Introduction
Chapter 1

Introduction

The study of differentiable manifolds besides being an interesting subject in itself has become useful in an ever increasing number of disciplines of pure as well as applied mathematics. In modern differential geometry, the study of manifolds with structures and their submanifolds has been engaging the attention of many contemporary leading mathematicians, as a consequence of which some very interesting and fruitful results have been obtained. The present thesis is an outcome of the inspiration that I received from the researches of these mathematicians.

1.1 A Glimpse of History

Differential geometry has been of increasing importance to Mathematical Physics due to Einstein’s general relativity postulation that the universe is curved. Contemporary differential geometry is intrinsic, meaning that the spaces it considers are smooth manifolds whose geometric structures are governed by Riemannian metric, which determines how distances are measured near each point, and not a priory part of some ambient flat Euclidean space.

Johnn Carl Friedrich Gauss was a German mathematician and physical scientist who contributed significantly to many fields, including Number theory, Statistics, Analysis, Differential Geometry, Geodesy, Geophysics, Electrostatics, Astronomy, and Optics. His more interest in differential geometry, a field of mathematics dealing with
curves and surfaces. Among other things, he came up with the notion of Gaussian curvature. This led in 1828 to an important theorem. The theorem says that curvature does not depend on how the surfaces might be embedded in three-dimensional spaces or two-dimensional spaces.

In the nineteenth century Gauss laid down the foundation of differential geometry of surface in three-dimensional Euclidean space. Now, in 1854, Riemann introduced the concept of differential geometry of more than three-dimensions. He has given the concept of tensors in defining the spaces, Riemannian metric and various results.

Later on mathematician like Beltrami (1868), Lipschitz (1869) and Christoffel (1868) etc. enriched the ideas of Riemann by introducing Beltrami parameters, covariant differentiation, Gauss equation and Christoffel symbols.

Schouten and Dantzing (1930, 1931) transferred the results of differential geometry of Riemannian spaces with affine connections to the case of space with complex structure. The opened an era of the Complex manifold. Ehresmann (1947) defined an almost complex type (1, 1) whose square is minus unity. Weyl (1847) pointed out that in a complex manifold, there exists a tensor of type (1, 1) whose square is minus unity \( f^2 = -I_n \), \( f \) is called an almost complex structure to \( M_n \). Nijenhuis in (1951) introduced a very important tensor, named as Nijenhuis tensor.

The study of an odd-dimensional manifold was initiated by Boothby and Wang in (1958) and Gray, J. W. in (1959) from the topological point of view and the structure introduced by them is called contract structure. Sasaki (1960) and Hsu (1962) made it possible to study the manifolds with the help of tensor analysis. These manifolds are called contact and almost contact manifolds. Yano, K. (1963) generalized the concept of an almost complex structure and defined \( f \)-structure as (1, 1) tensor field \( f(\neq 0) \) satisfying \( f^3 + f = 0 \).

\( \varphi(4 \pm 2) \)-structure on a \( C^\infty \) manifold defined by a non-null tensor field \( \varphi \) of type
(1, 1) was studied by Yano, Houh and Chen (1963). Tachibana (1959) introduced the concept of an almost analytic vector field in the almost complex spaces.

Other type of structures on the differentiable manifolds were defined and studied by Helgason (1962), Yano (1963), Duggal (1964), Yano and Ishihara (1965), Upadhyay and Dubey (1973), Kon (1973), Amur and Hedge (1974), Sinha and Singh (1975) and several others.

The differential geometry of tangent and cotangent bundles were studied by Sasaki (1958), Yano (1963), Ledger (1965). Yano and Kobayashi (1966) developed the theory of vertical and complete lifts of tensor fields and connections in the tangent bundles. Yano and Ishihara (1967) developed the theory of horizontal lift of tensor fields in tangent bundles. Yano and Patterson (1967) developed the theory of vertical, complete and horizontal lifts of tensor fields and connections to cotangent bundles.

