CHAPTER 4

TUNING AND STABILITY

4.1 INTRODUCTION

In chapter 3, the dynamic model of PMSM with two inputs was derived and linearized. The model considered had dominant linear and quadratic terms. But core loss, stray loss and unmodelled dynamics which may be of order greater than two has not been included in the model. Consideration of core loss involves product of squares of both current and angular velocity. These together with the stray losses and unmodelled dynamics are best accounted for by tuning the linearizing transformations against an actual PM machine on the lines similar to those proposed by Levin and Narendra (1993).

Stability of an autonomous control affine system having second and higher order nonlinearity is to be considered. Stability of PMSM model will then follow as a special case.

4.2 TUNING

Tuning rules are derived for the linearizing transformations to account for core loss, by back propagation of error between the outputs of quadratic linearized system with a normal form output.
4.2.1 Core loss

The core loss or iron loss, caused by the permanent magnet (PM) flux and armature reaction flux, is a significant component in the total loss of a PMSM, and thus, it can have a considerable effect on the PMSM modeling and performance prediction. The net core loss $P_{lc}$ as given in Ramin Monajemy (2000), for PMSM is computed as follows:

$$
P_{lc} = \left[ \frac{1.5 \omega_r^2 (L_d l_d)^2}{R_c} + \frac{1.5 \omega_r^2 (\lambda_{af} + L_d l_d)^2}{R_c} \right]
$$  \hspace{1cm} (4.1)

where $R_c$ represents core loss resistance, $\lambda_{af}$ represents magnet flux linkage and $\omega_r$ denotes rotor electrical speed. The mechanical torque equation including core losses is given by

$$
T_e = J \frac{d\omega_r}{dt} + T_l + T_{lc}
$$  \hspace{1cm} (4.2)

$$
T_{lc} = \frac{P_{lc}}{\omega_r}
$$  \hspace{1cm} (4.3)

The equations of the PMSM including core loss in state-space form is expressed as

$$
\frac{d\theta}{dt} = \omega_r
$$  \hspace{1cm} (4.4)

$$
\frac{d\omega_r}{dt} = \frac{1.5 p}{J} \left( \lambda_{af} i_q + (L_d - L_q) i_d \right) - \frac{T_l}{J} - \frac{T_{lc}}{J}
$$  \hspace{1cm} (4.5)

$$
\frac{d i_d}{dt} = \frac{v_q}{L_q} - \frac{R}{L_q} i_q - \frac{L_d}{L_q} p \omega_r i_d - \frac{\lambda_{af} p}{L_q} \omega_r
$$  \hspace{1cm} (4.6)
It is evident that the model contains linear terms, second order and third order terms. So, if core loss is included in the analysis, tuning of transformation (derived for quadratic linearization) has to be done for nullifying the effect of third order terms and also higher order terms which may not be modeled.

4.2.2 Tuning Formula

To account for the core loss as given in Equation (4.1), stray loss and unmodelled dynamics, tuning of the quadratic linearization transformations denoted by $N_1$ and $N_2$ can be done as given in Levin and Narendra (1993). Figure 4.1 shows the block diagram for tuning of linearizing transformations. The coefficients in the linearizing transformations $N_1$ (input state feedback) and $N_2$ (coordinate transformation) corresponding respectively to Equations (3.5) and (3.4) are updated based on the error between the outputs of quadratic linearized system and a linear canonical form of the machine model (normal form). $v(m)$ and $y(m)$ represent the input and output at the $m^{th}$ iteration. The updation laws are derived in subsequent subsections.

\[
\frac{d i_d}{d t} = \frac{v_d}{L_d} - \frac{R}{L_d} i_d + \frac{L_q}{L_d} p \omega_r i_q
\]  

(4.7)

**Figure 4.1** Block diagram for tuning of transformation
Error (E) can be calculated as

\[ E = (\varepsilon^T \varepsilon) = [(y - \hat{y})^T (y - \hat{y})]^{1/2} \tag{4.8} \]

where \( \varepsilon = (\varepsilon_1 \ \varepsilon_2 \ \varepsilon_3 \ \varepsilon_4)^T \)

