Chapter - 6

Reflection of Plane waves in Micropolar Thermoelastic solid half space with Impedance boundary condition

In this chapter, the governing differential equations of linear and homogeneous generalized micropolar thermoelasticity are formulated with in framework of Lord-Shulman [111] thermoelasticity. The solution of governing differential equations indicate the presence of four plane waves in which two are coupled longitudinal (CLD,CT) waves and two are coupled transverse (CTD,CTM) waves in the medium. The reflection problem at traction free thermally isolated/isothermal surface with impedance boundary condition is studied to get ratios of amplitudes and energy ratios. The amplitude ratios and energy ratios of reflected waves are calculated for relevant material parameters of an aluminium-epoxy composite. Effect of impedance parameters is depicted graphically on all reflected waves at each angle of incidence.

6.1 Basic equations

Following Eringen [200] and Nowacki [66], the fundamental equations for linear generalized micropolar thermoelastic half-space without external body and couple forces and sources of heat are:

(i) Equations of motion

\begin{align}
(\mu + \kappa) \nabla^2 \ddot{u} + (\lambda + \mu) \nabla (\nabla \cdot \ddot{u}) + \kappa \nabla \times \ddot{\phi} - \beta_1 \nabla T &= \rho \ddot{u}, \\
\gamma \nabla^2 \ddot{\phi} - 2\kappa \ddot{\phi} + (\alpha + \beta) \nabla (\nabla \cdot \ddot{\phi}) + \kappa \nabla \times \ddot{u} &= \rho j \ddot{\psi},
\end{align}

(ii) Heat equation

\begin{align}
(1 + \tau_0 \frac{\partial}{\partial t}) [\beta T_0 \nabla \ddot{u} + \rho C_E T] &= K \nabla^2 T,
\end{align}

(iii) Constitutive equations are

\begin{align}
\sigma_{ij} &= \kappa e_{ijk}(r_k - \phi_j) + \lambda e_{ik} \delta_{ij} + (2\mu + \kappa)e_{ij} - \beta_1 T \delta_{ij}, \\
m_{ij} &= \alpha \phi_{ik} \delta_{ij} + \beta \phi_{i,j} + \gamma \phi_{j,i}, \\
\rho S &= \beta_1 e_{ik} + \frac{\rho C_E}{T_0} T.
\end{align}
\[ r_i = \frac{1}{2}(\varepsilon_{ipq}u_{q,p}), \quad (6.7) \]
\[ e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad (6.8) \]

where, \( T = T^* - T_0 \), is increment in temperature assuming that \( \left| \frac{T}{T_0} \right| < 1 \), \( T^* \) is absolute temperature, \( T_0 \) is reference temperature, \( m_{ij} \) is components of couple stress and other symbols have their usual meanings as given in chapter 4. Consider the plane strain taking displacement components as \( \vec{u} = (u, 0, w) \) and microrotation components as \( \vec{\phi} = (0, \phi_2, 0) \). By Helmholtz representation of vectors the displacement components \( u \) and \( w \) are written as to \( q \) and \( \psi \) that is

\[
\begin{align*}
  u &= \frac{\partial q}{\partial x} - \frac{\partial \psi}{\partial z}, \\
  w &= \frac{\partial q}{\partial z} + \frac{\partial \psi}{\partial x}
\end{align*}
\]

With the use of equations (6.9) into equations (6.1) to (6.3), we have

\[ (c_1^2 + c_2^2)\nabla^2 q - \bar{\beta}^2 q - \bar{\beta} T - \bar{q} = 0, \quad (6.10) \]
\[ (1 + \tau_0 \frac{\partial}{\partial t})[ \beta T_0 \nabla^2 \dot{q} + \rho C_e \dot{T} ] = K\nabla^2 T, \quad (6.11) \]
\[ (c_2^2 + c_3^2)\nabla^2 \psi - c_3^2 \phi_2 - \psi = 0, \quad (6.12) \]
\[ c_4^2 \nabla^2 \phi_2 + \omega_0^2 \nabla^2 \psi - 2\omega_0^2 \phi_2 - \phi_2 = 0, \quad (6.13) \]

where, \( \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \), \( c_1^2 = \frac{\lambda + 2\mu}{\rho} \), \( c_2^2 = \frac{\mu}{\rho} \), \( c_3^2 = \frac{\kappa}{\rho} \), \( c_4^2 = \frac{\alpha + \beta}{\rho} \), \( \omega_0^2 = \frac{\kappa}{\rho} \), \( \bar{\beta} = \frac{\beta}{\rho} \).

The equations (6.10) and (6.11) are coupled in \( q, T \) (scalar potentials) and equations (6.12) and (6.13) are coupled in \( \phi_2 \) and \( \psi \) (vector potentials).

### 6.2 Plane wave propagation

To solve (6.10) and (6.11), considering the plane waves propagating in homogeneous isotropic micropolar thermoelastic half-space and taking the solution in the following form

\[ (q, T) = (\vec{q}, \vec{T}) e^{ik(\sin\theta z + \cos\theta - vt)} \quad (6.14) \]
where, $\bar{q}$ and $\bar{T}$ are the constants and $(\sin \theta, \cos \theta)$ represents the projection of wave normal onto xz plane. Inserting equation (6.14) into equations (6.10) to (6.11) we get

$$\omega^2 [v^2 - (c_i^2 + c_j^2)] \bar{q} - \bar{K} v^2 \bar{T} = 0,$$  \hspace{1cm} (6.15)

$$\beta T_0 \omega^2 \bar{q} + (\bar{K} - \rho C_E v^2) \bar{T} = 0,$$  \hspace{1cm} (6.16)