In 1972, K. Kenmotsu studied a certain class of an almost contact manifold. Janseen and Vanhecke (1981) named this structure as Kenmotsu structure and differentiable manifold equipped with this structure is called Kenmotsu manifold. In 1985, J. A. Oubina studied a new class of almost contact metric structure known as trans-Sasakian structure which generalizes both $\alpha$-Sasakian and $\beta$-Kenmotsu structure and nearly trans-Sasakian structure of type $(\alpha, \beta)$. Algere introduced and studied generalized Sasakian space forms.

The theory of submanifolds as a field of differential geometry is as old as differential geometry itself. CR-submanifold was introduced by Bejancu in 1978 and generalized CR-submanifolds of a Kähler manifold was introduced by Mihai in 1995. Chen, Blair, Shahid and some others have studied different properties of submanifolds.
1.2 Manifold

While there are many kinds of manifolds- for example, topological manifolds, $C^k$-manifolds, analytic manifolds, and complex manifolds. We are concerned mainly with smooth manifolds. Starting with topological manifolds, which are Hausdorff, second countable, locally Euclidean spaces, we introduce the concept of a maximal $C^\infty$ atlas, which makes a topological manifold into a smooth manifold [134].

1.3 Topological Manifold

A topological space is second countable if it has a countable basis. A neighborhood of a point $p$ in a topological space $M$ is an open set containing $p$. An open cover of $M$ is a collection $\{U_\alpha\}_{\alpha \in A}$ of open sets in $M$ whose union $\bigcup_{\alpha \in A} U_\alpha$ is $M$.

**Definition 1.3.1** A topological manifold is a Hausdorff, second countable, locally Euclidean space. It is said to be of dimension $n$ if it is locally Euclidean of dimension $n$.

A topological space is said to be Hausdorff (second countable) if each pair of distinct points has disjoint neighbourhoods (if and only if it has a countable basis of open sets). Locally Euclidean space means each point $p \in M$ has a neighbourhood $U$ in $M$ which is homeomorphic to an open subset of $\mathbb{R}^n$.

Let $M$ be an $n$-dimensional topological manifold. If $U$, an open set of $M$ containing $x \in M$, is homeomorphic to an open set $E$ of $\mathbb{R}^n$ by homeomorphism $\phi : U \to E$ such that $\phi(x) = x_i = (x_1, x_2, \ldots, x_n)$, then pair $(U, \phi)$ is called a coordinate chart, $U$ is called coordinate neighborhood, $\phi$ is called coordinate map and $x_i = (x_1, x_2, \ldots, x_n)$ are called local coordinate on $M$ at $x$. The adjective local is to indicate that the coordinates are defined only on the part $U$ of $M$.

$C^r$-Function:-
Let $f : U \subset \mathbb{R}^n \to \mathbb{R}^n$ be a real valued function defined on $U$. $f$ is said to be $C^r$-function defined on $U$ or equivalently we say that $f \in C^r(U)$, if all $k^{th}$ order partial derivatives exist and are continuous for $1 \leq k \leq r$, where $r$ is a positive integer. If a function is $C^r$ for every positive integer $r$ than $f$ is said to be $C^\infty$-function or smooth function defined on $U$ or equivalently we say that $f \in C^\infty(U)$.

1.4 Differentiable Manifold

"An $m$-dimensional manifold is a Hausdorff topological space which is connected and has the property that each point has a neighborhood homeomorphic to some open set in Cartesian $m$-space."

Suppose $(U, \varphi_U)$ and $(V, \varphi_V)$ are two coordinate charts of $M$. If $U \cap V \neq \varnothing$, then $\varphi_U(U \cap V)$ and $\varphi_V(U \cap V)$ are two non empty open sets in $\mathbb{R}^m$ and the map

$$\varphi_V \circ \varphi_U^{-1} : \varphi_U(U \cap V) \to \varphi_V(U \cap V),$$

defines a homeomorphism between these two open sets, with inverse given by

$$\varphi_U \circ \varphi_V^{-1} : \varphi_V(U \cap V) \to \varphi_U(U \cap V).$$

