Chapter 4

Lattice of Weakly Induced $T_1$-$L$ topologies

4.1 Introduction

In [11] Birkhoff proved that the set of all $T_1$ topologies, $\Lambda(X)$ is a complete sublattice of $\Sigma(X)$, the lattice of all topologies. $\Lambda(X)$ possess atoms, but is not atomic [67]. Also $\Lambda(X)$ is not a complemented lattice [57]. The concept of induced fuzzy topological space was introduced by Weiss [75]. Lowen called these spaces a topologically generated spaces. Martin [38] introduced a generalized concept, weakly induced spaces, which was called semi induced space by Mashhour et al.[40]. The notion of lower

Some results of this chapter are included in the following paper.

semicontinuous functions plays an important tool in defining the above concepts. In [24] Aygun et al. introduced a new class of functions from a topological space \((X, \tau)\) to a fuzzy lattice \(L\) with its scott topology called (completely) scott continuous functions as a generalization of (completely) lower-semi continuous functions from \((X, \tau)\) to \([0, 1]\). It is known that [30] lattice of \(L\)-topologies is complete, atomic and not complemented. In [31] Jose and Johnson generalised weakly induced spaces introduced by Martin [38] using the tool (completely) scott continuous functions and studied the lattice structure of the set \(W(X)\) of all weakly induced \(L\)-topologies on a given set \(X\). A related problem is to find subfamilies in \(W(X)\) having certain properties. The collection of all weakly induced \(T_1\)-\(L\) topologies \(W_1(X)\) form a lattice with natural order of set inclusion. In [64] Liu determined dual atoms in the lattice of \(T_1\) topologies and Frolich [18] proved this lattice is dually atomic. Here we study properties of the lattice \(W_{1\tau}\) of weakly induced \(T_1\)-\(L\) topologies defined by families of (completely) scott continuous functions with reference to a \(T_1\) topology \(\tau\) on \(X\). It has dual atoms if and only if the membership lattice \(L\) has dual atoms and it is not dually atomic in general. From the lattice \(W_{1\tau}\) we deduce the lattice \(W_1(X)\) of all weakly induced \(T_1\)-\(L\) topologies on a given set \(X\).

4.2 Preliminaries

Let \(X\) be a non empty ordinary set and \(L = L(\leq, \lor, \land, ')\) be a completely distributive lattice with smallest element 0 and largest element 1, \(0 \neq 1\) and with an order reversing involution \(a \rightarrow a'(a \in L)\). We identify the constant function from \(X\) to \(L\) with value \(\alpha\) by \(\alpha\). The fundamental
4.2. Preliminaries

Definition 4.2.1. [44] A fuzzy point $x_\lambda$ in a set $X$ is a fuzzy set in $X$ defined by

$$x_\lambda(y) = \begin{cases} 
\lambda & \text{if } y = x \\
0 & \text{if } y \neq x
\end{cases}$$

where $0 < \lambda \leq 1$

In an $L$-topological space $x_\lambda$ is called an $L$-point.

Definition 4.2.2. [44] An $L$-topological space $(X,F)$ is said to be a $T_1$-$L$ topological space if for every two distinct $L$-points $x_p$ and $y_q$, with distinct support, there exists an $f \in F$ such that $x_p \in f$ and $y_q \notin f$ and another $g \in F$ such that $y_q \in g$ and $x_p \notin g, \forall p, q \in L \setminus \{0\}$

Remark 4.2.1. We take the definition of $L$-points $x_\lambda, 0 < \lambda \leq 1$ so as to include all crisp singletons. Hence every crisp $T_1$ topology is a $T_1$-$L$ topology by identifying it with its characteristic function. If $\tau$ is any topology on a finite set, then $\tau$ is $T_1$, if and only if it is discrete. But in a $T_1$-$L$ topological space every $L$-point need not be closed.

Example 4.2.1. Let $X = \{a,b,c\}$ and $L = \wp(X)$, power set of $X$, then $F = \{\emptyset, \mu_{\{a\}}, \mu_{\{b\}}, \mu_{\{c\}}, \mu_{\{a,b\}}, \mu_{\{a,c\}}, \mu_{\{b,c\}}, 1\}$ is a $T_1$-$L$ topology. Let $a_\lambda, b_\lambda, c_\lambda, 0 \neq \lambda \in L$ are $L$-points. The complements of $a_\lambda, b_\lambda, c_\lambda$ are not open in $F$ so that $a_\lambda, b_\lambda, c_\lambda$ are not closed.

