Chapter 3

Lattice of $T_1$-$L$ topologies

3.1 Introduction

In this chapter we investigate the lattice structure of the collection of all $T_1$-$L$ topologies on a given set $X$. In [30], Johnson studied the lattice structure of the set of all $L$-topologies on a given set $X$. It is quite natural to find sublattices in the lattice of $L$-topologies and study their properties. The collection of all $T_1$-$L$ topologies on a given set $X$ forms one of the sublattice of the lattice of $L$-topologies on $X$. One distinguishing feature between these two lattices is that the lattice of $L$-topologies is atomic while the collection of all $T_1$-$L$ topologies is not. Lattice of $T_1$-$L$ topologies is a complete sublattice of lattice of $L$-topologies. Also, the collection of

\begin{flushright}
Some results of this chapter are included in the following paper.
\end{flushright}
all $T_1$-$L$ topologies is not modular. In [64] Liu determined dual atoms in the lattice of $T_1$ topologies and Frolich [18] proved this lattice is dually atomic. However, we prove that the collection of all $T_1$-$L$ topologies has dual atoms if and only if $L$ has dual atoms and that the collection of all $T_1$-$L$ topologies is not dually atomic in general.

3.2 Preliminaries

Let $X$ be a non empty ordinary set and $L = L(\leq, \lor, \land, ^\prime)$ be a $F$-lattice, i.e, a completely distributive lattice with a smallest element $0$ and a largest element $1((0 \neq 1)$ and with an order-reversing involution $a \rightarrow a'(a \in L)$ [34]. Assume $L$ has more than two elements. An $L$-fuzzy subset on $X$ is a mapping $f : X \rightarrow L$. The family of all $L$-fuzzy subsets on $X$ is denoted by $L^X$. We denote the constant function in $L^X$ taking the value $\alpha \in L$ by $\underline{\alpha}$. Here we call $L$-fuzzy subsets as $L$-subsets and $F \subseteq L^X$ is called an $L$-topology in the sense of Chang [13] and Goguen [23] as in [34], if

\begin{enumerate}[(i)]
  \item $0, 1 \in F$,
  \item $f, g \in F \Rightarrow f \land g \in F$,
  \item $f_i \in F$ for each $i \in I \Rightarrow \bigvee_{i \in I} f_i \in F$.
\end{enumerate}

\textbf{Definition 3.2.1.} [44] A fuzzy point $x_\lambda$ in a set $X$ is a fuzzy set in
X defined by
\[ x_\lambda(y) = \begin{cases} 
\lambda & \text{if } y = x \\
0 & \text{if } y \neq x 
\end{cases} \]
where \( 0 < \lambda \leq 1 \)

In an \( L \)-topological space \( x_\lambda \) is called an \( L \)-point.

**Definition 3.2.2.** [44] An \( L \)-topological space \((X, F)\) is said to be a \( T_1 - L \) topological space if for every two distinct fuzzy points \( x_p \) and \( y_q \), with distinct support, there exists an \( f \in F \) such that \( x_p \in f \) and \( y_q \notin f \) and another \( g \in F \) such that \( y_q \in g \) and \( x_p \notin g \), \( \forall p, q \in L \setminus \{0\} \).

**Remark 3.2.1.** We take the definition of \( L \)-points \( x_\lambda, 0 < \lambda \leq 1 \) so as to include all crisp singletons. Hence every crisp \( T_1 \) topology is a \( T_1-L \) topology by identifying it with its characteristic function. If \( \tau \) is any topology on a finite set, then \( \tau \) is \( T_1 \), if and only if it is discrete. However, the same is not true in \( L \)-topology.

**Example 3.2.1.** Let \( X = \{a, b, c\} \) and \( L = \{0, \alpha, \beta, 1\} \) be the diamond lattice, then \( F = \{0, \mu_{\{a\}}, \mu_{\{b\}}, \mu_{\{c\}}, \mu_{\{a,b\}}, \mu_{\{a,c\}}, \mu_{\{b,c\}}, 1\} \) is a \( T_1-L \) topology. Let \( a_\lambda, b_\lambda, c_\lambda, 0 \neq \lambda \in L \) are \( L \)-points. The complements of \( a_\lambda, b_\lambda, c_\lambda \) are not open in \( F \) so that \( a_\lambda, b_\lambda, c_\lambda \) are not closed.