These are two maps between open sets in a Euclidean space. Expressed in coordinates, $\varphi_V \circ \varphi_U^{-1}$ and $\varphi_U \circ \varphi_V^{-1}$ each represents $m$ real-valued functions on an open set of a Euclidean space. Let $r(r > 0)$ be an integer, a map $f$ from an open set $A \subset \mathbb{R}^m$ into $R$ is called $C^r$ on $A$ if it possesses continuous partial derivatives on $A$ of all orders $r$. If $f$ is $C^r$ on $A$ for all $r$, then $f$ is called $C^\infty$-map on $A$. Let $M$ be a set. An $m-$coordinate pair on $M$ is a pair $(U, \varphi_U)$ consisting of a subset $U$ of $\tilde{M}$ and one to one map $\varphi_U$ of $U$ onto an open set in $\mathbb{R}^m$. An $m$-coordinate pair $(U, \varphi_U)$ is $C^r$ related to another $(V, \varphi_V)$ if the maps $\varphi_U \circ \varphi_V^{-1}$ and $\varphi_V \circ \varphi_U^{-1}$ are $C^r$ maps. A maximal collection of $C^r$ related $m$-coordinate pairs is called a $C^r$ $m$-atlas. An $m$-
dimensional $C^r$ manifold is the set $M$ together with a $C^r$ $m$-atlas. A $C^\infty$ manifold is called differential manifold.

1.4.1 Tangent Vector

Let $M$ be a $C^\infty$ manifold of dimension $m$. Let $p$ be a point in $M$ and let $C^\infty(p)$ denote the set of real valued functions that are $C^\infty$ on some neighbourhood of $p$.

A tangent vector at point $p$ is a real valued function $X_p$ on $C^\infty(p)$ satisfying the following properties:

\[ X_p (f + g) = X_p f + X_p g, \]  
\[ X_p (af) = a X_p (f), \]  
\[ X_p (fg) = X_p (f) g(p) + f(p) X_p (g), \]

where $f$ and $g$ are $C^\infty$ functions in the neighbourhood of point $p$ and $a$ is a real number.

Taking $f$ as unit function and $g = f$, we obtain from above

\[ X_p (f) = f(p) X_p (1) + 1 X_p (f), \]

provided that $f(p) \neq 0$, this implies that $X_p = 0$. Since $X_p$ is a linear mapping, we have, for any constant function $\lambda$,

\[ X_p (\lambda) = X_p (\lambda . 1) = \lambda X_p (1) = 0, \]

and thus a tangent vector maps a constant function into zero.

The collection of all tangent vectors at a point forms a vector space over the field $R$. The vector space of tangent vectors at a point $P \in M$ is called the tangent space $TM$ at $P$. The collection of all tangent spaces of a manifold $M$ is called a tangent bundle of $M$ and denoted by $TM$. It is a vector space over the field of real numbers under the operations of vector addition and scalar multiplication given below

\[ (X_p + Y_p) f = X_p f + Y_p f, \]  

(1.4.4)
and

$$(aX_p) f = a(X_p f),$$

(1.4.5)

for $X_p \in T_p(M)$, $f \in C^\infty(p)$ and $a$ is a real number.

### 1.4.2 Vector Field

A vector field $X$ on an open set $U$ in $M$ is a mapping that assigns to each point $p$ in $U$ a vector $X_p \in T_p(M)$, where $T_p(M)$ is tangent space to manifold $M$ at $p$. If $U$ is open, the vector field $X$ is $C^\infty$ on $U$ and for each real valued function $f$ which is $C^\infty$ on $V$, the function $Xf$ is $C^\infty$ on $U \cap V$. Define $Xf$ as

$$(Xf)(p) = X_p f.$$

It is easy to verify that $Xf$ is again a $C^\infty$ vector field. The set of all $C^\infty$ vector fields on $M$, therefore, forms a module over the ring of real valued $C^\infty$ functions on $M$. Let us denote this by $\chi(M)$.

