CHAPTER 3

GENERALIZED QUADRATIC LINEARIZATION OF MACHINE MODELS

3.1 INTRODUCTION

In chapter 2, the explicit conditions for quadratic linearization of control affine systems with a single input are derived. Quadratic linearization being an approximate technique leaves behind third and higher order terms in the system in general (as given in Equation (2.4)).

In practical applications, such as, in PMSM model, this may not be desirable. For, when the PMSM model is quadratic linearized, third and higher order terms are introduced into the system by the process of quadratic linearization even though the system is not assumed to possess such higher order nonlinearity originally. This problem is approached in two ways. The first is theoretical and formal which is taken up in this chapter. The second approach is empirical, applicable to PMSM model and is taken up in the following chapter.

In this chapter, a new concept called 'generalized quadratic linearization' which seeks to remove the second order nonlinearity in the model without introducing third and higher order nonlinearities in the process is introduced. This is in contrast to the existing quadratic linearization techniques as given in Kang (1994a), Devanathan (2001 and 2004), Kang and
Krener (1990 and 1992) which introduce third and higher order terms in the process of removing second order terms. The generalized quadratic linearization, being a stronger condition than quadratic linearization, imposes additional constraints on the quadratic polynomials of the system than those imposed due to quadratic linearization alone. Sufficient conditions on the quadratic polynomials together with coordinate and state feedback transformations are derived for a class of systems for which generalized quadratic linearization is applicable. In particular, the results are shown to apply to the induction motor and permanent magnet synchronous motor models. The results can be extended to other machine models as well.

To summarize the rest of the chapter, in section 3.2, quadratic linearization of PMSM is given. In section 3.3, homological equations under multiple inputs for arbitrary order linearization are introduced as background. Section 3.4 helps to simplify the homological equations for quadratic linearization under multiple inputs, thus paving the way for the central result on generalized quadratic linearization presented in section 3.5. In section 3.6, generalized quadratic linearization is applied to two machine models viz. induction motor and PMSM models. In section 3.7, the chapter is summarized.

3.2 QUADRATIC LINEARIZATION OF PMSM

The PM machine model given in Bose (2002), Pillay and Krishnan (1989) can be derived as below

\[
\dot{x} = Ax + Bu + f^{(2)}(x) \tag{3.1}
\]

\[
x = [x_1 \ x_2 \ x_3 \ x_4]^T = [\theta \ \omega_r \ i_q \ i_d]^T
\]
where \( u_q, u_d, i_q, i_d \) represent the quadrature and direct axis voltages and currents respectively and \( \theta, \omega_r \) represent rotor position and rotor speed respectively. \( \lambda_{af} \) is the flux induced by the permanent magnet of the rotor in the stator phases. \( L_d, L_q \) are the direct and quadrature inductances respectively. \( R \) is the stator resistance, \( p \) is the number of pole pairs and \( J \) is the system moment of inertia.

The dynamic model of PMSM is largely dominated by linear and quadratic terms. Higher order terms appear only in extreme conditions of operation.

The model in Equation (3.1) can be reduced to normal form for two inputs as given in Brunovsky (1970), in a standard way using the following transformations as referred to Kuo (2001),
\[
x = \begin{pmatrix} a_1 c_1 & 0 & 0 & 0 \\ 0 & a_1 c_1 & 0 & 0 \\ 0 & 0 & c_1 & 0 \\ 0 & 0 & 0 & a_4 c_2 \\
\end{pmatrix} y
\]  \quad (3.2)

\[
u = \begin{pmatrix} 1 & 0 \\ 0 & a_4 / c_2 \\
\end{pmatrix} u' + \begin{pmatrix} 0 & -a_1 a_2 & -a_3 & 0 \\ 0 & 0 & 0 & -a_4^2 / c_2 \\
\end{pmatrix} y
\]

where

\[
a_1 = \frac{1.5 p \lambda}{J}; \quad a_2 = \frac{-\lambda p}{L_q}; \quad a_3 = \frac{-R}{L_q}; \quad a_4 = \frac{R}{L_d}; \quad c_1 = \frac{1}{L_q}; \quad c_2 = \frac{1}{L_d}
\]

The Brunovsky form for two inputs is given below (where \(x, u, A\) and \(B\) are retained for simplicity of notation).