The non-trivial solution of the equations (6.15) and (6.16) requires

$$v^4 - (c_i^2 + c_j^2 + \bar{K} + \varepsilon) v^2 + (c_i^2 + c_j^2) \bar{K} = 0,$$  \hspace{1cm} (6.17)

where, $\tau_0' = \tau_0 + \frac{i}{\omega}, \bar{K} = \frac{K}{\tau_0}, \bar{\bar{K}} = -\frac{K}{\rho C_E}, \varepsilon = \frac{\beta^2 T_0}{\rho^2 C_E}$, and $\omega = kv$ is the circular frequency, $k$ is wavenumber and $v$ is the phase velocity of wave and

$$v_{1,2}^2 = \frac{1}{2} \left\{ (c_i^2 + c_j^2 + \bar{K} + \varepsilon) \pm \sqrt{(c_i^2 + c_j^2 - \bar{K})^2 + 2 \varepsilon (c_i^2 + c_j^2 - \bar{K}) + \varepsilon^2} \right\},$$  \hspace{1cm} (6.18)

Here, the two complex roots $v_{s}^2$, ($s = 1, 2$) of equation (6.18) correspond to CLD (coupled longitudinal displacement) wave and CT (coupled thermal) wave respectively. If it is expressed as $v_{s}^{-1} = V_{s}^{-1} - i \omega_{s}^{-1} q_s$, ($s = 1, 2$) then clearly $V_s$ and $q_s$ are propagation speeds and coefficients of attenuation of CLD and CT waves.

The solution of equations (6.12) to (6.13) are sought in following form

$$(\phi_2, \psi) = (\bar{\phi}_2, \bar{\psi}) e^{ik(\xi x + \zeta z - \omega t)},$$  \hspace{1cm} (6.19)

where, $V$ is phase velocity and $\bar{\phi}_2, \bar{\psi}$ are constants, Inserting equation (6.19) in equations (6.12) to (6.13) it is obtained

$$\omega^2 [V^2 - (c_i^2 + c_j^2)] \bar{\psi} - c_i^2 V^2 \bar{\phi}_2 = 0,$$  \hspace{1cm} (6.20)

$$\omega^2 \omega_0^2 \bar{\psi} + (c_i^2 \omega^2 + 2 \omega_0^2 V^2 - \omega^2 V^2) \bar{\phi}_2 = 0,$$  \hspace{1cm} (6.21)

The non-trivial solution of the equations (6.20) and (6.21) requires

$$(1 - \chi)V^4 + \left\{ -(c_i^2 + c_j^2 + c_k^2) + (c_i^2 + \frac{1}{2} c_j^2) \chi \right\} V^2 + c_i^2 (c_i^2 + c_j^2) = 0,$$  \hspace{1cm} (6.22)
Parfitt and Eringen [72] have shown that solution of equation (6.22) corresponds to “coupled transverse waves” with propagation velocity $V_{3,4}$ written as

$$V_{3,4} = \frac{1}{2(1-\chi)} \left\{ c_s^2 + c_s^2 + c_s^2 - (c_s^2 + c_s^2 / 2) \chi \pm \sqrt{(c_s^2 - c_s^2 - c_s^2 + (c_s^2 + c_s^2 / 2) \chi)^2 + 2c_s^2 c_s^2 \chi} \right\},$$  \hspace{1cm} (6.23)

where, $\chi = \frac{2\omega_0^2}{\omega^2}$ the wave with speed $V_3$ corresponds to CTD (coupled transverse displacement) wave and the wave with speed $V_4$ corresponds to CTM (coupled transverse microrotational) wave and thermal parameters do not have any influence on wave speeds.

### 6.3 Reflection at free surface

Considering propagation of CLD wave in a micropolar thermoelastic half-plane ($z \leq 0$) at tractions-free thermally insulated boundary at $z = 0$ and the negative $z$-axis is along the direction of increasing depth into the half-plane. Incident of CLD wave at $z = 0$ with angle of incidence ($\theta_0$) with the normal will results in four reflected waves namely CLD, CT, CTD, and CTM waves in the half space as depicted in Figure 6.1 The relevant displacement, temperature and concentration potentials are

$$q = X_0 \exp \left\{ ik \left( x \sin \theta_0 + z \cos \theta_0 - V_i t \right) \right\} + \sum_{s=1}^{2} X_s \exp \left\{ ik_s \left( x \sin \theta_s - z \cos \theta_s - V_s t \right) \right\},$$  \hspace{1cm} (6.24)

$$T = \zeta_s X_0 \exp \left\{ ik \left( x \sin \theta_0 + z \cos \theta_0 - V_i t \right) \right\} + \sum_{s=1}^{2} \zeta_s X_s \exp \left\{ ik_s \left( x \sin \theta_s - z \cos \theta_s - V_s t \right) \right\},$$  \hspace{1cm} (6.25)

$$\psi = \sum_{l=3}^{4} B_l \exp \left\{ ik_l \left( x \sin \theta_l - z \cos \theta_l - V_l t \right) \right\},$$  \hspace{1cm} (6.26)

$$\phi_s = \sum_{l=3}^{4} \eta_l B_l \exp \left\{ ik_l \left( x \sin \theta_l - z \cos \theta_l - V_l t \right) \right\},$$  \hspace{1cm} (6.27)

where, $V_i$ is the velocity of the incident CLD and reflected CLD wave, $V_2, V_3$ and $V_4$ are the velocities of the reflected CT, CTD and CTM wave, $k_s, (s=0, 1, 2)$ and $k_l, (l=3, 4)$ are complex wave numbers respectively and the coupling coefficient $\frac{\zeta_s}{k_s}, (s=1, 2)$ and $\frac{\eta_l}{k_l}, (l=3, 4)$ are as
\[
\frac{c_s}{k_s^2} = \frac{V_s^2 - (c_s^2 + c_i^2)}{\beta_1}, \quad \frac{\eta_i}{k_i^2} = \frac{V_i^2 - (c_s^2 + c_i^2)}{c_s^2},
\]

