bounded lattice $L$ is said to be semi-
complemented if it is either join complemented or meet complemented.

Definition 7.2.9. [22] An element \( p \) of \( L \) is called prime if \( p \neq 1 \) and whenever \( a, b \in L \) with \( a \land b \leq p \), then \( a \leq p \) or \( b \leq p \). The set of all prime elements of \( L \) will be denoted by \( \text{Pr}(L) \).

Definition 7.2.10. [73] The Scott topology on \( L \) is the topology \( S \), generated by the sets of the form \( \{ t \in L : t \nleq p \} \) where \( p \in \text{Pr}(L) \). Let \( (X, \tau) \) be a topological space and \( f : (X, \tau) \to L \) be a function, where \( L \) has its Scott topology. We say that \( f \) is Scott continuous if for every \( p \in \text{Pr}(L) \), \( f^{-1}\{ t \in L : t \nleq p \} \in \tau \).

Remark 7.2.1. When \( L = [0, 1] \), the Scott topology coincides with the topology of topologically generated spaces of Lowen [35]. The set \( \omega_L(\tau) = \{ f \in L^X; f : (X, \tau) \to L \text{ is Scott continuous} \} \) is an \( L \)-topology. It is the largest element in \( W_\tau \), where \( W_\tau \) is the lattice of weakly induced \( L \)-topologies defined by families of Scott continuous functions with reference to \( \tau \) on \( X \). If \( \tau \) is a principal topology \( \omega_L(\tau) \) is a principal \( L \)-topology, we can denote it by \( \omega_{PL}(\tau) \). An \( L \)-topology \( F \) on \( X \) is called an induced principal \( L \)-topology if there exist a principal topology \( \tau \) on \( X \) such that \( F = \omega_{PL}(\tau) \).

Definition 7.2.11. ([24], [5]) Let \( (X, \tau) \) be a topological space and \( a \in X \). A function \( f : (X, \tau) \to L \), where \( L \) has its Scott topology, is said to be completely Scott continuous at \( a \in X \) if for every \( p \in \text{Pr}(L) \) with \( f(a) \nleq p \), there is a regular open neighbourhood \( U \) of \( a \) in \( (X, \tau) \) such that \( f(x) \nleq p \) for every \( x \in U \). That is \( U \subset f^{-1}\{ t \in L : t \nleq p \} \) and \( f \) is called completely Scott continuous on \( X \), if \( f \) is completely Scott continuous at every point of \( X \).
Note.
Let $F$ be a principal $L$-topology on the set $X$, let $F_c$ denote the 0-1 valued members of $F$, that is, $F_c$ is the set of all characteristic mappings in $F$. Then $F_c$ is a principal $L$-topology on $X$. Define $F_c^* = \{ A \subset X : \mu_A \in F_c \}$ where $\mu_A$ is the characteristic function of $A$. The principal $L$-topological space $(X, F_c)$ is same as the principal topological space $(X, F_c^*)$.

**Definition 7.2.12.** A principal $L$-topological space $(X, F)$ is said to be a weakly induced principal $L$-topological space, if for each $f \in F$, $f$ is a scott continuous function from $(X, F_c^*)$ to $L$.

**Definition 7.2.13.** If $F$ is the collection of all scott continuous functions from $(X, F_c^*)$ to $L$, then $F$ is an induced space and $F = \omega_{PL}(F_c^*)$.

### 7.3 Lattice of weakly induced principal $L$-topologies

For a given principal topology $\tau$ on $X$, the family $W_{P\tau}$ of all weakly induced principal $L$-topologies defined by families of scott continuous functions from $(X, \tau)$ to $L$ forms a lattice under the natural order of set inclusion. The least upperbound of a collection of weakly induced principal $L$-topologies belonging to $W_{P\tau}$ is the weakly induced principal $L$-topology which is generated by their union and the greatest lowerbound is their intersection. The smallest element is the indiscrete $L$-topology, denoted by $0$ and the largest element is denoted by $1 = \omega_{PL}(\tau)$.

Also for a principal topology $\tau$ on $X$, the family $CW_{P\tau}$ of all weakly
induced principal $L$ topologies defined by families of completely scott continuous function from $(X, \tau)$ to $L$ forms a lattice under the natural order of set inclusion. Since every completely scott continuous function is scott continuous, it follows that $CW_{P\tau}$ is a sublattice of $W_{P\tau}$. We note that $W_{P\tau}$ and $CW_{P\tau}$ coincide when each open set in $\tau$ is regular open.