\[
\dot{x} = Ax + Bu + f^{(2)}(x)
\]  \quad (3.3)

where

\[
A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\
\end{pmatrix}; \quad B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 1 \\
\end{pmatrix}; \quad f^{(2)}(x) = \begin{pmatrix} 0 \\ C_1 x_3 x_4 \\ C_2 x_2 x_4 \\ C_3 x_2 x_3 \\
\end{pmatrix}
\]

\[
C_1 = \frac{1.5 p (L_d - L_q) a_4}{J a_1}; \quad C_2 = \frac{-L_d p a_1 a_4}{L_q}; \quad C_3 = \frac{L_q c_1^2}{L_d a_4}
\]

\[
a_1 = \frac{1.5 p \lambda}{J}; \quad a_4 = \frac{-R}{L_d}; \quad c_1 = \frac{1}{L_q}
\]

3.2.1 Derivation of linearization transformations

Considering PMSM model given in Equation (3.3), and noting that \(A\) and \(B\) are matrices in the controller normal form for two inputs as
given in Brunovsky (1970), linearization transformations are derived. In order to linearize the model, a change of coordinates and feedback as given in Krener et al (1987 and 1988) of the following form is considered:

\[ y = x + \phi^{(2)}(x) \]  

\[ u = (l_2 + \beta^{(1)}(x))v + \alpha^{(2)}(x) \]  

where \( u = [u_1 \ u_2]^T \) represent the old inputs and \( v = [v_1 \ v_2]^T \) denote the new inputs.

Applying Equations (3.4) and (3.5), Equation (3.3) can be written as

\[ \dot{y} = Ay + Bv + O(y, v)^{(3)} \]  

provided the homological equations

\[ -A\phi^{(2)}(x) + B\alpha^{(2)}(x) + f^{(2)}(x) + \frac{\partial \phi^{(2)}(x)}{\partial x}A x = 0 \]  

\[ B\beta^{(1)}(x)v + \frac{\partial \phi^{(2)}(x)}{\partial x}Bv + g^{(1)}(x)v = 0; \ \forall v \]  

can be solved for \( \phi^{(2)}(x), \alpha^{(2)}(x) \) and \( \beta^{(1)}(x) \), where \( O(y, v)^{(3)} \) corresponds to terms of degree greater than or equal to 3. Noting that \( g^{(1)}(x) = 0 \) in the PMSM model, \( \phi^{(2)}(x), \alpha^{(2)}(x) \) and \( \beta^{(1)}(x) \) can be derived as

\[ \phi^{(2)}(x) = \begin{pmatrix} 0 \\
0 \\
0 \\
C_1 x_3 x_4 \\
0 \end{pmatrix} \]  

(3.9)
\[ \alpha^{(2)}(x) = \begin{pmatrix} -C_2 x_2 x_4 \\ -C_3 x_2 x_3 \end{pmatrix} \]  

(3.10)

\[ \beta^{(1)}(x) = (-1) \left( B^T \frac{\partial \phi^{(2)}(x)}{\partial x} - B \right) = - \begin{pmatrix} C_1 x_4 \\ 0 \end{pmatrix} \]  

(3.11)

A block diagram of the linearized PMSM model in closed loop is given below.

**Figure 3.1 Block Diagram of Linearized PMSM Model in Closed Loop**

**Remark 3.1**

As can be seen from Equation (3.5), this technique allows input transformation to be specified in implementable form (see Figure 3.1) without the need for matrix inversion as in the case of exact linearization. Hence the singularity issue of state feedback is avoided.

The resulting system after quadratic linearization of PMSM model in Equation (3.3) is Equation (3.6). It can be noted that third and higher order terms represented by \( O(y, v)^{(3)} \) are introduced into the system even though the PMSM model originally had only second order terms. A new technique called generalized quadratic linearization is introduced in this chapter which seeks to solve the issue.
3.3 BACKGROUND


\[
\dot{x} = Ax + Bu + f^{(2)}(x) + f^{(3)}(x) + \cdots + f^{(p)}(x) + g^{(1)}(x)u \\
+ \cdots g^{(p-1)}(x)u + \cdots 
\]

(3.12)

where \((A,B)\) are in Brunovsky normal form. That is, let \(k = \{k_1,k_2,\ldots,k_r\}\) be sequence of integers such that \(k_1 \geq k_2 \geq \cdots \geq k_r\) such that \(k_1 + k_2 + \cdots + k_r = n\)

\[
A = \begin{pmatrix}
A_{k_1} & 0 & \cdots & 0 \\
0 & A_{k_2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & A_{k_r}
\end{pmatrix}; \\
B = \begin{pmatrix}
b_{k_1} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & b_{k_2} & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & b_{k_r} & 0 & 0 & \cdots & 0
\end{pmatrix}
\]

(3.13)

where matrix \(A\) is partitioned into \(r^2\) blocks while matrix \(B\) is partitioned into \(\mu r\) blocks with \(r \leq \mu\). Each block \(A_{k_i}\) and \(b_{k_i}\) are of the form

\[
A_{k_i} = \begin{pmatrix}
0 & 1 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 \\
0 & 0 & \cdots & 0
\end{pmatrix}; b_{k_i} = \begin{pmatrix}
0 \\
\vdots \\
0 \\
0
\end{pmatrix}
\]