(6.28)

Following Singh and Kumar [214], Vinh and Hue [251] and Singh et al. [198], the boundary conditions at \( z = 0 \) are as

\[
\sigma_{zz} + \omega Z_3 w = 0, \quad \sigma_{xx} + \omega Z_2 \mu = 0, \quad m_{\gamma\gamma} + \omega Z_2 \phi_2 = 0, \quad \frac{\partial T}{\partial z} + hT = 0,
\]

(6.29)

where,

\[
\sigma_{xx} = \lambda \frac{\partial u}{\partial x} + (\lambda + 2\mu + \kappa) \frac{\partial w}{\partial z} - \beta T,
\]

(6.30)

\[
\sigma_{xx} = \mu \frac{\partial w}{\partial x} + (\mu + \kappa) \frac{\partial u}{\partial z} - \kappa \phi_2,
\]

(6.31)

\[
m_{\gamma\gamma} = \gamma \frac{\partial \phi_2}{\partial z},
\]

(6.32)

\( Z_1, Z_2, Z_3, \) are real valued impedance parameters, \( h \rightarrow 0, \ h \rightarrow \infty \) corresponds to thermally insulated and isothermal surface respectively. The potentials given by equations (6.24) to (6.27) must satisfy the boundary tractions (6.29). For each value of \( x \), the wave possesses the same phase and Snell’s law for the present problem is obtained as

\[
\frac{\sin \theta_0}{V_1} = \frac{\sin \frac{\theta}{V_s}}{(s = 1, 2, .., 4)},
\]

(6.33)

\[
k_3 V_i = \omega (say) (s = 1, 2, .., 4),
\]

(6.34)

Using potentials (6.24) to (6.27) in boundary conditions (6.29), and Snell’s law (6.33) it is obtained,

\[
\sum_{j=1}^{4} a_j R_j = b_i, \quad (i = 1, 2, .., 4),
\]

(6.35)

and,

\[
a_{i1} = -\lambda - (2\mu + \kappa) \cos^2 \theta = -\frac{\beta_1 \gamma_1}{k_2} - iZ_1 V_1 \cos \theta_0,
\]

\[
a_{i2} = \left[-\lambda - (2\mu + \kappa) \left(1 - \frac{V_2}{V_1} \right)^2 \sin^2 \theta_0 - \frac{\beta_2 \gamma_2}{k_2} \right] \left(\frac{V_1}{V_2} \right)^2 - iZ_3 \frac{V_2^2}{V_1} \sqrt{1-\left(\frac{V_2}{V_1}\right)^2} \sin^2 \theta_0,
\]
\( \alpha_{13} = (2\mu + \kappa) \left( \frac{V_1}{V_3} \right) \sin \theta_0 \sqrt{1 - \left( \frac{V_3}{V_1} \right)^2 \sin^2 \theta_0 + iZ_3 V_3 \sin \theta_0}, \)

\( \alpha_{14} = (2\mu + \kappa) \left( \frac{V_1}{V_4} \right) \sin \theta_0 \sqrt{1 - \left( \frac{V_4}{V_1} \right)^2 \sin^2 \theta_0 + iZ_4 V_4 \sin \theta_0}, \)

\( b_1 = \lambda + (2\mu + \kappa) \cos^2 \theta_0 + \frac{\beta \varepsilon_i}{k_i^2} - iZ_3 V_3 \cos \theta_0, \)

\( a_{21} = \frac{1}{2} (2\mu + \kappa) \sin 2\theta_0 + iZ_1 V_1 \sin \theta_0, \)

\( a_{22} = (2\mu + \kappa) \sin \theta_0 \left( \frac{V_1}{V_2} \right) \sqrt{1 - \left( \frac{V_2}{V_1} \right)^2 \sin^2 \theta_0 + iZ_1 V_1 \sin \theta_0}, \)

\( a_{23} = -\mu \sin^2 \theta_0 + (\mu + \kappa) \left( \frac{V_1}{V_3} \right)^2 \left[ 1 - \left( \frac{V_3}{V_1} \right)^2 \sin^2 \theta_0 \right] - \kappa \left( \frac{V_1}{V_3} \right)^2 \frac{\eta_3}{k_3^2} + iZ_1 \left( \frac{V_1}{V_3} \right) \sqrt{1 - \left( \frac{V_3}{V_1} \right)^2 \sin^2 \theta_0}, \)

\( a_{24} = -\mu \sin^2 \theta_0 + (\mu + \kappa) \left( \frac{V_1}{V_4} \right)^2 \left[ 1 - \left( \frac{V_4}{V_1} \right)^2 \sin^2 \theta_0 \right] - \kappa \left( \frac{V_1}{V_4} \right)^2 \frac{\eta_4}{k_4^2} + iZ_1 \left( \frac{V_1}{V_4} \right) \sqrt{1 - \left( \frac{V_4}{V_1} \right)^2 \sin^2 \theta_0}, \)

\( b_2 = \left[ (2\mu + \kappa) \cos \theta_0 - iZ_3 V_3 \right] \sin \theta_0, \)

\( a_{31} = 0, \ a_{32} = 0, \ a_{33} = -i\gamma \frac{\eta_3}{k_3^2} \left( \frac{V_1}{V_3} \right)^3 \sqrt{1 - \left( \frac{V_3}{V_1} \right)^2 \sin^2 \theta_0 + Z_3 V_3 \frac{\eta_3}{k_3^2} \left( \frac{V_1}{V_3} \right)^3}, \)

\( a_{34} = -i\gamma \frac{\eta_4}{k_4^2} \left( \frac{V_1}{V_4} \right)^3 \sqrt{1 - \left( \frac{V_4}{V_1} \right)^2 \sin^2 \theta_0 + Z_4 V_4 \frac{\eta_4}{k_4^2} \left( \frac{V_1}{V_4} \right)^3}, \)