When $\tau = D$, the discrete topology on $X$, these lattices coincide with lattice of weakly induced principal $L$-topologies on $X$.

**Theorem 7.3.1.** [18] The Ultra spaces on a set $E$ are exactly the topologies of the form $\mathcal{G}(x, \mathcal{U}) = \varnothing(E - x) \cup \mathcal{U}$, where $x \in E$, $\mathcal{U}$ is an ultrafilter on $E$ not containing $\{x\}$.

**Theorem 7.3.2.** The lattice $W_{P\tau}$ is complete.

**Proof.** Let $S$ be a subset of $W_{P\tau}$ and let $G = \bigcap_{F \in S} F$. Clearly $G$ is a principal $L$-topology. Let $g \in G$. Since each $F \in S$ is a weakly induced principal $L$ topology, $g$ is a scott continuous mapping from $(X, (F_c^*)$ to $L$, that is $g^{-1}\{t \in L : t \not< p \text{ where } p \in \Pr(L)\} \in F_c^*$ for each $F \in S$. Therefore $g^{-1}\{t \in L : t \not< p \text{ where } p \in \Pr L\} \in \bigcap_{F \in S} (F_c^*)$. Hence $g$ is a scott continuous function from $(X, G_c^*)$ to $L$, where $(X, G_c^*) = (X, \bigcap_{F \in S} F_c^*)$. That is $G \in W_{P\tau}$ and $G$ is the greatest lower bound of $S$. Let $K$ be the set of upperbounds of $S$. Then $K$ is non empty, since $1 = \omega_{PL}(\tau) \in K$.

Using the above argument $K$ has a greatest lowerbound, say $H$, then this $H$ is a least upper bound of $S$. Thus every subset $S$ of $W_{P\tau}$ has greatest lowerbound and least upperbound. Hence $W_{P\tau}$ is complete. \qed

**Theorem 7.3.3.** $W_{P\tau}$ is not atomic in general.
7.3. Lattice of weakly induced principal $L$-topologies

Proof. Atoms in $W_{P\tau}$ are either of the form $\{0, 1, \alpha\}$ or $\{0, 1, \mu_A\}$, where $\mu_A$ is the characteristic function of open subsets $A$ of $(X, \tau)$ and $\alpha \in (0, 1)$. Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{a, b\}\}$ and $F = \{0, 1, \mu_{\{a\}}, \mu_{\{a, b\}}\}$.

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| $F_c = \{0, 1, \mu_{\{a\}}, \mu_{\{a, b\}}\}$. $F_c^* = \{\phi, X, \{a\}, \{a, b\}\} = \tau$ and $F \in W_{P\tau}$. But this $F$ cannot be expressed as join of atoms. Hence $W_{P\tau}$ is not atomic. 

\[\square\]

**Theorem 7.3.4.** $W_{P\tau}$ is not distributive.

Proof. Since every distributive lattice is necessarily modular, we prove that $W_{P\tau}$ is not modular. This can be illustrated with an example. Let $X$ be an infinite set and $\tau$ be the discrete topology $D$ on $X$. Then $W_{P\tau}$ becomes lattice of all weakly induced principal $L$-topologies on $X$ and $\Pi(X)$, the lattice of principal topologies on $X$ (identifying its characteristic functions) is a sublattice of $W_{PD}$. We know that $\Pi(X)$ is not modular and hence not distributive. Thus $W_{P\tau}$ is not distributive in general. \[\square\]

**Theorem 7.3.5.** If $L$ is a finite pseudo complemented chain or a complemented $F$-lattice, then $W_P(X)$ has dual atoms.

Proof. case 1.

Let $X$ be a non empty set and $L$ be a finite pseudo complemented chain.

Since $L$ is a finite pseudo complemented chain, it has atom and dual atom. Let $\tau$ be a dual atom in the lattice of principal topologies on $X$. 