Definition 4.2.3. [22] An element $p \in L$ is called prime if $p \neq 1$ and whenever $a, b \in L$ with $a \land b \leq p$, then $a \leq p$ or $b \leq p$. The set of all prime elements of $L$ will be denoted by $P_r(L)$.

Definition 4.2.4. [73] The scott topology on $L$ is the topology generated by the sets of the form $\{t \in L : t \nleq p\}$, where $p \in P_r(L)$. Let
(X, \tau) be a topological space and f : (X, \tau) \to L be a function where L has its scott topology, we say that f is scott continuous if for every \( p \in P_r(L), f^{-1}\{t \in L : t \not\in p\} \in \tau \).

**Remark 4.2.2.** When L = [0, 1], the scott topology coincides with the topology of topologically generated spaces of Lowen[35]. Every Scott continuous function need not be lower semi continuous.

**Remark 4.2.3.** The set \( \omega_L(\tau) = \{ f \in L^X ; f : (X, \tau) \to L \text{ is scott continuous} \} \) is an L-topology. It is the largest element in \( W_\tau \). If \( \tau \) is a T1 topology \( \omega_L(\tau) \) is a T1-L topology, we can denote it by \( \omega_{1L}(\tau) \). An L-topology F on X is called an induced T1-L topology if there exists a T1 topology \( \tau \) on X such that \( F = \omega_{1L}(\tau) \).

**Definition 4.2.5.** [24] Let (X, \tau) be a topological space and \( \alpha \in X \). A function f : (X, \tau) \to L, where L has its scott topology, is said to be completely scott continuous at \( \alpha \in X \) if for every \( p \in P_r(L) \) with \( f(\alpha) \not\in p \), there is a regular open neighbourhood U of \( \alpha \in (X, \tau) \) such that \( f(x) \not\in p \) for every \( x \in U \). That is \( U \subset f^{-1}\{t \in L : t \not\in p\} \) and f is called completely scott continuous on X, if f is completely scott continuous at every point of X.

**Note 1.**
Let F be a T1-L topology on the set X, let \( F_c \) denote the 0 – 1 valued members of F, that is, \( F_c \) is the set of all characteristic mappings in F. Then \( F_c \) is a T1-L topology on X. Define \( F_c^* = \{ A \subset X : \mu_A \in F_c, \text{ where } \mu_A \text{ is the characteristic function of } A \} \). The T1-L topological space \( (X, F_c) \) is same as the T1 topological spaces\( (X, F_c^*) \).

**Definition 4.2.6.** A T1-L topological space \( (X, F) \) is said to be a
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weakly induced $T_1$-$L$ topological space, if for each $f \in F, f$ is a scott continuous function from $(X, F^*_c)$ to $L$.

**Definition 4.2.7.** If $F$ is the collection of all scott continuous functions from $(X, F^*_c)$ to $L$, then $F$ is an induced space and $F = \omega_1(L(F^*_c))$.

**Definition 4.2.8.** [34] An element of a lattice $L$ is called an atom if it is the minimal element of $L \setminus \{0\}$.

**Definition 4.2.9.** [34] An element of a lattice $L$ is called a dual atom if it is the maximal element of $L \setminus \{1\}$.

**Definition 4.2.10.** [34] A bounded lattice is said to be complemented if for all $x$ in $L$ there exists $y$ in $L$ such that $x \lor y = 1$ and $x \land y = 0$.

### 4.3 Lattice of weakly induced $T_1 - L$ topologies

For a given $T_1$-topology $\tau$ on $X$, the family $W_1{\tau}$ of all weakly induced $T_1$-$L$ topologies defined by families of scott continuous functions from $(X, \tau)$ to $L$ forms a lattice under the natural order of set inclusion. The least upper bound of a collection of weakly induced $T_1$-$L$ topologies belonging to $W_1{\tau}$ is the weakly induced $T_1$-$L$ topology which is generated by their union and their greatest lower bound is their intersection. The smallest element is the crisp cofinite topology denoted by $0$ and the largest element is $\omega_1(L(\tau))$.