(3.14)

\(A_{k_i}\) is of order \(k_i \times k_i\), \(b_{k_i}\) is of order \(k_i \times 1\) and \(B\) is \(n \times \mu\) matrix. \(x = [x_1, x_2, \cdots x_n]^T\) and \(u = [u_1, u_2, \cdots u_\mu]^T\) is a vector input such that \(\mu < n\). \(f^{(p)}(x), g^{(p-1)}(x)\) are vector polynomials of order \(p\) and \((p-1)\) respectively, \(p = 2,3,\cdots\)
In order to linearize the system, change of coordinate and state feedback as given in Kang and Krener (1992) of the following form is considered

\[ y = x + \phi(x) \]  
(3.15)

\[ u = (I_\mu + \beta(x))v + \alpha(x) \]  
(3.16)

where \( \phi(x) = \sum_{j=2,3,\ldots} \phi^{(j)}(x) \); \( \alpha(x) = \sum_{j=2,3,\ldots} \alpha^{(j)}(x) \); \( \beta(x) = \sum_{j=2,3,\ldots} \beta^{(j-1)}(x) \) and \( I_\mu \) is the identity matrix of order \( \mu \). Applying the transformations in Equations (3.15) and (3.16), Equation (3.12) is reduced to

\[ \dot{y} = Ay + Bv \]  
(3.17)

provided the following equations, called homological equations as given in Kang and Krener (1992), are satisfied for \( m \geq 2 \).

\[- A \phi^{(m)}(x) + B \alpha^{(m)}(x) + f^{(m)}(x) + \frac{\partial \phi^{(m)}(x)}{\partial x}Ax = 0 \]  
(3.18)

\[ B \beta^{(m-1)}(x)v + \frac{\partial \phi^{(m)}(x)}{\partial x}Bv + g^{(m-1)}(x)v = 0, \forall v \]  
(3.19)

where \( f^{(m)}(x) = f^{(m)}(x) \) for \( m = 2 \) and \( f^{(m)}(x) \) is expressed in terms of \( f^{(m-i)}(x); i = 0,1,2,\ldots,(m-2) \) and \( f^{(m-j)}(x); j = 1,2,\ldots,(m-2); m > 2 \); \( g^{(m-1)}(x) = g^{(1)}(x) \); \( m = 2 \) and \( g^{(m-1)}(x) \) is expressed in terms of \( g^{(m-i)}(x); i = 1,2,\ldots,(m-1) \) and \( f^{(m-j)}(x); j = 1,2,\ldots,(m-2); m > 2 \).
3.4 QUADRATIC LINEARIZATION WITH MULTIPLE INPUTS

Consider the specialized case of system in Equation (3.12) containing only quadratic terms in \( x \) as in

\[
\dot{x} = Ax + Bu + f^{(2)}(x) \tag{3.20}
\]

Quadratic linearization of Equation (3.20) involves specialization of the result discussed in section 3.3 for \( m = 2 \) and simplifying it by applying \( g^{(1)}(x) = 0 \). That is, applying the following coordinate transformation and feedback,

\[
y = x + \phi^{(2)}(x) \tag{3.21}
\]

\[
u = \left( I + \beta^{(1)}(x) \right) v + \alpha^{(2)}(x) \tag{3.22}
\]

Equation (3.20) is reduced to

\[
\dot{y} = Ay + Bv + O(y, v)^{(3)} \tag{3.23}
\]

where \( O(y, v)^{(3)} \) corresponds to terms of degree 3 or higher, provided the following equations are satisfied.

\[
-A\phi^{(2)}(x) + B\alpha^{(2)}(x) + f^{(2)}(x) + \frac{\partial \phi^{(2)}(x)}{\partial x}Ax = 0 \tag{3.24}
\]

\[
B\beta^{(1)}(x)v + \frac{\partial \phi^{(2)}(x)}{\partial x}Bv = 0, \forall v \tag{3.25}
\]