**Definition 3.2.3.** [22] An element \( p \in L \) is called prime if \( p \neq 1 \) and whenever \( a, b \in L \) with \( a \wedge b \leq p \), then \( a \leq p \) or \( b \leq p \). The set of all prime elements of \( L \) will be denoted by \( P_r(L) \).

**Definition 3.2.4.** [73] Scott topology on \( L \) is the topology generated by the sets of the form \( \{t \in L : t \nleq p\} \), where \( p \in P_r(L) \). Let \((X, \tau)\) be a topological space and \( f : (X, \tau) \to L \) be a function where \( L \) has
its Scott topology, we say that $f$ is Scott continuous if for every $p \in P_r(L)$, $f^{-1}(t \in L : t \not\leq p) \in \tau$. (Some authors used the notation $f^-$ instead of $f^{-1}$, for example in [48], [49], [50], [51]).

**Remark 3.2.2.** When $L = [0, 1]$, the Scott topology coincides with the topology of topologically generated spaces of Lowen [35]. Every Scott continuous function need not be lower semi continuous.

**Example 3.2.2.** Suppose $k$ is a large positive integer. Let $D_k$ be the set of all devisors of $k$. Give the order $a/b$ in $D_k$; $a, b \in D_k$ such that $a \wedge b = \gcd(a, b)$, $a \vee b = \text{lcm}(a, b)$ and the corresponding Scott topology. Consider $X = D_k$ with the Scott topology, $L = D_k$ Then $f : X \to L$ defined as $f(x) = x$ is Scott continuous since $f^-(p, \infty) = (p, \infty)$, which is open in $X$ for any prime $p$. But not lower semi continuous since $f^-(n, \infty) = (n, \infty)$, where $n$ is not a prime is not open in $X$.

**Remark 3.2.3.** The set $\omega_L(\tau) = \{ f \in L^X; f : (X, \tau) \to L \text{ is scott continuous } \}$ is an $L$-topology. An $L$-topology $F$ on $X$ is called an induced $L$-topology if there exists a topology $\tau$ on $X$ such that $F = \omega_L(\tau)$. If $\tau$ is a $T_1$ topology, $\omega_L(\tau)$ is a $T_1-L$ topology.

**Note 1.**
A lattice $L$ is modular if and only if, it has no sublattice isomorphic to $N_5$, where $N_5$ is a standard non modular lattice [20].

**Definition 3.2.5.** [34] An element of a lattice $L$ is called an atom if it is the minimal element of $L \setminus \{0\}$.

**Definition 3.2.6.** [34] An element of a lattice $L$ is called a dual atom if it is the maximal element of $L \setminus \{1\}$.
3.3 Lattice of $T_1$-L topologies

For any set $X$, the set $\Omega(X)$ of all $T_1$-L topologies on $X$ forms a lattice with natural order of set inclusion. The least upper bound of a collection of $T_1$-L topologies belonging to $\Omega(X)$ is the $T_1$-L topology generated by their union and the greatest lower bound is their intersection. The smallest $T_1$-L topology is the cofinite topology denoted by 0 and largest $T_1$-L topology is the discrete $L$-topology denoted by 1.

**Theorem 3.3.1.** [18] The Ultra spaces on a set $E$ are exactly the topologies of the form $\mathcal{G}(x, \mathcal{U}) = \varphi(E - x) \cup \mathcal{U}$, where $x \in E$, $\mathcal{U}$ is an ultrafilter on $E$ not containing $\{x\}$.

**Theorem 3.3.2.** [62] Let $(X, F)$ and $(X, G)$ be two fuzzy topological spaces on $X$. Then $G$ covers $F$ if and only if $G = F(g)$ for every $g \in G - F$, where $F(g)$ is the simple extension of $F$ by $g$.

**Theorem 3.3.3.** The lattice $\Omega(X)$ is complete.

**Proof.** Let $S$ be a subset of $\Omega(X)$ and $G = \bigcap_{\delta \in S} \delta$. Then $G$ is a $T_1$-L topology and $G$ is the greatest lower bound of $S$. Since any join(resp. meet) complete lattice with a smallest (resp.largest) element is complete, $\Omega(X)$ is complete. 