If $X, Y$ are $C^\infty$ vector fields on $M$, their Lie bracket $[X, Y]$ is defined as

$$[X, Y] f = X(Y f) - Y(X f).$$

(1.4.6)

If $f, g$ are $C^\infty$ functions then

$$[X, Y](f + g) = [X, Y]f + [X, Y]g,$$

(1.4.7)

and

$$[X, Y]af = a[X, Y] f,$$

(1.4.8)

where $a$ is the real number.
Further,

\[ [X, Y](fg) = X\{Y(fg)\} - Y\{X(fg)\}, \]
\[ = X\{f(Yg) + g(Yf)\} - Y\{f(Xg) + g(Xf)\}, \]
\[ = fX(Yg) + gX(Yf) - fY(Xg) - gY(Xf), \]
\[ = f\{X(Yg) - Y(Xg)\} + g\{X(Yf) - Y(Xg)\}, \]
\[ = f[X, Y]g + g[X, Y]f, \]

\[ [X, Y](fg) = f[X, Y]g + g[X, Y]f. \] \hspace{1cm} (1.4.9)

Thus, \([X, Y]\) is vector field.

We also have

\[ [X, Y] = -[Y, X], \] \hspace{1cm} (1.4.10)
\[ [X, X] = 0, \] \hspace{1cm} (1.4.11)
\[ [fx, gy] = f(Xg)Y - YfX + fgy[X, Y], \] \hspace{1cm} (1.4.12)
\[ [X + Y, Z] = [X, Z] + [Y, Z], \] \hspace{1cm} (1.4.13)

for all \(X, Y, Z \in \chi(M)\).

The Lie bracket satisfies the Jacobi identity

\[ [X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0. \] \hspace{1cm} (1.4.14)

### 1.4.3 Tensors and Forms

Let \(M\) be an \(m\)-dimensional \(C^\infty\) manifold and let \(p\) be any point in \(M\). The tangent space \(T_p(M)\) at \(p\) is an \(m\)-dimensional vector space. For an integer \(q > 0\), a \(q\)-cotensor at a point \(p\) of \(M\) is a real valued \(q\)-linear function on \(T_p(M) \times T_p(M) \times \ldots \ldots T_p(M)\) \((q\text{-copies})\). Thus, in particular, \(F\) is a \(Z\)-cotensor at \(p\) if

\[ F(X_p, Y_p) \text{ is real.} \] \hspace{1cm} (1.4.15)
\[
F(X_p + Y_p, Z_p) = F(X_p, Z_p) + F(Y_p, Z_p),
\]
\[
F(X_p, Y_p + Z_p) = F(X_p, Y_p) + F(X_p, Z_p),
\]
(1.4.16)

and

\[
F(bX_p, Y_p) = F(X_p, bY_p) = bF(X_p, Y_p),
\]
(1.4.17)

for all \(X_p, Y_p, Z_p \in T_p(M)\) and \(b \in R\), \(R\) is the set of real numbers.

Let \(T^*_p(M)\) be the dual space of \(T_p(M)\). Thus \(T^*_p(M)\) is the set of all linear functions on \(T_p(M)\) into the set of real numbers. An element of \(T^*_p(M)\) is called covariant vector or 1-form at the point \(p\). For \(q > 0\), a \(q\)-contravariant tensor or a tensor of type \((q, 0)\), at \(p \in M\) is a real valued \(q\)-linear function on \(T^*_p(M) \times T^*_p(M) \times \ldots \times T^*_p(M)\) \((q\)-copies).

Thus \(T\) is a 2-contravariant tensor on \(M\) if it satisfies

\[
T(\alpha, \beta) \text{ is real.}
\]
(1.4.18)

\[
T(\alpha + \beta, \gamma) = T(\alpha, \gamma) + T(\beta, \gamma),
\]
\[
T(\alpha, \beta + \gamma) = T(\alpha, \beta) + T(\alpha, \gamma),
\]
(1.4.19)
\[
T(b\alpha, \beta) = T(\alpha, b\beta) = bT(\alpha, \beta),
\]
(1.4.20)

for all \(\alpha, \beta, \gamma \in T^*_p(M)\) and \(b\) is a real number. This process can be generalized for the tensor of type \((r, s)\).

\section{1.5 Riemannian Manifold}

The branch of differential geometry which studies the structures associated with a field of non-singular, symmetric, second-order covariant tensors is called Riemannian geometry.