Then by theorem 7.3.1, $\tau$ must be of the form $G(a, \mathcal{U}) = \varphi(X - a) \cup \mathcal{U}$, where $a \in X, \mathcal{U}$ is an ultrafilter not containing $\{a\}$. Since $\tau$ is a principal topology, $\mathcal{U}$ is a principal ultra filter so that $\tau = G(a, \mathcal{U}(b)) = \varphi(X - a) \cup \mathcal{U}(b)$. Then $\omega_{\mathcal{P}L}(\tau) = \{f \geq b_\lambda | f : (X, \tau) \to L \text{ is a scott continuous function from } (X, \tau) \text{ to } L\}, b \in X$ and $\lambda$ is an atom in $L$. Then $a_\alpha \notin \omega_{\mathcal{P}L}(\tau)$ for $0 \neq \alpha \in L$. Let $\beta$ be the dual atom in $L$ and $F(a_\beta) = \omega_{\mathcal{P}L}(\tau) \lor a_\beta$. Then $F(a_\beta)$ is the ultra $L$-topology $G(a_\beta)$ in $\beta(X)$ since the simple extension of $F(a_\beta)$ by $a_1$ is the discrete $L$-topology. Let $G = F(a_\beta)$, $G_c = \{0 - 1 \text{ valued functions in } G\}$ and $G^*_c = \{A \subset X | \mu_A \in G_c\}$. Then the weakly induced principal $L$-topology defined by Scott continuous functions from $(X, (G^*_c))$ to $L$ is a dual atom in $W_P(X)$.

**Case 2.**

Let $X$ be a non empty set and $L$ is a finite complemented $F$-lattice.

Since $L$ is a complemented $F$-lattice, it has atoms and dual atoms. Let $\tau$ be a dual atom in the lattice of principal topologies on $X$. Then by theorem 7.3.1, $\tau$ must be of the form $G(a, \mathcal{U}) = \varphi(X - a) \cup \mathcal{U}$, where $a \in X, \mathcal{U}$ is an ultrafilter not containing $\{a\}$. Since $\tau$ is a principal topology, $\mathcal{U}$ is a principal ultra filter so that $\tau = G(a, \mathcal{U}(b)) = \varphi(X - a) \cup \mathcal{U}(b)$. Then $\omega_{\mathcal{P}L}(\tau) = \{f \geq b_\lambda | f : (X, \tau) \to L \text{ is a scott continuous function}\}, b \in X$ and $\lambda$ is an atom. Then $a_\alpha \notin \omega_{\mathcal{P}L}(\tau)$ for $0 \neq \alpha \in L$. Let $\beta_1, \beta_2, \ldots, \beta_m$ are dual atoms in $L$ and $F(a_{\beta_1}) = \omega_{\mathcal{P}L}(\tau) \lor a_{\beta_1}, F(a_{\beta_2}) = \omega_{\mathcal{P}L}(\tau) \lor a_{\beta_2}, \ldots, F(a_{\beta_m}) = \omega_{\mathcal{P}L}(\tau) \lor a_{\beta_m}$. Let $F_{\beta_j}$ is the principal $L$-topology generated by $(m - 1) F(a_{\beta_i})$ from $m F(a_{\beta_i}), i = 1, 2, \ldots, m, j = 1, 2, \ldots, m, i \neq j$. Then $F_{\beta_j}$ is the ultra $L$-topology $G_{\beta_j}$ in $\beta(X)$ since the simple extension of $F_{\beta_j}$ by $a_{\beta_j}$ or $a_1$ is the discrete $L$-topology. Take $G = F_{\beta_j}$ and let $G_c = \{0 - 1 \text{ valued functions in } G\}$ and $G^*_c = \{A \subset X | \mu_A \in G_c\}$. Then the weakly induced principal $L$-topology defined by Scott continuous functions from
7.3. Lattice of weakly induced principal $L$-topologies

$(X, G^*_c)$ to $L$ is a dual atom in $W_P(X)$
In both cases, $W_P(X)$ has dual atoms. Hence the theorem.

**Theorem 7.3.6.** If $L$ has neither dual atoms nor atoms, then $W_P(X)$ has no dual atoms.

**Proof.** Let $F$ be any weakly induced principal $L$-topology other than 1. Then we claim that there exists at least one weakly induced principal $L$-topology finer than $F$. Since $F$ is a weakly induced principal $L$-topology different from 1, $F$ cannot contain all characteristic functions of subsets of $X$. Since $L$ has neither dual atoms nor atoms, the collection $S$ of Scott continuous functions not belonging to $F$ is infinite. If $g \in S$, then $F(g)$, the simple extension of $F$ by $g$ is a principal $L$-topology. Take $G = F(g)$. Let $G_c$ denote the 0–1 valued members of $G$ and $G^*_c = \{A \subset X | \mu_A \in G_c\}$, where $\mu_A$ is the characteristic function of $A$. Then there exists a weakly induced principal $L$-topology $H$ defined by Scott continuous functions from $(X, G^*_c)$ to $L$. Thus for any weakly induced principal $L$-topology $F$ there exists a weakly induced principal $L$-topology $H$ such that $F \subset H \neq 1$. Hence the proof of the theorem is completed.