**Note 2.**
Let CFT denote the crisp cofinite topology, where CFT $= \{\chi_A|A \text{ is a subset of } X \text{ whose complement is finite } \}$ together with $\emptyset$, $\chi_A$ is the characteristic function of $A$. 


Theorem 3.3.4. $\Omega(X)$ is not atomic.

Proof. Atoms in $\Omega(X)$ are the $T_1$-$L$ topologies generated by $\text{CFT} \cup \{x_\lambda\}$, $0 < \lambda \leq 1$, or $\text{CFT} \cup A$, $0 < \lambda < 1$, where $x_\lambda$ is an $L$-point. Let $P = \{f \in L^X : f(x) > 0 \text{ for all but finite number of points of } X\}$ together with $\emptyset$. Then $P$ is a $T_1$-$L$ topology and $P$ cannot be expressed as join of atoms. Hence $\Omega(X)$ is not atomic. \qed

Theorem 3.3.5. $\Omega(X)$ is not modular.

Proof. Let $x_1, x_2, x_3 \in X$ and $\alpha, \beta, \gamma \in (0, 1)$. Let $F$ be the $T_1$-$L$ topology generated by $\text{CFT} \cup \{f_1, f_2, f_3\}$ where $f_1, f_2, f_3$ are $L$ subsets defined by

$$f_1(y) = \begin{cases} \alpha & \text{when } y = x_1 \\ 0 & \text{when } y \neq x_1 \end{cases}$$

$$f_2(y) = \begin{cases} \alpha & \text{when } y = x_1 \\ \beta & \text{when } y = x_2 \\ \gamma & \text{when } y = x_3 \\ 0 & \text{when } y \neq x_1, x_2, x_3 \end{cases}$$

$$f_3(y) = \begin{cases} \beta & \text{when } y = x_2 \\ \gamma & \text{when } y = x_3 \\ 0 & \text{when } y \neq x_2, x_3 \end{cases}$$

Let $F_1$ be the $T_1$-$L$ topology generated by $\text{CFT} \cup \{f_1\}$. Let $F_2$ be the $T_1$-$L$ topology generated by $\text{CFT} \cup \{f_1, f_2\}$.
Let $F_3$ be the $T_1$-$L$ topology generated by $CFT \cup \{f_3\}$.

Then, we notice that $F_2 \lor F_3 = F$ and $F_1 \lor F_3 = F$ so that $\{CFT, F_1, F_2, F_3, F\}$ forms a sublattice of $\Omega(X)$ isomorphic to $N_5$, where $N_5$ is the standard non-modular lattice. Hence $\Omega(X)$ is not modular. \qed

**Theorem 3.3.6.** $\Omega(X)$ is not complemented.

**Proof.** Let $F$ be the $T_1$-$L$ topology generated by $CFT \cup \{x_\lambda\}$. Then $1$ is not a complement of $F$ since $F \land 1 \neq 0$. Let $H$ be any $T_1$-$L$ topology other than $1$, the discrete $L$-topology. If $F \subset H$, then $H$ cannot be the complement of $F$. Suppose that $F \not\subseteq H$, then $H$ cannot contain simultaneously all characteristic functions of open sets in $\tau$ and all constant $L$-subsets. Then the set $K = \{k : k$ is a function from $(X, \tau)$ to $L$ and $k \not\in H\}$ is non empty. Let $F \lor H = G$ and $G$ has the subbase $\{f \land h | f \in F, h \in H\}$. Then $G$ cannot be equal to the discrete $L$-topology, since there exists at least one subset of $K$ which is not contained in $G$. Hence $H$ is not a complement of $F$. \qed

**Theorem 3.3.7.** If $L$ has dual atoms, then $\Omega(X)$ has dual atoms.

**Proof.** Case 1.

Let $X$ be a non empty set and $L$ be a finite pseudo complemented chain.