\( b_3 = 0, \)

(a) For thermally insulated
\[ a_{41} = \frac{\xi_1}{k_1^2} \cos \theta_0, \quad a_{42} = \frac{\xi_2}{k_2^2} \left( \frac{V_1}{V_2} \right)^2 \sqrt{1 - \left( \frac{V_2}{V_1} \right)^2} \sin^2 \theta_0, \quad a_{43} = 0, \quad a_{44} = 0, \]
\[ b_x = \frac{\xi_1}{k_1^2} \cos \theta_0, \]

(b) For isothermal
\[ a_{41} = \frac{\xi_1}{k_1^2}, \quad a_{42} = \frac{\xi_2}{k_2^2} \left( \frac{V_1}{V_2} \right)^2, \quad a_{43} = 0, \quad a_{44} = 0, \]
\[ b_x = -\frac{\xi_1}{k_1^2}, \]

and \( R_1 = \frac{X_1}{X_0}, \quad R_2 = \frac{X_2}{X_0}, \quad R_3 = \frac{B_3}{X_0} \) and \( R_4 = \frac{B_4}{X_0} \) are the amplitude ratios of reflected CLD, CT, CTD and CTM waves, respectively.

6.4 Energy ratios

Now we consider the partition of energy among various reflected waves. Following Achenbach [11] and Gutenberg [24], the time average of power per unit area \( P^* \) at traction free surface of a micropolar thermoelastic half-plane with impedance boundary condition is
\[ P^* = \sigma_{xx} \ddot{u}_x + \sigma_{xx} \ddot{u}_y + m_y \ddot{\phi}_2, \quad (6.36) \]
Using equations (6.9) in equation (6.36) it is obtained
\[ \langle P^* \rangle = \sigma_{xx} \left( \frac{\partial^2 q}{\partial t \partial z} + \frac{\partial^2 \psi}{\partial t \partial x} \right) + \sigma_{xx} \left( \frac{\partial^2 q}{\partial t \partial x} - \frac{\partial^2 \psi}{\partial t \partial z} \right) + m_y \frac{\partial^2 \phi_2}{\partial t^2}, \quad (6.37) \]
Using potentials (6.24) to (6.27) in equation (6.37) and with the help of equation (6.30) to (6.32) it is obtained
\[ \langle P^*_{\text{inc. CLD}} \rangle = -\left[ \lambda + 2\mu + \kappa + \beta_1 \frac{\xi_1}{k_1^2} \right] \cos \theta_0 k_1 V_1 X_0^2, \quad (6.38) \]
\[ \langle P^*_{\text{ref. CLD}} \rangle = \left[ \lambda + 2\mu + \kappa + \beta_1 \frac{\xi_1}{k_1^2} \right] \cos \theta_0 k_1 V_1 X_1^2, \quad (6.39) \]
\[ \langle P^*_{\text{ref. CT}} \rangle = \left[ \lambda + 2\mu + \kappa + \beta_2 \frac{\xi_2}{k_2^2} \right] \cos \theta_0 k_2 V_2 X_2^2, \quad (6.40) \]
\[ \langle P_{\text{ref. CTD}}^* \rangle = \left[ \mu + \kappa - \kappa \frac{\eta_3}{k_3^2} - \gamma k_3^2 \left( \frac{\eta_3}{k_3^2} \right)^2 \right] \cos \theta_3 k_3^2 V_3 B_3^2, \]  

(6.41)

\[ \langle P_{\text{ref. CTM}}^* \rangle = \left[ \mu + \kappa - \kappa \frac{\eta_4}{k_4^2} - \gamma k_4^2 \left( \frac{\eta_4}{k_4^2} \right)^2 \right] \cos \theta_4 k_4^2 V_4 B_4^2, \]  

(6.42)

Making use of Snell’s law (6.33) in equations (6.38) and (6.42), energy ratios \( |E_1| \) and \( |E_2| \) of reflected coupled longitudinal, CLD and CT wave and energy ratios \( |E_3| \) and \( |E_4| \) of reflected coupled transverse waves CTD and CTM wave are obtained as

\[ |E_1| = \frac{\langle P_{\text{ref. CLD}}^* \rangle}{\langle P_{\text{inc. CLD}}^* \rangle} = \frac{X_1^2}{X_0^2}, \]  

(6.43)

\[ |E_2| = \frac{\langle P_{\text{ref. CT}}^* \rangle}{\langle P_{\text{inc. CLD}}^* \rangle} = -\frac{\left[ \lambda + 2 \mu + \kappa + \beta \frac{\xi_1}{k_2^2} \right]}{\lambda \cos \theta_2 \left( \frac{V_2}{V_1} \right)^2 \sin^2 \theta_0 \left( \frac{V_1}{V_2} \right)^3 \left( \frac{X_2}{X_0} \right)^2}, \]  

(6.44)

\[ |E_3| = \frac{\langle P_{\text{ref. CTD}}^* \rangle}{\langle P_{\text{inc. CLD}}^* \rangle} = -\frac{\left[ \mu + \kappa - \kappa \frac{\eta_3}{k_3^2} - \gamma k_3^2 \left( \frac{\eta_3}{k_3^2} \right)^2 \right]}{\lambda \cos \theta_3 \left( \frac{V_3}{V_1} \right)^2 \sin^2 \theta_0 \left( \frac{V_1}{V_3} \right)^3 \left( \frac{B_3}{X_0} \right)^2}, \]  

(6.45)

\[ |E_4| = \frac{\langle P_{\text{ref. CTM}}^* \rangle}{\langle P_{\text{inc. CLD}}^* \rangle} = -\frac{\left[ \mu + \kappa - \kappa \frac{\eta_4}{k_4^2} - \gamma k_4^2 \left( \frac{\eta_4}{k_4^2} \right)^2 \right]}{\lambda \cos \theta_4 \left( \frac{V_4}{V_1} \right)^2 \sin^2 \theta_0 \left( \frac{V_1}{V_4} \right)^3 \left( \frac{B_4}{X_0} \right)^2}, \]  

(6.46)

**6.5 Particular cases**

**6.5.1 Thermoelastic case with impedance:** If micropolar parameters are neglected (i.e. \( \alpha = \beta = \gamma = \kappa = 0 \)) the corresponding field equations for isotropic thermo elastic half space with impedance boundary reduces as.