The homological equations given by Equations (3.24) and (3.25) are simplified by representing them in partitioned form, using Brunovsky normal form introduced in section 3.3.
Let

\[
\phi^{(2)}(x) = \begin{pmatrix}
\phi_{k_1}^{(2)}(x) \\
\phi_{k_2}^{(2)}(x) \\
\vdots \\
\phi_{k_r}^{(2)}(x)
\end{pmatrix}
\]  \hspace{1cm} (3.26)

\[
x = \begin{pmatrix}
x_{k_1} \\
x_{k_2} \\
\vdots \\
x_{k_r}
\end{pmatrix}
\]  \hspace{1cm} (3.27)

\[
f^{(2)}(x) = \begin{pmatrix}
f_{k_1}^{(2)}(x) \\
f_{k_2}^{(2)}(x) \\
\vdots \\
f_{k_r}^{(2)}(x)
\end{pmatrix}
\]  \hspace{1cm} (3.28)

\[
\alpha^{(2)}(x) = \begin{pmatrix}
\alpha_1^{(2)}(x) \\
\alpha_2^{(2)}(x) \\
\vdots \\
\alpha_{\mu}^{(2)}(x)
\end{pmatrix}
\]  \hspace{1cm} (3.29)

where putting \(p_i = \sum_{j=1}^{i} k_j \); \(i = 1, 2, \ldots, r\)

\[
\phi_{k_i}^{(2)}(x) = \begin{pmatrix}
\phi_{p_{i-1}+1}^{(2)}(x) \\
\vdots \\
\phi_{p_{i-1}+t}^{(2)}(x) \\
\phi_{p_{i-1}}^{(2)}(x) \\
\phi_{p_i}^{(2)}(x)
\end{pmatrix}
\]  \hspace{1cm} (3.30)
\[ x_{ki} = \begin{pmatrix} 
\vdots 
\end{pmatrix} \]

(3.31)

\[ f_{k1}^{(2)}(x) = \begin{pmatrix} 
\phi_{k1,1}^{(2)}(x) \\
\vdots \\
\phi_{k1,r}^{(2)}(x) \\
\phi_{k2,1}^{(2)}(x) \\
\vdots \\
\phi_{k2,r}^{(2)}(x) \\
\vdots \\
\phi_{kr,1}^{(2)}(x) \\
\vdots \\
\phi_{kr,r}^{(2)}(x) 
\end{pmatrix} \]

(3.32)

\[ \frac{\partial \phi^{(2)}(x)}{\partial x} = \begin{pmatrix} 
\phi_{k1,1}^{(2)} & \phi_{k1,2}^{(2)} & \cdots & \phi_{k1,r}^{(2)} \\
\phi_{k2,1}^{(2)} & \phi_{k2,2}^{(2)} & \cdots & \phi_{k2,r}^{(2)} \\
\vdots & \vdots & \ddots & \vdots \\
\phi_{kr,1}^{(2)} & \phi_{kr,2}^{(2)} & \cdots & \phi_{kr,r}^{(2)} 
\end{pmatrix} \]

(3.33)

where

\[ \phi_{k_i,k_j}^{(2)} = \begin{pmatrix} 
\frac{\partial \phi^{(2)}_{k_i,k_j}}{\partial x_{k_{i-1}+1}} & \cdots & \frac{\partial \phi^{(2)}_{k_i,k_j}}{\partial x_{k_{i-1}+1}} \\
\frac{\partial \phi^{(2)}_{k_i,k_j}}{\partial x_{k_{i-1}+t}} & \cdots & \frac{\partial \phi^{(2)}_{k_i,k_j}}{\partial x_{k_{i-1}+t}} \\
\vdots & \vdots & \vdots \\
\frac{\partial \phi^{(2)}_{k_i,k_j}}{\partial x_{k_{i-1}+1}} & \cdots & \frac{\partial \phi^{(2)}_{k_i,k_j}}{\partial x_{k_{i-1}+1}} \\
\frac{\partial \phi^{(2)}_{k_i,k_j}}{\partial x_{k_{i-1}+t}} & \cdots & \frac{\partial \phi^{(2)}_{k_i,k_j}}{\partial x_{k_{i-1}+t}} \\
\vdots & \vdots & \vdots \\
\frac{\partial \phi^{(2)}_{k_i,k_j}}{\partial x_{k_{i-1}+1}} & \cdots & \frac{\partial \phi^{(2)}_{k_i,k_j}}{\partial x_{k_{i-1}+1}} \\
\frac{\partial \phi^{(2)}_{k_i,k_j}}{\partial x_{k_{i-1}+t}} & \cdots & \frac{\partial \phi^{(2)}_{k_i,k_j}}{\partial x_{k_{i-1}+t}} \\
\end{pmatrix} ; \\
i, j = 1, 2 \ldots r 
\]

(3.34)
\[
\beta^{(1)}(x) = \begin{pmatrix}
\beta^{(1)}_1(x) \\
\beta^{(1)}_2(x) \\
\vdots \\
\beta^{(1)}_\mu(x)
\end{pmatrix}
\]  \hspace{1cm} (3.35)

where

\[
\beta^{(1)}_i(x) = [\beta^{(1)}_{i,1}(x), \beta^{(1)}_{i,2}(x), \ldots, \beta^{(1)}_{i,\mu}(x)]; i = 1,2,\ldots,\mu \hspace{1cm} (3.36)
\]