Let $\tau$ be a dual atom in the lattice of $T_1$ topologies on $X$. Then by theorem 3.3.1, $\tau$ must be of the form $\mathcal{G}(a, \mathcal{U}) = \wp(X - a) \cup \mathcal{U}$, where $a \in X, \mathcal{U}$ is a non principal ultrafilter not containing $\{a\}$. Then $\omega_{\mathcal{A}}(\tau) = \{f | f : (X, \tau) \rightarrow L$ is a scott continuous function$\}$. Then $a_\lambda \not\in \omega_{\mathcal{A}}(\tau), \lambda \in L$. Let $\beta$ be the dual atom in $L$ and $F = \omega_{\mathcal{A}}(\tau) \lor a_\beta$ and then $F$ is the ultra $L$-topology $\mathcal{G}(a_\beta)$ in $\Omega(X)$ since the simple extension of $F$
by \(a_1\) is the discrete \(L\)-topology.

**Case 2.**

Let \(X\) be a non empty set and \(L\) is not a finite pseudo complemented chain.

Let \(\tau\) be a dual atom in the lattice of \(T_1\) topologies on \(X\). Then by theorem 3.3.1, \(\tau\) must be of the form \(\mathcal{G}(a, \mathcal{U}) = \varphi(X - a) \cup \mathcal{U}\), where \(a \in X\), \(\mathcal{U}\) is non principal ultrafilter not containing \(\{a\}\). Then \(\omega_{1L}(\tau) = \{f|f : (X, \tau) \to L\text{ is a scott continuous function}\}\). Then \(a_{\lambda} \notin \omega_{1L}(\tau), \lambda \in L\).

Let \(\beta_1, \beta_2, \ldots, \beta_m\) are dual atoms in \(L\) and \(F(\beta_i) = \omega_{1L}(\tau) \vee a_{\beta_1}, F(\beta_2) = \omega_{1L}(\tau) \vee a_{\beta_2}, \ldots, F(\beta_m) = \omega_{1L}(\tau) \vee a_{\beta_m}\). Let \(F_{\beta_j}\) is the \(L\)-topology generated by \((m - 1) F(\beta_i)\) from \(F(\beta_i), i = 1, 2, \ldots m, j = 1, 2, \ldots m, i \neq j\).

Then as in case 1. \(F_{\beta_j}\) is the ultra \(L\)-topology \(\mathcal{G}_{\beta_j}\) in \(\Omega(X)\) since the simple extension of \(F_{\beta_j}\) by \(a_{\beta_j}\) is the discrete \(L\)-topology.

In both cases since \(L\) has dual atoms, \(\Omega(X)\) has dual atoms. Hence the theorem. \(\Box\)

**Note 3.**

Let \(\tau\) be a dual atom in the lattice of \(T_1\) topologies on \(X\), \(\beta\) be the dual atom in \(L\) and \(A \subset X\) not in \(\tau\). Then \(\omega_{1L}(\tau) \vee \beta = \omega_{1L}(\tau) \vee \mu_{A}^{\beta}, \mu_{A}^{\beta}\) is defined by \(\mu_{A}^{\beta}(x) = \begin{cases} \beta & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}\)

**Theorem 3.3.8.** If \(\Omega(X)\) has dual atoms, then \(L\) has dual atoms.

**Proof. Case 1.**
Let $X$ be a nonempty set and $L$, a finite pseudocomplemented chain.

Suppose that $F$ is a dual atom in $\Omega(X)$. Then $F$ is of the form $\mathfrak{S}(a_\beta)$ and $\beta$ must be the dual atom in $L$. Otherwise there exists an element $G$ greater than $F$ and less than 1. Which is a contradiction to the hypothesis.

**Case 2.**
Let $X$ be a non-empty set and $L$ is not a finite pseudo complemented chain.

Suppose that $F$ is a dual atom in $\Omega(X)$. Then $F$ is of the form $\mathfrak{S}_{\beta_j}$ and $\beta_1, \beta_2, ..., \beta$ must be dual atoms in $L$. Otherwise there exists an element $G$ greater than $F$ and less than 1. Which is a contradiction to the hypothesis.

So in either case if $\Omega(X)$ has dual atoms, then $L$ has dual atoms. Hence the proof of the theorem is completed.

Combining theorem 3.3.7 and theorem 3.3.8, we have

**Theorem 3.3.9.** The lattice of $T_1$-$L$ topologies $\Omega(X)$ has dual atoms if and only if $L$ has dual atoms.

**Theorem 3.3.10.** $\Omega(X)$ is not dually atomic in general.

Proof. This follows from Theorem 3.3.7.