Equation motion (6.1) for this case reduces to

\[ \mu \nabla^2 \dddot{u} + (\lambda + \mu) \nabla (\nabla \dddot{u}) - \beta \nabla T = \rho \dddot{u}, \]  

(6.47)

\[ \mu \nabla^2 \dddot{\psi} - \rho \dddot{\psi} = 0, \]  

(6.48)
Heat equation (6.3) reduces to
\[
(1 + \tau_0 \frac{\partial}{\partial t})[\beta T_0 \nabla \mathbf{u} + \rho C_v \mathbf{T}] = K \nabla^2 \mathbf{T},
\]  
(6.49)

For this case equations (6.15) to (6.16) and (6.20) reduces to
\[
\omega^2 [v^2 - c_s^2] \bar{q} - \bar{B}_1 v^2 \bar{T} = 0,
\]  
(6.50)
\[
\beta_1 T_0 \omega^2 \bar{q} + (\bar{K} - \rho C_v v^2) \bar{T} = 0,
\]  
(6.51)
\[
(\nabla^2 - c_s^2) \bar{\psi} = 0,
\]  
(6.52)

The non trivial solution of equation (6.50) and (6.51) require
\[
v^4 - (c_i^2 + \bar{K} + \varepsilon)v^2 + \bar{K}c_i^2 = 0,
\]  
(6.53)

From equation (6.53), it is obtained
\[
v_{1,2}^3 = \frac{1}{2} \left\{ (c_i^2 + \bar{K} + \varepsilon) \pm \sqrt{(c_i^2 + \bar{K} + \varepsilon)^2 - 4\bar{K}c_i^2} \right\},
\]  
(6.54)

The real parts \((V_1 = \text{Re}(v_1), \text{ and } V_2 = \text{Re}(v_2))\) of the two roots of equation (6.54) corresponds to \(P\) and \(T\) waves speed. Roots of equation (6.52) indicate the existence of \(SV\) waves with speed \(v_3 = \sqrt{\frac{\mu}{\rho}}, \text{ here } \beta_i = (3\lambda + 2\mu)\alpha_i.\)

In this case the for incident \(P\) wave, there will be \(P\), \(T\) and \(SV\) waves reflected in the half-plane \((z \leq 0)\). The appropriate potentials are taken as

\[q = X_0 \exp \{ik_i (x \sin \theta_0 + z \cos \theta_0 - V_i t)\} + \sum_{s=1}^3 X_s \exp \{ik_s (x \sin \theta_s - z \cos \theta_s - V_s t)\},\]  
(6.55)
\[T = F_i^* X_0 \exp \{ik_i (x \sin \theta_0 + z \cos \theta_0 - V_i t)\} + \sum_{s=1}^3 F_s^* X_s \exp \{ik_s (x \sin \theta_s - z \cos \theta_s - V_s t)\},\]  
(6.56)
\[\psi = B_i e^{ik_i (x \sin \theta_i, x \cos \theta_i, z - V_i t)},\]  
(6.57)

and, \[\frac{F_i^*}{k_s} = \frac{V_i^2 - c_i^2}{\beta_i}, \text{ (s = 1, 2)},\]

For this case boundary conditions (6.29) at \(z = 0\) reduces as
\[
\sigma_{zz} + \omega Z_3 w = 0, \quad \sigma_{zx} + \omega Z_3 u = 0, \quad \frac{\partial T}{\partial z} + hT = 0,
\]
where,
\[
\sigma_{zz} = \lambda \frac{\partial u}{\partial x} + (\lambda + 2\mu) \frac{\partial w}{\partial z} - \beta T = \lambda \left( \frac{\partial^2 q}{\partial x^2} + \frac{\partial^2 q}{\partial z^2} \right) + 2\mu \frac{\partial^2 q}{\partial z^2} + 2\mu \frac{\partial^2 \psi}{\partial x \partial z} - \beta T,
\]
\[
\sigma_{zx} = \mu \frac{\partial w}{\partial x} + \mu \frac{\partial u}{\partial z} = 2\mu \frac{\partial^2 q}{\partial x^2} + \mu \left( \frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial z^2} \right).
\]