The error can be written as

\[ E = (\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_4^2)^{1/2} \tag{4.9} \]

To tune \( N_1 \) and \( N_2 \), it can be noticed that \( \phi^{(2)}(x) \) given by Equation (3.9) and \( \beta^{(1)}(x) \) given by Equation (3.11) are both functions of \( C_1 \). \[
\phi^{(2)}(x) = \begin{pmatrix}
0 \\
0 \\
C_1 x_3 x_4 \\
0
\end{pmatrix}
\] and \( \beta^{(1)}(x) = - \begin{pmatrix} C_1' x_4 \\
0 \\
C_1' x_3 \\
0
\end{pmatrix} \) are redefined so that \( \phi^{(2)}(x) \) and \( \beta^{(1)}(x) \) can be independently tuned by tuning \( C_1 \) and \( C_1' \) respectively and \( \alpha^{(2)}(x) \) is not varied.

**4.2.2.1 Updation of \( N_2 \) transformation coefficients**

Tuning of \( N_2 \) transformation implies the tuning of \( \phi^{(2)}(x) \). As \( \phi^{(2)}(x) \) is a function of only \( C_1 x_3 x_4 \), the coefficient \( C_1 \) has to be updated based on the error between the outputs of quadratic linearized system and normal form. The updation law is derived as follows.

\[
\Delta C_1 = \frac{\partial E}{\partial C_1} = \frac{\partial E}{\partial y} \frac{\partial y}{\partial C_1} \tag{4.10}
\]

From Equation (4.8), it is seen that for \( y = (y_1 \ y_2 \ y_3 \ y_4)^T \),
\[
\frac{\partial E}{\partial y_i} = \frac{\epsilon_i}{E}; \quad i = 1, 2, 3, 4 \tag{4.11}
\]

Hence

\[
\frac{\partial E}{\partial y} = \begin{pmatrix} \frac{\epsilon_1}{E} & \frac{\epsilon_2}{E} & \frac{\epsilon_3}{E} & \frac{\epsilon_4}{E} \end{pmatrix} \tag{4.12}
\]

\[
\therefore \Delta C_1 = \begin{pmatrix} \frac{\epsilon_1}{E} & \frac{\epsilon_2}{E} & \frac{\epsilon_3}{E} & \frac{\epsilon_4}{E} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ x_3 x_4 \\ 0 \end{pmatrix} = \frac{\epsilon_3}{E} x_3 x_4 \tag{4.13}
\]

Update of \( \phi^{(2)}(x) \) is done by using the formula:

\[
C_1(m) = C_1(m - 1) - \rho_1 \Delta C_1(m), \quad 0 < \rho_1 < 1 \tag{4.14}
\]

where \( m \) corresponds to the updating step and \( \rho_1 \) corresponds to the accelerating factor.

### 4.2.2.2 Updation of \( N_1 \) transformation coefficients:

Tuning of \( N_1 \) transformation is achieved by tuning of \( \beta^{(1)}(x) \). As \( \beta^{(1)}(x) \) is a function of \( C_1'x_3 \) and \( C_1'x_4 \), the coefficient \( C_1' \) has to be updated based on the error between the outputs of quadratic linearized system and normal form. The updation law is derived as follows.

\[
\Delta C_1' = \frac{\partial E}{\partial C_1'} = \frac{\partial E}{\partial y} \frac{\partial y}{\partial x} \frac{\partial x}{\partial u_1} \frac{\partial u_1}{\partial C_1'} \tag{4.15}
\]
where

\[
\frac{\partial E}{\partial y} = \begin{bmatrix} \varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \varepsilon_4 \\ E & E & E & E \end{bmatrix}
\]  \hspace{1cm} (4.16)

\[
\frac{\partial y}{\partial x} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & (1 + C_1 x_4) & C_1 x_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\]  \hspace{1cm} (4.17)