Also for a $T_1$ topology $\tau$ on $X$, the family $CW_1{\tau}$ of all weakly induced $T_1$-$L$ topologies defined by families of completely scott continuous functions from $(X, \tau)$ to $L$ forms a lattice under the natural order of set inclusion.
Since every completely scott continuous function is scott continuous, it follows that $CW_1\tau$ is a sublattice of $W_1\tau$. We note that $W_1\tau$ and $CW_1\tau$ coincide when each open set in $\tau$ is regular open. When $\tau = D$, the discrete topology on $X$, these lattices coincide with lattice of weakly induced $T_1$-L topologies on $X$.

**Theorem 4.3.1.** [18] The Ultra spaces on a set $E$ are exactly the topologies of the form $\mathfrak{G}(x, \mathcal{U}) = \varnothing(E - x) \cup \mathcal{U}$, where $x \in E$, $\mathcal{U}$ is an ultrafilter on $E$ not containing $\{x\}$.

**Theorem 4.3.2.** [62] Let $(X, F)$ and $(X, G)$ be two fuzzy topological spaces on $X$. Then $G$ covers $F$ if and only if $G = F(g)$ for every $g \in G - F$, where $F(g)$ is the simple extension of $F$ by $g$.

**Theorem 4.3.3.** The lattice $W_1\tau$ is complete.

Proof. Let $S$ be a subset of $W_1\tau$ and let $G = \bigcap_{F \in S} F$. Clearly $G$ is a $T_1$-L topology. Let $g \in G$. Since each $F \in S$ is a weakly induced $T_1$-L topology, $g$ is a scott continuous mapping from $(X, F_c^*)$ to $L$. That is $g^{-1}(\{t \in L : t \nleq p, \text{where } p \in P_r(L)\}) \subseteq F_c^*$ for each $F \in S$. Therefore $g^{-1}(\{t \in L : t \nleq p \text{ where } p \in P_r(L)\}) \subseteq \bigcap_{F \in S} F_c^*$. Hence $g$ is a scott continuous function from $(X, G_c^*)$ to $L$, where $(X, G_c^*) = (X, \bigcap_{F \in S} F_c^*)$. That is $G \in W_1\tau$ and $G$ is the greatest lower bound of $S$. Let $K$ be the set of upper bounds of $S$. Then $K$ is non empty since $1 = \omega_{1L}(\tau) \in K$. Using the above argument $K$ has a greatest lower bound say $H$. Then this $H$ is a least upper bound of $S$. Thus every subset $S$ of $W_1\tau$ has greatest lower bound and least upper bound. Hence $W_1\tau$ is complete. $\square$
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Note 2.
Let $CFT$ denote the crisp cofinite topology, where $CFT = \{ \mu_A : A \text{ is a subset of } X \text{ whose complement is finite} \}$ together with 0, $\mu_A$ is the characteristic function of $A$.

**Theorem 4.3.4.** $W_{1\tau}$ is not atomic in general.

**Example 4.3.1.** Take $\tau =$ Ccountable topology on the set $R$ of real numbers and $L = [0, 1]$. The smallest element in $W_{1\tau}$ is the crisp cofinite topology $CFT$ denoted by 0 and largest element is $\omega_{1L}(\tau) = \{ f | f : (X, \tau) \to L \text{ is a scott continuous function} \}$. Then the $T_1$-$L$ topologies of the form $CFT \cup \alpha$, $CFT \cup \mu_A$ where $X - A$ is countably infinite are weakly induced. But $T_1$-$L$ topology of the form $CFT \cup h$ where $h$ is a scott continuous function which is neither constant function nor a characteristic function is not weakly induced. Hence weakly induced $T_1$-$L$ topologies of the form $CFT \cup f$ are atoms in $W_{1\tau}$ and hence $\omega_{1L}(\tau)$ cannot be expressed as join of atoms. Thus $W_{1\tau}$ is not atomic.

