Chapter 5

Feasibility and Optimality of the Solution

In this chapter, the feasibility and optimality of the solution of transshipment problem with mixed constraints has been discussed, in detail.

5.1 Feasibility of the Solution

It is important to prove that the optimal solution obtained by transformed transportation problem $\hat{x}_{ij}$ is a feasible solution to the transshipment problem with mixed constraints, to justify the method described above in section 4.2.

If the transshipment problem with mixed constraints has a feasible solution, then the optimal solution to the transformed standard transportation problem will have:

$$\hat{y}_{i,m+n+1} = 0, \quad \forall \ i \in Q \cup S$$

$$\hat{y}_{m+n+1,j} = 0, \quad \forall \ j \in Q \cup S$$

For $i \in S$, we have

$$\sum_{j=1}^{m+n} \hat{x}_{ij} = \sum_{j=1}^{m+n} \hat{w}_{ij} = \sum_{j=1}^{m+n} \hat{y}_{ij} + \hat{y}_{i,m+n+1}$$

$$= a'_i = a_i + T$$

$$= a_i + s_i + \hat{x}_{ii}$$

$$\sum_{j=1, j \neq i}^{m+n} \hat{x}_{ij} = a_i + s_i$$

$$\therefore \sum_{j=1, j \neq i}^{m+n} \hat{x}_{ij} - \sum_{j=1, j \neq i}^{m+n} \hat{x}_{ji} = a_i, \quad \forall \ i \in S \quad (\because s_i = \sum_{j=1, j \neq i}^{m+n} \hat{x}_{ij})$$

Similarly for $i \in Q$, we have

$$\sum_{j=1}^{m+n} \hat{x}_{ij} = \sum_{j=1}^{m+n} \hat{w}_{ij} = \sum_{j=1}^{m+n} \hat{y}_{ij} + \hat{y}_{i,m+n+1}$$

$$\geq a'_i = a_i + T$$

$$\geq a_i + s_i + \hat{x}_{ii}$$

$$\sum_{j=1, j \neq i}^{m+n} \hat{x}_{ij} \geq a_i + s_i$$

$$\therefore \sum_{j=1, j \neq i}^{m+n} \hat{x}_{ij} - \sum_{j=1, j \neq i}^{m+n} \hat{x}_{ji} \geq a_i, \quad \forall \ i \in Q \quad (\because s_i = \sum_{j=1, j \neq i}^{m+n} \hat{x}_{ij})$$
Whilst for $i \in T$

$$\sum_{j=1}^{m+n} \hat{x}_{ij} = \sum_{j=1}^{m+n} \hat{w}_{ij} = \sum_{j=1}^{m+n} \hat{y}_{ij} + \hat{y}_{i,m+n+1}$$

$$\leq a_i = a_i + T$$

$$\leq a_i + s_i + \hat{x}_{ii}$$

$$\sum_{j=1, j \neq i}^{m+n} \hat{x}_{ij} \leq a_i + s_i$$

$$\therefore \sum_{j=1, j \neq i}^{m+n} \hat{x}_{ij} \leq a_i, \quad \forall \ i \in T$$

$$\therefore s_i = \sum_{j=1, j \neq i}^{m+n} \hat{x}_{ij}$$

$$\therefore \hat{y}_{i,m+n+1} \geq 0$$

Similarly, the column constraints are satisfied.

Hence, we have $\hat{x}_{ij} \leq \hat{w}_{ij} \leq \hat{y}_{ij} \geq 0, \quad \forall \ i, j = 1,2, \ldots , m+n$

which completes the proof.

Thus, our solution is feasible to the transshipment problem with mixed constraints.

### 5.2 Optimality of the Solution

The proof of optimality is based upon showing that a particular optimal dual solution to the transformed transportation problem contains an optimal dual solution to the transshipment problem with mixed constraints.

The dual of the transshipment problem with mixed constraints can be written as:

$$\max Z = \sum_{i=1}^{m} a_i p_i + \sum_{j=m+1}^{m+n} b_j q_j$$

s.t. $p_i - q_j \leq c_{ij}, \quad \forall \ i = 1,2, \ldots , m$

$$p_i \geq 0, \quad \forall \ i \in Q; \quad p_i \leq 0, \quad \forall \ i \in T$$

whilst the dual of the transformed transportation problem is:

$$\max Z = \sum_{i=1}^{m+n+1} a_i^* u_i + \sum_{j=1}^{m+n+1} b_j^* v_j$$

s.t. $u_i + v_j \leq c_{ij}^*, \quad \forall \ i, j = 1,2, \ldots , m+n+1$

Now, we first prove two lemmas:
Lemma 5.2.1: Suppose the transshipment problem has a feasible solution, then the dual of the transformed transportation problem has an optimal solution \((u, v)\) with

\[ U_{m + n + 1} = V_{m + n + 1} = 0 \]

Proof: As we know that the optimal dual solution to any transportation problem is not unique and we may arrange \(U_{m + n + 1} = 0\)

Since, \(a'_{m + n + 1} = \left(\sum_{j=m+1}^{m+n} b_j + mT\right)\) and \(b'_{m + n + 1} = \left(\sum_{i=1}^{m} a'_i + nT\right)\) so, it is clear that the optimal primal solution has \(\hat{y}_{m + n + 1, m + n + 1} > 0\)

Now we have,

\[ U_{m + n + 1} + V_{m + n + 1} = c'_{m + n + 1, m + n + 1} \]
\[ U_{m + n + 1} + V_{m + n + 1} = 0 \quad (\because \ c'_{m + n + 1, m + n + 1} = 0) \]

which gives,

\[ V_{m + n + 1} = 0 \quad (\because \ U_{m + n + 1} = 0) \]

This completes the proof.