**Theorem 7.3.7.** The lattice $W_P(X)$ of all weakly induced principal $L$-topologies on any set $X$ is not dually atomic in general.

**Proof.** This follows from theorem 7.3.6.
7.4 Complementation problem in the lattice of weakly induced principal $L$-topologies

Proposition 7.4.1
If $L$ has no dual atoms, then atoms in $W_{P_{\tau}}$ of the form $\{0, 1, \alpha\}$ have no complements in $W_{P_{\tau}}$.

Proof. Let $F = \{0, 1, \alpha\}$ be atom in $W_{P_{\tau}}$. We claim that $F$ has no complement. 1 is not a complement of $F$ since $1 \land F \neq 0$. Let $P$ be a weakly induced principal $L$-topology in $W_{P_{\tau}}$ other than 1. If $F \subseteq P$, then $P$ cannot be the complement of $F$, since $F \land P \neq 0$. If $F \not\subseteq P$, let $F \lor P = G$ and $G$ has the subbase $\{f \land p | f \in F, p \in P\}$. Then $G$ cannot be equal to 1. Hence $P$ is not a complement of $F$. \qed

Remark 7.4.1. The above proposition is not true for an arbitrary lattice $L$. For example, take $L = \{0, \alpha, 1\}$ ordered by $0 < \alpha < 1$. If $(X, \tau)$ is a principal $L$-topological space and $K = \{0, 1, \alpha\}$, then clearly $K$ is an atom in $W_{P_{\tau}}$, when $\alpha$ is not a characteristic function. Let $H = \{0, 1\} \cup \{\mu_A : A \in \tau\}$. Then $H$ is an element of $W_{P_{\tau}}$ and $K \land H = 0$ and $K \lor H = 1$. Hence $H$ is a complement of $K$.

Theorem 7.4.1. $W_{P_{\tau}}$ is not complemented.

Proof. This follows from the Proposition 7.4.1. \qed

Remark 7.4.2. When $\tau = D$, the discrete topology on $X$ then $W_{P_D} = W_{P}(X)$, the collection of all weakly induced principal $L$-topologies on $X$. Let $\Delta$ denote the family of all weakly induced principal $L$-topologies
7.4. Complementation problem in the lattice of weakly induced principal $L$-topologies defined by scott continuous functions where each scott continuous function is a characteristic function. Then $\Delta$ is a sublattice of $W_P(X)$ and is a lattice isomorphic to the lattice of all principal topologies on $X$. The elements of $\Delta$ are called crisp principal topologies.

**Theorem 7.4.2.** The lattice of weakly induced principal $L$-topologies $W_P(X)$ is not complemented.

*Proof.* This follows from theorem 7.4.1. □

**Theorem 7.4.3.** Every atom in $W_P(X)$ of the form $\{0, 1, \mu_A\}$ has complement.

*Proof.* Let $F = \{0, 1, \mu_A\}$. Then $F$ is an element of $\Pi$, lattice of principal topologies on $X$. Since $\Pi$ is complemented there exists $\tau$ in $\Pi$ such that $\tau \vee F$ equal to the discrete principal topology and $\tau \wedge F$ equal to the indiscrete principal topology on $X$. Then $F \vee \omega_{PL}(\tau) = 1 = \omega_{PL}(D)$ and $F \wedge \omega_{PL}(\tau) = 0$. □

**Theorem 7.4.4.** The lattice $W_P(X)$ of all weakly induced principal $L$-topologies on any set $X$ is semi complemented.

*Proof.* Let $F \in W_P(X)$. Since $F$ is weakly induced principal $L$-topology, there is a principal topology $\tau = F^*_c$ on $X$ such that each element $f \in F$ is a scott continuous function from $(X, F^*_c)$ to $L$. Since the lattice of principal topologies is complemented, we can find a principal topology $\tau'$ such that $F \vee \omega_{PL}(\tau') = 1 = \omega_{PL}(D)$ where $D$ is a discrete topology and $F \wedge \omega_{PL}(\tau')$ need not be equal to 0, the indiscrete principal $L$-topology.
on $X$. Thus every $F$ in $W_P(X)$ has a join complement. Hence $W_P(X)$ is semi complemented. $\square$