**Theorem 4.3.5.** [4] $\Lambda(X)$ is not modular and hence not distributive.

**Theorem 4.3.6.** $W_{1\tau}$ is not distributive in general.

**Proof.** Since every distributive lattice is necessarily modular, we prove that $W_{1\tau}$ is not modular. This can be illustrated with an example. Take $X$ as any infinite set and $\tau = D$, discrete topology on $X$. Then $W_{1\tau}$ becomes lattice of all weakly induced $T_1$-$L$ topology on $X$ and $\Lambda(X)$, the lattice of $T_1$ topologies on $X$(identifying by its characteristic functions) is a sublattice of $W_{1D}$. We know that by theorem 4.3.5 $\Lambda(X)$ is not modular and hence not distributive. Thus $W_{1\tau}$ is not distributive in general. \qed
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**Theorem 4.3.7.** If $L$ has dual atoms, then $W_1\tau$ has dual atoms.

**Proof.** Case 1.

Let $X$ be a non empty set and $L$ be a finite pseudo complemented chain.

Let $\tau$ be a dual atom in the lattice of $T_1$ topologies on $X$. Then by theorem 4.3.1, $\tau$ must be of the form $\mathcal{G}(a, \mathcal{U}) = \varnothing(X - a) \cup \mathcal{U}$, where $a \in X$, $\mathcal{U}$ is non principal ultrafilter not containing $\{a\}$. Then $\omega_{1L}(\tau) = \{f|f : (X, \tau) \to L \text{ is a scott continuous function}\}$. Then $a_\lambda \notin \omega_{1L}(\tau), 0 \neq \lambda \in L$. Let $\beta$ be the dual atom in $L$ and $F = \omega_{1L}(\tau) \lor a_\beta$ Then $F$ is the ultra $L$-topology $\mathcal{G}(a_\beta)$ in $\Omega(X)$ since the simple extension of $F$ by $a_1$ is the discrete $L$-topology. Let $F_c = 0 \to 1$ valued functions in $F$ and $F_c^* = \{A \subset X|\mu_A \in F_c\}$. Then the weakly induced $T_1$-$L$ topology defined by Scott continuous functions from $(X, (F_c^*))$ to $L$ is a dual atom in $W_1\tau$.

Case 2.

Let $X$ be a non empty set and $L$ is not a finite pseudo complemented chain.

Let $\tau$ be a dual atom in the lattice of $T_1$ topologies on $X$. Then by theorem 4.3.1, $\tau$ must be of the form $\mathcal{G}(a, \mathcal{U}) = \varnothing(X - a) \cup \mathcal{U}$, where $a \in X$, $\mathcal{U}$ is non principal ultrafilter not containing $\{a\}$. Then $\omega_{1L}(\tau) = \{f|f : (X, \tau) \to L \text{ is a scott continuous function}\}$. Then $a_\lambda \notin \omega_{1L}(\tau), 0 \neq \lambda \in L$. Let $\beta_1, \beta_2, ..., \beta_m$ are dual atoms in $L$ and $F(a_{\beta_1}) = \omega_{1L}(\tau) \lor a_{\beta_1}, F(a_{\beta_2}) = \omega_{1L}(\tau) \lor a_{\beta_2}, ......., F(a_{\beta_m}) = \omega_{1L}(\tau) \lor a_{\beta_m}$. Let $F_{\beta_j}$ is the $L$-topology generated by $(m - 1) F(a_{\beta_i})$ from $m F(a_{\beta_i}), i = 1, 2, ..., m, j = 1, 2, ..., m, i \neq j$ Then as in case 1. $F_{\beta_j}$ is the ultra $L$-topology $\mathcal{G}_{\beta_j}$ in $\Omega(X)$ since the simple extension of $F_{\beta_j}$ by $a_{\beta_j}$ is the discrete $L$-topology. Let $G = F_{\beta_j}, G_c = \{0 \to 1$ valued functions in $G$
and $G^*_c = \{ A \subset X | \mu_A \in G_C \}$. Then the weakly induced $T_1$-$L$ topology defined by Scott continuous functions from $(X, G^*_c)$ to $L$ is a dual atom in $W_{1\tau}$.

In both cases since $L$ has dual atoms, $W_{1\tau}$ has dual atoms. Hence the theorem.