Lemma 5.2.2: Suppose the transshipment problem with mixed constraints has a feasible solution then

\[ p_i = u_i, \quad \forall \ i = 1, 2, \ldots, m \]
\[ q_j = v_j, \quad \forall \ j = m + 1, m + 2, \ldots, m + n \]

is a feasible solution to the dual of the transshipment problem with mixed constraints.

Proof: It is known that \(u_i + v_j \leq c'_{ij}\)

Since \(c'_{ij} = c_{ij}, \quad \forall \ i, j = 1, 2, \ldots, m + n\) so, \(u_i + v_j = c_{ij}\)

Now for \(i \in Q\)

either \(\hat{y}_{i, m + n + 1} > 0, \) when \(u_i + V_{m + n + 1} = c'_{i, m + n + 1}\)

\[ u_i + V_{m + n + 1} \geq 0 \quad (\because \ c'_{i, m + n + 1} \geq 0 \ for \ i \in Q) \]

so,

\[ u_i \geq 0 \quad (\because \ V_{m + n + 1} = 0) \]

or, \(\hat{y}_j > 0, \) for some \(j = 1, 2, \ldots, m + n\)

when \(u_i + v_j = c_{ij}\)

Now, \(u_{m + n + 1} + v_j \leq c'_{m + n + 1, j}\)

\[ u_{m + n + 1} + v_j \leq c_{ij} \quad (\because \ c'_{m + n + 1, j} = c_{ij}) \]
\[ u_{m + n + 1} + v_j \leq u_i + v_j \]

which gives,
\[ u_i \geq 0 \quad (\because u_{m + n + 1} = 0) \]

For \( i \in T \)
\[ u_i + v_{m + n + 1} \leq c_i, m + n + 1 \]
\[ u_i + v_{m + n + 1} \leq 0 \quad (\because c_i, m + n + 1 = 0 \text{ for } i \in T) \]

so,
\[ u_i \leq 0 \quad (\because v_{m + n + 1} = 0) \]

Similarly, it can be shown that
\[ v_j \geq 0, \; \forall \; j \in U \]
and \[ v_j \leq 0, \; \forall \; j \in V \]

This completes the proof.

**Theorem 5.2.1:** The optimal solution obtained by transformed transportation problem \( x_{ij}, \; \forall \; i, j = 1, 2, \ldots, m + n \) is an optimal solution to the original transshipment problem with mixed constraints.

**Proof:** We certainly have the optimal solution to the transformed transportation problem and so the primal objective equals the dual objective i.e.

\[ \sum_{i=1}^{m+n+1} \sum_{j=1}^{m+n+1} c'_{ij} y_{ij} = \sum_{i=1}^{m+n+1} a'_{i} u_{i} + \sum_{j=1}^{m+n+1} b'_{j} v_{j} \]

......(1)

As it is shown that

\[ \sum_{i=1}^{m+n+1} \sum_{j=1}^{m+n+1} c'_{ij} y_{ij} = \sum_{i=1}^{m+n} \sum_{j=1}^{m+n} c_{ij} x_{ij} \]

whilst the right hand side of eq. (1) equals

\[ = \sum_{i=1}^{m+n} a'_{i} u_{i} + \sum_{j=1}^{m+n} b'_{j} v_{j} \]

\[ = \sum_{i=1}^{m+n} a'_{i} u_{i} + a'_{m + n + i} u_{m + n + i} + b'_{j} v_{j} + b'_{m + n + i} v_{m + n + i} \]

\[ = \sum_{j=1}^{m+n} a'_{j} u_{j} + \sum_{j=1}^{m+n} b'_{j} v_{j} \quad (\because u_{m + n + 1} = v_{m + n + 1} = 0) \]

Now it can be written as,
since, \( a'_i = a_i + T, \quad \forall \ i = 1,2,\ldots, m \); \( a'_i = T, \quad \forall \ i = m+1, m+2,\ldots, m+n \)

\( b'_j = T, \quad \forall \ i = 1,2,\ldots, m \); \( b'_j = b_j + T, \quad \forall \ j = m+1, m+2,\ldots, m+n \)

\[
\sum_{i=1}^{m} (a_i + T)u_i + \sum_{j=m+1}^{m+n} T v_j + \sum_{j=m+1}^{m+n} (b_j + T)v_j
\]

since \( T \) is not available in original transshipment problem with mixed constraints so, setting \( T \) as 0, we get

\[
\sum_{i=1}^{m} au_i + \sum_{j=m+1}^{m+n} bv_j
\]

which is a dual objective of the original transshipment problem with mixed constraints.

Thus, the objective functions of primal and dual of the transshipment problem with mixed constraints are equal and hence this is optimal.