Using potentials (6.55) to (6.57) in boundary conditions (6.58) and using Snell’s law (6.33) and (6.34), the equation (6.35) reduces in this case to
\[
\sum_{m=1}^{3} a_{im}^* R_m^* = b_i^*, \quad (i = 1, 2, 3),
\]
where,
\[
a_{11}^* = \lambda + 2\mu \cos^2 \theta_0 + \frac{\beta F_{11}^*}{k_1^2} - iV_i Z_3 \cos \theta_0,
\]
\[
a_{12}^* = \left[ \lambda + 2\mu \left( 1 - \left( \frac{V_2}{V_1} \right)^2 \sin^2 \theta_0 \right) + \frac{\beta F_{12}^*}{k_2^2} \right] \left( \frac{V_1}{V_2} \right)^2 - iZ_3 \frac{V_1^2}{V_2} \sqrt{1 - \left( \frac{V_2}{V_1} \right)^2} \sin^2 \theta_0,
\]
\[
a_{13}^* = -2\mu \sin \theta_0 \frac{V_1}{V_3} \sqrt{1 - \left( \frac{V_2}{V_1} \right)^2} \sin^2 \theta_0 + iZ_3 V_1 \sin \theta_0,
\]
\[b_1^* = -\left[ \lambda + 2\mu \cos^2 \theta_0 + \frac{\beta F_{11}^*}{k_1^2} + iV_i Z_3 \cos \theta_0 \right],
\]
\[a_{21}^* = 2\mu \sin \theta_0 \cos \theta_0 - iZ_1 V_1 \sin \theta_0,
\]
\[a_{22}^* = 2\mu \sin \theta_0 \frac{V_1}{V_2} \sqrt{1 - \left( \frac{V_2}{V_1} \right)^2} \sin^2 \theta_0 - iZ_1 V_1 \sin \theta_0,
\]
\[a_{23}^* = \mu \left( \frac{V_1}{V_3} \right)^2 \left[ 1 - 2 \left( \frac{V_3}{V_1} \right)^2 \sin^2 \theta_0 \right] - iZ_1 \frac{V_1^2}{V_3} \sqrt{1 - \sin^2 \theta_0} \left( \frac{V_2}{V_1} \right)^2,
\]
\[b_2^* = 2\mu \sin \theta_0 \cos \theta_0 + iZ_1 V_1 \sin \theta_0,
\]
(a) thermally insulated case
\[a_{31}^* = \frac{F_{11}^*}{k_1^2} \cos \theta_0, \quad a_{32}^* = \frac{F_{12}^*}{k_2^2} \left( \frac{V_1}{V_2} \right)^3 \sqrt{1 - \left( \frac{V_2}{V_1} \right)^2} \sin^2 \theta_0, \quad a_{33}^* = 0,\]
\[ b^*_3 = \frac{F^*_1}{k_1^2} \cos \theta_0, \]

(b) isothermal case

\[ a^*_{31} = \frac{F^*_1}{k_1^2}, \quad a^*_{32} = \frac{F^*_1}{k_2^2} \left( \frac{V_1}{V_2} \right)^2, \quad a^*_{33} = 0, \]

\[ b^*_3 = -\frac{F^*_1}{k_1^2}, \]

and \( R_1^* = \frac{X_1}{X_0}, R_2^* = \frac{X_2}{X_0} \) and \( R_3^* = \frac{B}{X_0} \) are reflection coefficient of reflected \( P \) wave, reflected \( T \) wave and reflected \( SV \) waves, respectively.

Using potentials (6.55) to (6.77), equation (6.59) to (6.60) and equation (6.9) in equation (6.36), for this case the instantaneous rate of power per unit area \( \langle P^* \rangle \) reduces as

\[
\langle P^*_{inc.\_P} \rangle = (\lambda + 2\mu + \beta_1 \frac{F^*_1}{k_1^2}) \cos \theta_0 k_1^4 V_1 X_0^2, \tag{6.62}
\]

\[
\langle P^*_{ref.\_P} \rangle = -(\lambda + 2\mu + \beta_1 \frac{F^*_1}{k_1^2}) \cos \theta_1 k_1^4 V_1 X_1^2, \tag{6.63}
\]

\[
\langle P^*_{ref.\_T} \rangle = -(\lambda + 2\mu + \beta_1 \frac{F^*_1}{k_1^2}) \cos \theta_2 k_2^4 V_2 X_2^2, \tag{6.64}
\]

\[
\langle P^*_{ref.\_SV} \rangle = -\mu \cos \theta_3 k_3^4 V_3 B_3^2, \tag{6.65}
\]

using Snell’s law (6.33) and (6.34) in equation (6.62) to (6.65), the energy ratios of reflected waves expressed in equation (6.43) to (6.46) reduces for this case as

\[
|E_1| = \frac{\langle P^*_{ref.\_P} \rangle}{\langle P^*_{inc.\_P} \rangle} = -\left| R_1^* \right|^2, \tag{6.66}
\]

\[
|E_2| = \frac{\langle P^*_{ref.\_T} \rangle}{\langle P^*_{inc.\_P} \rangle} = \frac{-\left(\lambda + 2\mu + \beta_1 \frac{F^*_1}{k_1^2}\right) \sqrt{1 - \left(\frac{V_2}{V_1}\right)^2 \sin^2 \theta_0}}{\left(\lambda + 2\mu + \beta_1 \frac{F^*_1}{k_1^2}\right) \cos \theta_0} \left(\frac{V_1}{V_2}\right)^3 \left| R_2^* \right|^2, \tag{6.67}
\]
\[ |E_1| = \frac{\langle P_{\text{ref, sv}}^* \rangle}{\langle P_{\text{inc, r}}^* \rangle} = \frac{-\mu \sqrt{1 - \left(\frac{V_3}{V_1}\right)^2 \sin^2 \theta_0}}{(\lambda + 2\mu + \beta_1 \frac{F_1^*}{k_1^2}) \cos \theta_0} \left(\frac{V_1}{V_3}\right)^3 |R_1^*|^2. \]  

(6.68)

The equations (6.66) to (6.68) gives the energy ratios \(|E_1|, |E_2|\) and \(|E_3|\) of reflected \(P\), \(T\) and \(SV\) waves.

**6.5.2 Thermoelastic case:** If micropolarity and impedance are neglected (i.e. \(\alpha = \beta = \gamma = \kappa = 0\), \(Z_1 = Z_2 = Z_3 = 0\)) here, \(\beta_1 = (3\lambda + 2\mu)\alpha_i\) and the fundamental equations (6.47) to (6.57) for isotropic thermoelastic solid half space without impedance are same as in case 6.5.1.