Using Equation (3.13), Equations (3.26) – (3.29) and (3.33), \(i^{th}\) block row of Equation (3.24) (homological equation) can be written as

\[-A_{k_l}(x)\phi^{(2)}_{k_l}(x) + b_{k_l}\alpha^{(2)}_{k_l}(x) + f^{(2)}_{k_l}(x) + \sum_{j=1}^{r} \phi^{(2)}_{k_l,k_j}A_{k_j}x_{k_j} = 0; \]

\[i = 1,2,\ldots,r \hspace{1cm} (3.37)\]

The second homological equation viz., Equation (3.25) can be simplified as follows. Since the effect of control input \(u\) in Equation (3.20) is dependent on the \(B\) matrix only, without loss of generality, \(\mu = r\) can be assumed, because the terms in \(B\) matrix are non zero only for the first \(r\) inputs. So, the revised \(B\) matrix can be written as

\[
B = \begin{pmatrix}
b_{k_1} & 0 & \cdots & 0 \\
0 & b_{k_2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & b_{k_r}
\end{pmatrix}
\]  \hspace{1cm} (3.38)

Equation (3.25) is now put in partitioned form. Using Equation (3.13), Equations (3.33) – (3.36), \(i^{th}\) block row of Equation (3.25) can be written as
\[ b_{k_i} \beta_i^{(1)}(x) + \left[ \phi_{k_i,k_i}^{(2)} b_{k_i} \phi_{k_i,k_2}^{(2)} b_{k_2} \cdots \phi_{k_i,k_r}^{(2)} b_{k_r} \right] v = 0 \forall v \]
\[ i = 1,2 \cdots r \quad (3.39) \]

Since Equation (3.39) is to be an identity in \( v \), \( v \) can be dropped and Equation (3.39) can be rewritten as

\[
\begin{pmatrix}
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 \\
\end{pmatrix}
\begin{pmatrix}
\beta_{l,1}^{(1)}(x) \\
\beta_{l,2}^{(1)}(x) \\
\vdots \\
\beta_{l,r}^{(1)}(x) \\
\end{pmatrix}
+ \begin{pmatrix}
\frac{\partial \phi_{p_{l-1}+1}^{(2)}(x)}{\partial x_{p_1}} & \cdots & \frac{\partial \phi_{p_{l-1}+1}^{(2)}(x)}{\partial x_{p_r}} \\
\vdots & \ddots & \vdots \\
\frac{\partial \phi_{p_{l-1+t}}^{(2)}(x)}{\partial x_{p_1}} & \cdots & \frac{\partial \phi_{p_{l-1+t}}^{(2)}(x)}{\partial x_{p_r}} \\
\vdots & \ddots & \vdots \\
\frac{\partial \phi_{p_l}^{(2)}(x)}{\partial x_{p_1}} & \cdots & \frac{\partial \phi_{p_l}^{(2)}(x)}{\partial x_{p_r}} \\
\end{pmatrix}
= 0 ; \quad (3.40)
\]
\[ i = 1,2 \cdots r \]

Since \( \beta_{l,l}^{(1)}(x) ; l = 1,2 \cdots r \) are arbitrary, it is clear that a necessary condition for Equation (3.25) to be satisfied is

\[
\frac{\partial \phi_{p_{l-1+t}}^{(2)}(x)}{\partial x_{p_q}} = 0 ; q = 1,2 \cdots r ; t = 1,2 \cdots (k_i - 1); \]
\[ i = 1,2, \cdots r \quad (3.41) \]

### 3.5 GENERALIZED QUADRATIC LINEARIZATION

In Theorem 3.1 below, it is shown that a special class of systems with only quadratic nonlinear terms can be linearized to remove the quadratic terms without introducing third and higher order terms.
Theorem 3.1:

Consider the system

\[
\dot{x} = Ax + Bu + f^{(2)}(x)
\]  

(3.42)

where \( x = [x_1, x_2, \cdots x_n]^T \) and \( u = [u_1, u_2, \cdots u_r]^T \); \( n > r \). \((A,B)\) is in Brunovsky normal form as given in Equations (3.13), (3.14) and (3.38). \( f^{(2)}(x) \) is given by Equation (3.28), where

\[
f_{k_i}(x) = \begin{cases} 
\begin{pmatrix} 0 \\ \vdots \\ f^{(2)}_{p_i-1}(x') \\ f^{(2)}_{p_i}(x) \end{pmatrix}; & k_i > 2 \\
\begin{pmatrix} f^{(2)}_{p_i-1}(x') \\ f^{(2)}_{p_i}(x) \end{pmatrix}; & k_i = 2 \\
\begin{pmatrix} f^{(2)}_{p_i}(x) \end{pmatrix}; & k_i = 1 
\end{cases} 
\]\n
(3.43)