Also,

\[
\frac{\partial x}{\partial u_1} = \begin{pmatrix} 0 \\ -1/C_2 x_4 \\ 0 \\ -1/C_2 x_2 \end{pmatrix}
\]  \hspace{1cm} (4.18)

assuming that the steady state of the Simulink model of Equation (3.3) is reached within the tuning period. From Equation (3.5) considering \( u = (u_1, u_2)^T \),

\[
\frac{\partial u_1}{\partial C_1} = -v_1 x_4 - v_2 x_3
\]  \hspace{1cm} (4.19)

Thus

\[
\Delta C_1' = \frac{(v_1 x_4 + v_2 x_3)}{E} \left( \frac{\varepsilon_2}{C_2 x_4} + \frac{\varepsilon_3 C_1 x_3 + \varepsilon_4}{C_2 x_2} \right)
\]  \hspace{1cm} (4.20)

Updation of \( \beta^{(1)}(x) \) is done by using the formula:

\[
C_1'(m) = C_1'(m - 1) - \rho_2 \Delta C_1'(m); 0 < \rho_2 < 1
\]  \hspace{1cm} (4.21)

where \( m \) corresponds to the updating step and \( \rho_2 \) corresponds to the accelerating factor.
4.3 STABILITY ANALYSIS

In order to check the asymptotic stability of PMSM model, the stability analysis of a general system having second order nonlinearity and a system having third and higher order nonlinearities typically resulting from a quadratic linearization exercise is considered.

4.3.1 System having Second Order Nonlinearity

Consider the system

$$\dot{x} = Ax + Bu + f^{(2)}(x)$$  \hspace{1cm} (4.22)

where $x = [x_1 \ x_2 \ \ldots \ x_n]^T$; $u = [u_1 \ u_2 \ \ldots \ u_m]^T$; $f^{(2)}(x) = [f_1^{(2)}(x) \ f_2^{(2)}(x) \ \ldots \ f_i^{(2)}(x) \ \ldots \ f_n^{(2)}(x)]^T$. For autonomous system we put $u = 0$, which results in

$$\dot{x} = Ax + f^{(2)}(x)$$  \hspace{1cm} (4.23)

Assuming $A$ is strictly Hurwitz as given in Fang et al (1997) and Greenberg (1998), there exist, symmetric and positive – definite matrices $P$ and $Q$ which satisfy

$$A^TP + PA = -2Q$$  \hspace{1cm} (4.24)

Consider the following candidate Lyapunov function

$$V(x) = 1/2 \ x^TPx$$. The time derivative of $V$ is given by

$$\dot{V}(x) = -x^TPx + \frac{1}{2}x^TPf^{(2)}(x) + \frac{1}{2}(f^{(2)}(x))^TPx$$  \hspace{1cm} (4.25)
To verify that \( f^{(2)}(x) \) is locally Lipschitz (Fang et al. (1997)), \( f^{(2)}_i(x) \) is considered. Expanding by Taylor series as given in Greenberg (1998),

\[
f^{(2)}_i(x + \Delta x) = f^{(2)}_i(x) + \Delta x^T D f^{(2)}_i(x) + \frac{1}{2} \Delta x^T \{ D^2 f^{(2)}_i(x) \} \Delta x
\]  \tag{4.26}

where \( D f^{(2)}_i(x) \) is the gradient of \( f^{(2)}_i(x) \) evaluated at \( x \) and \( D^2 f^{(2)}_i(x) \) is the Hessian matrix.