**Theorem 4.3.8.** If $L$ has no dual atoms, then $W_{1\tau}$ has no dual atoms.

**Proof.** Let $F$ be any weakly induced $T_1$-$L$ topology other than $1 = \omega_{1L}(\tau)$. Then we claim that there exists at least one weakly induced $T_1$-$L$ topology finer than $F$. Since $F$ is a weakly induced $T_1$-$L$ topology different from $\omega_{1L}(\tau)$, $F$ cannot contain at the same time all characteristic functions of open sets in $\tau$ and all constant $L$-subsets. Since $L$ has no dual atoms, the collection $S$ of $L$ subsets not belonging to $F$ is infinite. Since $F$ is a $T_1$-$L$ topology, $g \in S$, we have $F(g)$, the simple extension of $F$ by $g$ is also a $T_1$-$L$ topology. Let $G = F(g)$, $G_c$ denote the $0-1$ valued members of $G$ and $G^*_c = \{ A \subset X | \mu_A \in G_c \}$, where $\mu_A$ is the characteristic function of $A$. Then there exists a weakly induced $T_1$-$L$ topology $K$ defined by Scott continuous functions from $(X, G^*_c)$ to $L$. Thus for any weakly induced $T_1$-$L$ topology $F$ there exists a weakly induced $T_1$-$L$ topology $K$ such that $F \subset K \neq 1$. Hence the proof of the theorem is completed. 

Combining Theorem 4.3.7 and Theorem 4.3.8 we have

**Theorem 4.3.9.** The lattice of weakly induced $T_1$-$L$ topologies $W_{1\tau}$ has dual atoms if and only if $L$ has dual atoms.

**Theorem 4.3.10.** $W_{1\tau}$ is not dually atomic in general.
Proof. This follows from Theorem 4.3.8. 

4.4 Complementation in the lattice of weakly induced $T_1$-$L$ topologies

Theorem 4.4.1. $W_{1\tau}$ is not complemented in general.

Example 4.4.1. Let $X = R$, set of real numbers and $\tau =$ Ccoountable topology on $R$. Take $L = [0, 1]$. The weakly induced $T_1$-$L$ topology of the form $F = CFT \cup \alpha$ where $\alpha \in (0, 1)$ has no complement. For, clearly 1 is not a complement of $F$, since $F \wedge 1 \neq 0$. Let $G$ be any weakly induced $T_1$-$L$ topology in $W_{1\tau}$ other than 1. If $F \subset G$, then $G$ is not a complement of $F$. Hence suppose that $F$ is not contained in $G$. Since $G \neq 1$, $G$ cannot contain simultaneously all constant $L$-subsets and all characteristic functions of open sets in $\tau$. Then $F \vee G = H \neq 1$ and so $G$ is not a complement of $F$.

Remark 4.4.1. When $\tau = D$, the discrete topology on $X, W_{1D} = W_1(X)$, the collection of all weakly induced $L$-topologies on $X$. The family of all weakly induced $T_1$-$L$ topologies is defined by scott continuous functions where each scott continuous function is a characteristic function, is a sublattice of $W_1(X)$ and is a lattice isomorphic to the lattice of all $T_1$ topologies on $X$. The elements of this lattice are called crisp $T_1$ topologies.

Theorem 4.4.2. The lattice of weakly induced $T_1$-$L$ topologies $W_1(X)$ is not complemented.
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Proof. This follows from theorem 4.4.1.

Note 3.
Several types of $T_1$ topologies have complements in $\Lambda(X)$ [57].

Theorem 4.4.3. An induced $T_1$-$L$ topology has complement if the corresponding $T_1$ topology has complement.

Proof. Let $F$ be an induced $T_1$-$L$ topology. Since $F$ is induced, $F$ is the collection of all Scott continuous functions from $(X,F^*_c)$ to $L$. Let $F^*_c = \tau$. If $\tau$ has complement, there exists $\tau'$ such that $\tau \land \tau'$ equal to the cofinite topology and $\tau \lor \tau'$ equal to the discrete topology on $X$. Then $F \land \tau'$ is CFT and $F \lor \tau'$ is the discrete $L$-topology.