Let \(M\) be a manifold. Then a function

\[
g: M_p \times M_p \to R,
\]
defined at each point $p \in M_p$ is called Riemannian metric if it satisfies following conditions:

\begin{align*}
(1)(i) \quad g(X + Y, Z) &= g(X, Z) + g(Y, Z), \\
(ii) \quad g(X, Y + Z) &= g(X, Y) + g(X, Z), \\
(iii) \quad g(aX, Y) &= ag(X, Y) = g(X, aY),
\end{align*}

i.e., $g$ is bilinear form on $M_p$, for all $p \in M_p$.

\begin{equation}
(2) \quad g(X, Y) = g(Y, X),
\end{equation}

i.e., $g$ is symmetric.

\begin{equation}
(3) \quad g(X, X) > 0, \text{ iff } X \neq 0 \quad \& \quad g(X, X) = 0, \text{ iff } X = 0,
\end{equation}

i.e., $g$ is positive definite.

Thus Riemannian metric is a $C^\infty$ real valued, bilinear symmetric & positive definite function $g$ on ordered pair of tangent vectors at each point of the manifold.

"A manifold $M$ with Riemannian metric $g$ is called Riemannian manifold, denoted by $(M, g)$." 

If condition (3) is replaced by

$$g(X, Y) = 0, \text{ for all } X \text{ implies } Y = 0,$$

i.e., $g$ is non-singular, then the metric is called pseudo Riemannian metric or semi-Riemannian metric.

### 1.5.1 Affine Connection

Let $M$ be a $C^\infty$ manifold. An affine connection $\nabla$ on manifold $M$ is an operator, that assigns to each pair of $C^\infty$ vector fields $X$ and $Y$, a $C^\infty$ vector filed $\nabla_X Y$, i.e.,

$$\nabla : (X, Y) \rightarrow \nabla_X Y,$$
which satisfies the following conditions:

\[
\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z, \tag{1.5.1}
\]

\[
\nabla_{X+Y}Z = \nabla_X Z + \nabla_Y Z, \tag{1.5.2}
\]

\[
\nabla_{fX}Y = f\nabla_X Y, \tag{1.5.3}
\]

and

\[
\nabla_X(fY) = (Xf)Y + f\nabla_X Y, \tag{1.5.4}
\]

where \( f \) is a \( C^\infty \) real valued function on \( M \). The operator \( \nabla_X \) is called the covariant differentiation with respect to \( X \).

An affine connection \( \nabla_X \) on \( M \) is said to be the Riemannian connection, if

\[
\nabla_X Y - \nabla_Y X = [X, Y], \tag{1.5.5}
\]

and

\[
Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z), \tag{1.5.6}
\]

where \( g \) is the Riemannian metric on \( M \) and \([X, Y] \) is a vector field defined by

\[
[X, Y]f = X(Yf) - Y(Xf).
\]

Further, if the Riemannian metric \( g \) is such that:

\[
\nabla g = 0.
\]

Then the connection is called metric connection otherwise it is non-metric.

### 1.5.2 Torsion Tensor

The torsion tensor of connection \( \nabla \) is a vector valued tensor \( T \) that assigns to each pair of vector fields \( X, Y \), a vector field \( T(X, Y) \) given by

\[
T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]. \tag{1.5.7}
\]

The above equation shows that \( T(X, Y) \) is skew-symmetric.
1.5.3 Curvature Tensor

Let $\nabla$ be a connection on a differentiable manifold $M$. The curvature tensor $R$ of the connection $\nabla$ is a tensor of type $(1, 3)$ given by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z,$$  \hspace{1cm} (1.5.8)

where $X, Y$ and $Z$ are arbitrary vector fields in $M$.

The curvature tensor $R$ of the connection $\nabla$ satisfies the following properties:

$$R(X, Y)Z + R(Y, X)Z = 0,$$  \hspace{1cm} (1.5.9)

i.e., $R$ is skew-symmetric

$$R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0,$$  \hspace{1cm} (1.5.10)

it is called Bianchi’s first identity

$$\nabla_X R(Y, Z)W + \nabla_Y R(Z, X)W + \nabla_Z (X, Y)W = 0,$$  \hspace{1cm} (1.5.11)

it is known as Bianchi’s second identity

$$R(X, Y)(Z + U) = R(X, Y)Z + R(X, Y)U,$$

$$R(X + Y, Z)U = R(X, Z)U + R(Y, Z)U,$$

$$R(fX, Y)Z = fR(X, Y)Z = R(X, Y)fZ,$$  \hspace{1cm} (1.5.12)

i.e., it is trilinear.
1.5.4 Ricci Tensor

Let $M$ be a Riemannian manifold with Riemannian connection $\nabla$, then Ricci tensor field $S$ is a covariant tensor field of degree 2 defined as

$$\text{Ric}(Y, Z) = S(Y, Z) = \text{tr}\{X \to R(X, Y)Z\} = g(QY, Z),$$

(1.5.13)

where $Q$ is the Ricci operator.