In this case for the incident \(P\) wave, there will be \(P\), \(T\) and \(SV\) waves reflected in the half-plane \((z \leq 0)\). The relevant potentials are taken as same as in equation (6.55) to (6.57) and the boundary conditions at \(z = 0\) reduces to

\[ \sigma_{zz} = 0, \sigma_{zx} = 0, \frac{\partial T}{\partial z} + hT = 0, \]  

(6.69)

Using potentials (6.55) to (6.57) in boundary conditions (6.70) and using Snell’s law (6.33) and (6.34), the equation (6.35) reduces to

\[ \sum_{j=1}^{3} a'_i R'_j = b'_i, \quad (i = 1, 2, 3), \]  

(6.70)

where, \(a'_{11} = \lambda + 2\mu \cos^2 \theta_0 + \frac{\beta_1 F_1^*}{k_1^2}\),

\[ a'_{12} = \left[ \lambda + 2\mu \left\{ 1 - \left(\frac{V_2}{V_1}\right)^2 \sin^2 \theta_0 \right\} + \frac{\beta_1 F_2^*}{k_2^2} \right] \left(\frac{V_1}{V_2}\right)^2, \]

\[ a'_{13} = -2\mu \sin \theta_0 \frac{V_1}{V_3} \sqrt{1 - \left(\frac{V_3}{V_1}\right)^2 \sin^2 \theta_0}, \]

\[ b'_1 = -[\lambda + 2\mu \cos^2 \theta_0 + \frac{\beta_1 F_1^*}{k_1^2}], \]

\[ a'_{21} = 2\mu \sin \theta_0 \cos \theta_0, \]
\[ a'_{22} = 2\mu \sin \theta \sqrt{\frac{V_1}{V_2}} \left[ 1 - \left( \frac{V_2}{V_1} \right)^2 \sin^2 \theta \right], \]

\[ a'_{33} = \mu \left( \frac{V_1}{V_3} \right)^2 \left[ 1 - 2 \left( \frac{V_3}{V_1} \right)^2 \sin^2 \theta \right], \]

\[ b'_{2} = \mu \sin 2\theta, \]

(a) thermally insulated case

\[ a'_{31} = \frac{F_1^*}{k_1^*} \cos \theta, \quad a'_{32} = \frac{F_2^*}{k_2^*} \left( \frac{V_1}{V_2} \right)^2 \sqrt{1 - \left( \frac{V_2}{V_1} \right)^2 \sin^2 \theta}, \quad a'_{33} = 0, \]

\[ b'_{3} = \frac{F_1^*}{k_1^*} \cos \theta, \]

(b) isothermal case

\[ a'_{31} = \frac{F_1^*}{k_1^*}, \quad a'_{32} = \frac{F_2^*}{k_2^*} \left( \frac{V_1}{V_2} \right)^2, \quad a'_{33} = 0, \]

\[ b'_{3} = -\frac{F_1^*}{k_1^*}, \]

\[ R'_1 = \frac{X_1}{X_0}, \quad R'_2 = \frac{X_2}{X_0}, \quad R'_3 = \frac{B_1}{X_0}, \]

\( R'_1, R'_2 \) and \( R'_3 \) are ratios of amplitude of reflected wave. The energy ratios of reflected waves for this case reduces as

\[ |E_1| = \frac{\langle P_{\text{ref}.r}^* \rangle}{\langle P_{\text{inc}.r} \rangle} = -\left| R'_1 \right|^2, \]

(6.71)

\[ |E_2| = \frac{\langle P_{\text{ref}.r}^* \rangle}{\langle P_{\text{inc}.r} \rangle} = \frac{-\left( \lambda + 2\mu + \beta \right) \frac{F_1^*}{k_1^*} \sqrt{1 - \left( \frac{V_2}{V_1} \right)^2 \sin^2 \theta}}{(\lambda + 2\mu + \beta) \frac{F_1^*}{k_1^*} \cos \theta \left( \frac{V_1}{V_2} \right)^3 \left| R'_2 \right|^2}, \]

(6.72)
\[ |E_s| = \frac{\langle P_{\text{ref, SV}} \rangle}{\langle P_{\text{inc, P}} \rangle} = \frac{-\mu \sqrt{1 - \left( \frac{V_3}{V_1} \right)^2 \sin^2 \theta_0}}{\left( \lambda + 2\mu + \beta \frac{F_1}{k_2} \right) \cos \theta_0} \left( \frac{V_1}{V_3} \right)^3 |R_s|^2. \] (6.73)

The mathematical expression given in equations (6.71) to (6.73) corresponds to the energy ratios \(|E_1|, |E_2|\) and \(|E_3|\) of reflected \(P, T\) and \(SV\) waves.

### 6.6 Discussion of results

To observe the influence of impedance on energy ratios of reflected CLD, CT, CTD and CTM waves, the following material constants of aluminium-epoxy composite at \(T_0 = 27^\circ C\) are taken from Gauthier [74]

\[
\begin{align*}
\lambda &= 7.59 \times 10^{11} \text{ dyne cm}^{-2}, & \mu &= 1.89 \times 10^{11} \text{ dyne cm}^{-2}, & \rho &= 2.7 \times 10^{3} \text{ gm cm}^{-3}, \\
C_E &= 0.2361 \text{ cal gm}^{-1} \text{ C}^{-1}, & \kappa &= 0.0149 \times 10^{11} \text{ dyne cm}^{-2}, & K &= 0.492 \text{ cal cm}^{-1} \text{ s}^{-1} \text{ C}^{-1}, \\
\alpha_1 &= 0.005, & \tau_0 &= 0.04 \text{s}, & \omega &= 5 \text{ Hz}.
\end{align*}
\]