\( f_{k_i}(x) \) is of order \( k_i \times 1 \) and \( x' \) is given by the \( r \)-tuple \( x' = (x_{p_1}, x_{p_2}, \cdots, x_{p_r}) \). Then the transformations given by Equations (3.15) and (3.16) where \( \phi(x) = \phi^{(2)}(x) \) is given by Equation (3.26) where

\[
\phi_{k_i}(x) = \begin{cases} 
\begin{pmatrix} \phi^{(2)}_{p_i-1+1}(x) \\ \vdots \\ \phi^{(2)}_{p_i-1}(x) \\ \phi^{(2)}_{p_i}(x) \end{pmatrix}; & k_i > 2 \\
\begin{pmatrix} \phi^{(2)}_{p_i}(x') \end{pmatrix}; & \phi^{(2)}_{p_i}(x') \text{ arbitrary}; \ k_i = 1 
\end{cases} 
\]

(3.44)
\[\alpha(x) = \alpha^{(2)}(x) \] given by Equation (3.29) where

\[\alpha_i^{(2)}(x) = -f_i^{(2)}(x); i = 1, 2, \ldots r \tag{3.45}\]

and

\[\beta^{(m-1)}(x) = (-1)^{m-1} \{B^T \frac{\partial \phi^{(2)}(x)}{\partial x} B\}^{m-1}; m \geq 2 \tag{3.46}\]

reduces the system given by Equation (3.42) to

\[\dot{y} = Ay + Bv \tag{3.47}\]

**Proof:** See Appendix 5.

### 3.6 GENERALIZED QUADRATIC LINEARIZATION OF MACHINE MODELS

#### 3.6.1 Squirrel Cage Induction Motor

The squirrel cage induction motor model given in Bose (2002) can be expressed as

\[\dot{x} = Ax + Bu + f^{(2)}(x) \tag{3.48}\]

where \(x = [x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5]^T = [i_{ds}^s \quad i_{qs}^s \quad \psi_{dr}^s \quad \psi_{qr}^s \quad \omega_r]^T\)

\[u = [u_1 \quad u_2 \quad u_3 \quad u_4]^T = [v_{ds}^s \quad v_{qs}^s \quad \psi' \quad \psi_{dr}^s]^T \]

where \(\psi' = i_{qs}^s \psi_{dr}^s - i_{ds}^s \psi_{qr}^s\) where \(v_{qs}^s, v_{ds}^s, i_{qs}^s, i_{ds}^s\) represent the quadrature and direct axis voltages and currents respectively, \(\psi_{dr}^s, \psi_{qr}^s\) represent direct and quadrature axis fluxes respectively. \(\psi_{dr}^s, \psi_{qr}^s\) represent the estimated direct
and quadrature flux respectively and \( \omega_r \) represents the rotor speed. \( i_{dq}^s \) and \( i_{ds}^s \) represent the reference values of quadrature and direct axis currents respectively. Superscript refers to variables in the stationary frame.

\[
A = \begin{pmatrix}
-(L_m^2 R_r + L_s^2 R_s) / \sigma L_s L_r^2 & 0 & L_m R_r / \sigma L_s L_r^2 & 0 & 0 \\
0 & -(L_m^2 R_r + L_s^2 R_s) / \sigma L_s L_r^2 & 0 & L_m R_r / \sigma L_s L_r^2 & 0 \\
L_m R_r / L_r & 0 & L_m R_r / L_r & 0 & -R_r / L_r \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

\[
B = \begin{pmatrix}
1 / \sigma L_s & 0 & 0 & 0 \\
0 & 1 / \sigma L_s & 0 & 0 \\
0 & 0 & 0 & -R_r / L_r \\
0 & 0 & -3P^2 L_m / 8J L_r & 0
\end{pmatrix}
\]

\[
f^{(2)}(x) = \begin{pmatrix}
L_m \omega_r \psi_{qR}^s / \sigma L_s L_r \\
-L_m \omega_r \psi_{qR}^s / \sigma L_s L_r \\
-\omega_r \psi_{qR}^s \\
\omega_r \psi_{dR}^s \\
0
\end{pmatrix} = \begin{pmatrix}
C''_1 x_4 x_5 \\
-C''_1 x_3 x_5 \\
x_4 x_5 \\
x_3 x_5 \\
0
\end{pmatrix}
\]

where \( L_m, L_r \) and \( L_s \) represent the magnetizing, rotor and stator inductances respectively. \( R_r \) and \( R_s \) represent the rotor and stator resistances respectively. \( J \) represents the system moment of inertia and \( P \) represents the number of poles. \( \sigma = 1 - L_s^2 / L_m L_r \) and \( C''_1 = L_m / \sigma L_s L_r \). Equation (3.48) can be reduced to normal form in a standard way using linear transformation of state variables given in Kuo (2001) and linear state feedback as follows, where \( x, u, A, B \) and \( f^{(2)}(x) \) are retained for simplicity of notation.