\[
D f^{(2)}_i(x) = \nabla f^{(2)}_i(x) = \left( \frac{\partial f^{(2)}_i(x)}{\partial x_1}, \ldots, \frac{\partial f^{(2)}_i(x)}{\partial x_n} \right)^T \tag{4.27}
\]

\[
D^2 f^{(2)}_i(x) = H_i(x) = \begin{pmatrix}
\frac{\partial^2 f^{(2)}_i(x)}{\partial x_1^2} & \frac{\partial^2 f^{(2)}_i(x)}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f^{(2)}_i(x)}{\partial x_1 \partial x_n} \\
\frac{\partial^2 f^{(2)}_i(x)}{\partial x_2 \partial x_1} & \frac{\partial^2 f^{(2)}_i(x)}{\partial x_2^2} & \cdots & \frac{\partial^2 f^{(2)}_i(x)}{\partial x_2 \partial x_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial^2 f^{(2)}_i(x)}{\partial x_n \partial x_1} & \frac{\partial^2 f^{(2)}_i(x)}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f^{(2)}_i(x)}{\partial x_n^2}
\end{pmatrix} \tag{4.28}
\]

Assuming that the stability of the system is verified about the origin, without loss of generality, \( x = 0 \) and \( \Delta x = x \) can be substituted.

Equation (4.26) can be rewritten as

\[
f^{(2)}_i(x) = f^{(2)}_i(0) + x^T D f^{(2)}_i(0) + \frac{1}{2} x^T \{ D^2 f^{(2)}_i(0) \} x \tag{4.29}
\]
Notice that \( f_i^{(2)}(0) = 0 \) and \( f_i^{(2)}(x) \) being a quadratic function, 
\[
\nabla f_i^{(2)}(x) = \sum_{i=1}^{n} k_i x_i . \text{ Hence } D f_i^{(2)}(0) = 0. \text{ Hence } f_i^{(2)}(x) \text{ can be written as }
\]
\[
f_i^{(2)}(x) = \frac{1}{2} x^T \{ D^2 f_i^{(2)}(0) \} x = \frac{1}{2} x^T \{ H_i(0) \} x
\]
(4.30)

Taking norms on both sides of Equation (4.30), to get
\[
\| f_i^{(2)}(x) - f_i^{(2)}(0) \| = \frac{1}{2} \| H_i(0) \| \| x \|^2
\]
(4.31)

Assuming
\[
\text{Sup } \| x \| < \delta > 0 \frac{\| f_i^{(2)}(x) - f_i^{(2)}(0) \|}{\| x \|} = \frac{1}{2} \| H_i(0) \| \| x \| ;
\]
(4.32)

Equation (4.32) becomes
\[
\| f_i^{(2)}(x) - f_i^{(2)}(0) \| \leq \frac{1}{2} \| H_i(0) \| \delta
\]
(4.33)

Hence \( f_i^{(2)}(x) \) is locally Lipschitz for the assumption Sup \( \| x \| < \delta > 0 \). To verify that \( f^{(2)}(x) \) is locally Lipschitz.

\[
f^{(2)}(x) = \begin{pmatrix}
f_1^{(2)}(x) \\
f_2^{(2)}(x) \\
\vdots \\
f_i^{(2)}(x) \\
\vdots \\
f_n^{(2)}(x)
\end{pmatrix};
\]
Considering the valid $L_2$ norm for the determination of $\|f^{(2)}(x)\|$, we get

$$\|f^{(2)}(x)\| = \left[ \sum_{i=1}^{n} |f_i^{(2)}(x)|^2 \right]^{1/2}$$

$$\leq \left[ \sum_{i=1}^{n} \|f_i^{(2)}(x)\|^2 \right]^{1/2} \quad (4.34)$$

Substituting for $\|f_i^{(2)}(x)\|$ using Equation (4.33) and since $f_i^{(2)}(0) = 0$, Equation (4.34) becomes

$$\|f^{(2)}(x)\| \leq \|x\| \frac{\delta}{2} \left( \sum_{i=1}^{n} \|H_i(0)\|^2 \right)^{1/2} \quad (4.35)$$

Since $f^{(2)}(0) = 0$, we can write

$$\|f^{(2)}(x) - f^{(2)}(0)\| \leq \frac{\delta}{2} \|x\| \left( \sum_{i=1}^{n} \|H_i(0)\|^2 \right)^{1/2},$$

proving that $f^{(2)}(x)$ is locally Lipschitz.