Using above physical constants, energy ratios \(|E_1|, |E_2|, |E_3|\) and \(|E_4|\) of reflected CLD, CT, CTD and CTM waves, given by equations (6.43) to (6.46) are computed by Fortran Program for thermally insulated case. The values of \(|E_s|\), \((s = 1, 2, \ldots, 4)\) of different waves are plotted versus the incident angle \((0^\circ < \theta_0 \leq 90^\circ)\) in Figures 6.2(a) to 6.2(d) by solid lines (\(Z_1 = Z_2 = Z_3 = 0\)), dotted lines with circle (\(Z_1 = Z_2 = Z_3 = 10\)) and dotted lines with star (\(Z_1 = Z_2 = Z_3 = 50\)). For \(Z_1 = Z_2 = Z_3 = 0\) (without impedance parameters), the value \(|E_1|\) of CLD wave is near to one at \(\theta_0 = 1^\circ\). It decreases to 0.9792 at \(\theta_0 = 56^\circ\) and then it increases to value one \(\theta_0 = 90^\circ\). This variation is shown by solid line. For (\(Z_1 = Z_2 = Z_3 = 10\)) and (For \(Z_1 = Z_2 = Z_3 = 50\)), the change of energy ratio of reflected CLD wave are also similar with minimum values at \(58^\circ\) and \(73^\circ\), respectively. For \(Z_1 = Z_2 = Z_3 = 0\) (without impedance parameters), the maximum value of energy ratio of reflected CT wave is \(0.3895e-05\) at \(\theta_0 = 1^\circ\). It decreases to \(0.2941e-12\) at \(\theta_0 = 90^\circ\). This variation is shown by solid line. The variations of energy ratios \(|E_2|\) of reflected CT wave for (\(Z_1 = Z_2 = Z_3 = 10\)) and (For \(Z_1 = Z_2 = Z_3 = 50\)) are also similar. For \(Z_1 = Z_2 = Z_3 = 0\) (without impedance parameters), \(|E_3|\) and \(|E_4|\) of CTD and CTM waves is near to zero at \(\theta_0 = 1^\circ\) and \(\theta_0 = 90^\circ\). They attain their respective maximum values at \(\theta_0 = 79^\circ\) and \(\theta_0 = 57^\circ\). These
variations are shown by solid lines. For \((Z_1 = Z_2 = Z_3 = 10)\) and \((Z_1 = Z_2 = Z_3 = 50)\), the amount of change of energy ratio of reflected CTD and CTM waves are also similar. Comparing the variations shown by solid and dotted lines in Figures 6.2(a) to 6.2(d), the effect of impedance parameters are shown significantly at each incident angle except normal \((\theta_0 = 0^\circ)\) and grazing \((\theta_0 = 90^\circ)\) incidences. Influences of impedance parameters on reflected CLD, CTD and CTM waves is more at angles near to the grazing \((\theta_0 = 90^\circ)\) incidence, whereas it is more at angles near to the normal incidence \((\theta_0 = 0^\circ)\) for reflected CT wave.

For \(\theta_0 = 45^\circ\), the values of \(|E_s|\), \((s = 1, 2, \ldots, 4)\) of various waves reflected in the medium are plotted versus of impedance parameters \(Z_1, Z_2\) and \(Z_3\) in Figures 6.3(a) to 6.5(d). From these figures, the influence of impedance parameters on energy ratios of wave reflected in the medium is observed significantly at a particular value of incidence.
Figure 6.1. Geometry of the problem.
Figure 6.2(a) Variations of $|E_1|$ versus incident angle $\theta_0$ of CLD wave for three different sets of impedance parameters.

Figure 6.2(b) Variations of $|E_2|$ versus incident angle $\theta_0$ of CLD wave for three different sets of impedance parameters.
Figure 6.2(c) Variations of $|E_3|$ versus incident angle $\theta_o$ of CLD wave for three different sets of impedance parameters.

Figure 6.2(d) Variations of $|E_4|$ versus incident angle $\theta_o$ of CLD wave for three different sets of impedance parameters.
Figure 6.3(a) Variations of $|E_1|$ of reflected CLD waves versus the impedance parameter $Z_1$ when $\theta_0 = 45^\circ$.

Figure 6.3(b) Variations of $|E_2|$ of reflected CT waves versus the impedance parameter $Z_1$ when $\theta_0 = 45^\circ$.
Figure 6.3(c) Variations of $|E_3|$ of reflected CTD waves versus the impedance parameter $Z_1$ when $\theta_0 = 45^\circ$.

Figure 6.3(d) Variations of $|E_4|$ of reflected CTM waves versus the impedance parameter $Z_1$ when $\theta_0 = 45^\circ$. 
Figure 6.4(a) Variations of $|E_1|$ of reflected CLD waves versus the impedance parameter $Z_2$ when $\theta_0 = 45^\circ$.

Figure 6.4(b) Variations of $|E_2|$ of reflected CT waves versus the impedance parameter $Z_2$ when $\theta_0 = 45^\circ$. 
Figure 6.4(c) Variations of $|E_3|$ of reflected CTD waves versus the impedance parameter $Z_2$ when $\theta_o = 45^\circ$.

Figure 6.4(d) Variations of $|E_4|$ of reflected CTM waves versus the impedance parameter $Z_2$ when $\theta_o = 45^\circ$. 
Figure 6.5(a) Variations of $|E_1|$ of reflected CLD waves versus the impedance parameter $Z_3$ when $\theta_o = 45^\circ$.

Figure 6.5(b) Variations of $|E_2|$ of reflected CT waves versus the impedance parameter $Z_3$ when $\theta_o = 45^\circ$. 
Figure 6.5(c) Variations of $|E_3|$ of reflected CTD waves versus the impedance parameter $Z_3$ when $\theta_0 = 45^\circ$.

Figure 6.5(d) Variations of $|E_3|$ of reflected CTM waves versus impedance parameter $Z_3$ when $\theta_0 = 45^\circ$. 