\[
\dot{x} = Ax + Bu + f^{(2)}(x) \quad (3.49)
\]

where \( A \) and \( B \) given by Equation (3.13) with \( r = \mu = 4 \) and \( k_1 = 2, k_2 = k_3 = k_4 = 1 \) are
$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}; \quad B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \quad f^{(2)}(x) = \begin{pmatrix} f^{(2)}_1(x) \\ f^{(2)}_2(x) \\ f^{(2)}_3(x) \\ f^{(2)}_4(x) \\ f^{(2)}_5(x) \end{pmatrix}$

where $f^{(2)}_1(x) = f_{1,35} x_3 x_5 + f_{1,45} x_4 x_5; \quad f^{(2)}_2(x) = f_{2,35} x_3 x_5 + f_{2,45} x_4 x_5; \quad f^{(2)}_3(x) = f_{3,35} x_3 x_5 + f_{3,45} x_4 x_5; \quad f^{(2)}_4(x) = f_{4,35} x_3 x_5 + f_{4,45} x_4 x_5; \quad f^{(2)}_5(x) = f_{5,35} x_3 x_5 + f_{5,45} x_4 x_5.$

$f^{(2)}(x)$ can be partitioned using Equation (3.28) as

$$f^{(2)}(x) = \begin{pmatrix} f^{(2)}_{k_1}(x) \\ f^{(2)}_{k_2}(x) \\ f^{(2)}_{k_3}(x) \\ f^{(2)}_{k_4}(x) \end{pmatrix}$$

where

$$f^{(2)}_{k_1}(x) = \begin{pmatrix} f^{(2)}_1(x) \\ f^{(2)}_2(x) \end{pmatrix} = \begin{pmatrix} f_{1,35} x_3 x_5 + f_{1,45} x_4 x_5 \\ f_{2,35} x_3 x_5 + f_{2,45} x_4 x_5 \end{pmatrix}$$

$$f^{(2)}_{k_2}(x) = \begin{pmatrix} f^{(2)}_3(x) \end{pmatrix} = \begin{pmatrix} f_{3,35} x_3 x_5 + f_{3,45} x_4 x_5 \end{pmatrix}$$

$$f^{(2)}_{k_3}(x) = \begin{pmatrix} f^{(2)}_4(x) \end{pmatrix} = \begin{pmatrix} f_{4,35} x_3 x_5 + f_{4,45} x_4 x_5 \end{pmatrix}$$

$$f^{(2)}_{k_4}(x) = \begin{pmatrix} f^{(2)}_5(x) \end{pmatrix} = \begin{pmatrix} f_{5,35} x_3 x_5 + f_{5,45} x_4 x_5 \end{pmatrix}$$
As $f_1^{(2)}(x)$ is a function of $x_3, x_4$ and $x_5$ only and hence a function of $x' = (x_2, x_3, x_4, x_5)$, one can write $f_1^{(2)}(x) = f_1^{(2)}(x')$. The system given by Equations (3.48) and (3.49) agree with conditions of Theorem 3.1 as given in Equation (3.43). Hence Theorem 3.1 can be applied to system given by Equation (3.49) to obtain the solution of generalized quadratic linearization which is given in Corollary 3.1.

**Corollary 3.1**:

The system given by Equation (3.49) can be quadratic linearized in the generalized sense using the transformation

$$y = x + \phi(x) \quad \text{(3.51)}$$

$$u = (I + \beta(x))v + \alpha(x) \quad \text{(3.52)}$$

where

$$\phi(x) = \phi^{(2)}(x) = \begin{pmatrix} 0 \\ f_1^{(2)}(x') \\ \phi_3^{(2)}(x') \\ \phi_4^{(2)}(x') \\ \phi_5^{(2)}(x') \end{pmatrix}$$

$$\alpha(x) = \alpha^{(2)}(x) = \begin{pmatrix} -f_2^{(2)}(x) \\ -f_3^{(2)}(x) \\ -f_4^{(2)}(x) \\ -f_5^{(2)}(x) \end{pmatrix}$$

$$\beta^{(m-1)}(x) = (-1)^{m-1}\{B^T \frac{\partial \phi^{(2)}(x)}{\partial x} B\}^{m-1}$$

where $\phi_3^{(2)}(x'), \phi_4^{(2)}(x'), \phi_5^{(2)}(x')$ are completely arbitrary, and $x' = (x_2, x_3, x_4, x_5)$. The system is reduced to
Proof: The result follows directly by applying the result of Theorem 3.1 to Equation (3.49) where \( n = 5, r = 4, \mu = 4, k_1 = 2, k_2 = k_3 = k_4 = 1; p_1 = 2, p_2 = 3, p_3 = 4, p_4 = 5. \)