Let $\frac{\delta}{2} \left( \sum_{i=1}^{n} \|H_i(0)\|^2 \right)^{1/2} = M$, then $\dot{V}(x) \leq -\|Q - MP\|\|x^2\|$

On that account, if the condition $Q - MP$ is positive definite ie.

$$x^TQx > x^T MPx \ \forall \ x \neq 0, \|x\| < \delta > 0,$$

$$Q > MP \quad (4.36)$$

is satisfied, $x = 0$ is an asymptotically stable equilibrium.
4.3.2 System having Higher Order Nonlinearity

The result is extended to the case of system

\[ \dot{x} = Ax + Bu + O(x)^{(3)} \]  \hspace{1cm} (4.37)

resulting after quadratic linearization. As discussed in chapter 3, approximate linearization, while removing nonlinearities up to a certain order, introduces nonlinearities into the system of order higher than those removed. Here \( O(x)^{(3)} \) represents higher order nonlinearities that are introduced into the system due to quadratic linearization.

To verify the stability of the autonomous system \( u = 0 \) is substituted in Equation (4.37) to get

\[ \dot{x} = Ax + O(x)^{(3)} \]  \hspace{1cm} (4.38)

Assuming \( A \) is strictly Hurwitz as given in Fang et al (1997), Greenberg (1998) and Zhu et al (2001), there exist, symmetric and positive–definite matrices \( P \) and \( Q \) which satisfy

\[ A^T P + PA = -2Q \]  \hspace{1cm} (4.39)

Consider the following candidate Lyapunov function

\[ V(x) = \frac{1}{2} x^T P x. \]  \hspace{1cm} The time derivative of \( V \) is given by

\[ \dot{V}(x) = -x^T Q x + \frac{1}{2} x^T P O(x)^{(3)} + \frac{1}{2} (O(x)^{(3)})^T P x \]  \hspace{1cm} (4.40)

\( O(x)^{(3)} \) involves an infinite series of vector polynomials. Hence its satisfaction of Lipschitz condition cannot be explicitly derived as in the case
of \( f^{(2)}(x) \). Hence it is assumed that \( O(x)^{(3)} \) is locally Lipschitz. The following theorem provides the result on the stability of the system given by Equation (4.38).

**Theorem 4.1:**

The system described by Equation (4.38) is asymptotically stable at \( x = 0 \), if \( O(x)^{(3)} \) is locally Lipschitz in a neighborhood containing the origin and \( Q > M_oP \) is satisfied for some constant \( M_o > 0 \).

**Proof:** It is assumed that \( O^{(3)}(x) \) is locally Lipschitz and hence there exists a positive constant \( M_o \) such that \( \|O(x_1)^{(3)} - O(x_2)^{(3)}\| \leq M_o\|x_1 - x_2\| \) for \( x_1, x_2 \) in a neighborhood \( R_o \) containing the origin. If \( x \in R_o \), then the time derivative of \( V(x) \) is given by

\[
\dot{V}(x) \leq -\|Q - M_oP\|\|x^2\| \quad (4.41)
\]

On that account, if the condition if the condition \( Q - M_oP \) is positive definite i.e. \( x^TQx > x^TM_oPx \) \( \forall x \neq 0 \)

\[
Q > M_oP \quad (4.42)
\]

is satisfied, \( x = 0 \) is an asymptotically stable equilibrium. Hence the result.

### 4.4 SUMMARY

In this chapter, an empirical approach in the form of tuning of the transformation parameters is presented by comparing the output of the linearized PMSM model with a normal form output. This is to account for the
core loss, unmodelled dynamics and third and higher order nonlinearities introduced into the PMSM model due to quadratic linearization.

Stability analysis is also presented for a control affine system containing only second order nonlinearity and the result is extended for a system containing third and higher order terms. This condition typically occurs when quadratic linearization is applied. Stability analysis of PMSM model follows as a special case.