### 3.6.2 Permanent Magnet Synchronous Motor

The PM machine model given in Bose (2002) which was discussed in section 3.2 is considered once again. The PMSM machine model is given by Equation (3.1) and the model reduced to the Brunovsky form for two inputs is given by Equation (3.3).

\[ f^{(2)}(x) \] can be partitioned using Equation (3.28) as

\[
f^{(2)}(x) = \begin{pmatrix} f_{k_1}^{(2)}(x) \\ f_{k_2}^{(2)}(x) \end{pmatrix} \tag{3.54}
\]

where

\[
f_{k_1}^{(2)}(x) = \begin{pmatrix} f_1^{(2)}(x) \\ f_2^{(2)}(x) \\ f_3^{(2)}(x) \end{pmatrix} = \begin{pmatrix} 0 \\ C_1 x_3 x_4 \\ C_2 x_2 x_4 \end{pmatrix}
\]

\[
f_{k_2}^{(2)}(x) = \begin{pmatrix} f_4^{(2)}(x) \end{pmatrix} = (C_3 x_2 x_3)
\]

It can be noted that \( f_1^{(2)}(x) = 0, f_2^{(2)}(x) \) is a function of \( x' = (x_3, x_4) \) and hence \( f_2^{(2)}(x) = f_2^{(2)}(x') \). The system given by Equation (3.3) is in line with conditions of Theorem 3.1 as given in Equation (3.43). Hence Theorem 3.1 can be applied to system given by Equation (3.3) to obtain the
solution of generalized quadratic linearization which is given by Corollary 3.2.

**Corollary 3.2:**

The system given by Equation (3.3) can be quadratic linearized in the generalized sense using the transformation

\[ y = x + \phi(x) \quad (3.55) \]

\[ u = (I_2 + \beta(x))v + \alpha(x) \quad (3.56) \]

where

\[ \phi(x) = \phi^{(2)}(x) = \begin{pmatrix} 0 \\ 0 \\ f_2^{(2)}(x') \\ f_4^{(2)}(x') \end{pmatrix} \]

\[ \alpha(x) = \alpha^{(2)}(x) = \begin{pmatrix} -f_3^{(2)}(x) \\ -f_4^{(2)}(x) \end{pmatrix} \]

\[ \beta^{(m-1)}(x) = (-1)^{m-1} \{ B^T \frac{\partial \phi^{(2)}(x)}{\partial x} B \}^{m-1} \]

where \( \phi_4^{(2)}(x') \) is completely arbitrary and \( x' = (x_3, x_4) \). The system then is reduced to

\[ \dot{y} = Ay + Bv \quad (3.57) \]

**Proof:** The result follows directly by applying the result of Theorem 3.1 to Equation (3.3) where \( n = 4, r = 2, \mu = 2, k_1 = 3, k_2 = 1; p_1 = 3, p_2 = 4 \).
3.7 SUMMARY

Quadratic linearization of PMSM model introduces higher order terms in the process of quadratic linearization. In this chapter, the concept of generalized quadratic linearization technique is introduced, wherein the higher order terms introduced during the process of quadratic linearization are removed even as the quadratic term is removed.

A solution to the problem of generalized quadratic linearization is given for a class of control affine systems. Induction motor and PM motor models involving quadratic nonlinearity are considered and are quadratic linearized in the generalized sense. The proposed method can also be extended to wound rotor and synchronous machine models as well.

The generalized quadratic linearization technique proposed in this chapter can also be considered as the approximate feedback linearization equivalent of exact linearization of a class of systems of the form given by Equation (3.42). Since the solution of the generalized quadratic linearization involves removing third and higher order nonlinearities, the result of the chapter can also been seen as a contribution towards a special case of the open problem on arbitrary order linearization of control affine systems as given by Devanathan (2003).

Finally, as per the result of Theorem 3.1, $\beta(x) = \sum_{m=2,3} \beta^{(m-1)}(x)$ whose convergence needs to be established. Also, Theorem 3.1 of section 3.5 deals only with sufficient condition for generalized quadratic linearization. The question arises as to what are the necessary conditions for Equation (3.20) to be quadratic linearizable in the generalized sense. These are questions beyond the scope of the present thesis and may contribute to future work.