Also,

$$\text{Ric}(Y, Z) = (C_1^R)(Y, Z) = -(C_2^R)(Y, Z) = \text{Ric}(Z, Y),$$

(1.5.14)

i.e., Ricci tensor is symmetric.

1.5.5 Geodesics

Basically, geodesics are seen in the study of Riemannian geometry and more generally metric geometry. In relativistic physics, geodesics describe the motion of point particles under the influence of gravity alone. In particular, the path taken by a falling rock, an orbiting satellite, or the shape of a planetary orbit is all geodesics in curved space-time. More generally, the topic of sub-Riemannian geometry deals with the paths that objects may take when they are not free, and their movement is constrained in various ways. As the special case of the surfaces in any $n$-dimensional Riemannian space, there are special intrinsic curves called geodesics which are curves of shortest distance.

A submanifold $M$ of $\tilde{M}$ is known to be totally geodesic if second fundamental form denoted by $\sigma$ is zero or $A_N$ (which is defined later) is zero. $M$ is totally umbilical in $\tilde{M}$, then

$$A_N = \zeta I,$$

where $\zeta$ is Kählerian metric.
$M$ is minimal in $\tilde{M}$ if mean curvature

$$\mu = \frac{\text{trace} \sigma}{\dim \tilde{M}} = 0.$$  

Any submanifold $M$ which is minimal and totally umbilical is totally geodesic. For a distribution $D$ on $M$, $M$ is said to be $D$-totally geodesic if, for all $X, Y \in D$, we have $\sigma(X, Y) = 0$. If for all $X, Y \in D$, we have $\sigma(X, Y) = g(X, Y)\hat{N}$ for some normal vector $\hat{N}$, then $M$ is called $D$-totally umbilical. For two distributions $D_1$ and $D_2$ defined on $M$, $M$ is said to be $(D_1, D_2)$-mixed totally geodesic if, for all $X \in D_1$ and $Y \in D_2$, we have

$$\sigma(X, Y) = 0 . \tag{1.5.15}$$

A semi-Riemannian submanifold $M$ of $\tilde{M}$ is called totally geodesic if $\sigma(X, Y) = 0$. Thus a totally geodesic submanifold $M$ is intrinsically flat. A point $P$ of $M$ in $\tilde{M}$ is umbilical, provided there is a normal vector $Z \in T_P^\perp M$, such that

$$\sigma(X, Y) = g(X, Y)Z \quad \forall X, Y \in TM, \tag{1.5.16}$$

A semi-Riemannian submanifold $M$ of $\tilde{M}$ is totally umbilical provided every point of $M$ is umbilical. Then there is a smooth normal vector field $Z$ of $M$ called the normal curvature vector field of $M$ such that

$$\sigma(X, Y) = g(X, Y)Z . \tag{1.5.17}$$

1.5.6 Integrable Distribution

When a differentiable manifold $M$ admits a distribution of $r$-planes, it may happen that these $r$-planes (or contact elements) fit together in such a way that they determine an $r$-dimensional submanifold of $M$, the $r$-plane at $P$ being the tangent space at $P$ to this submanifold. In this case, the distribution is said to be integrable or involutive. The condition of integrability may be expressed as follows:
The distribution $D$ is integrable when for any two vector fields $X, Y$ belonging to $D$, the vector field $[X, Y]$ also belongs to $D.$

### 1.6 Semi-Riemannian Manifold

**Definition 1.6.1** A metric tensor $g$ on a smooth manifold $M$ is a symmetric non-degenerate $(0,2)$ tensor field on $M$ of constant index. In other words, $g$ smoothly assigns to each point $p$ of $M$ a scalar product $g_p$ on the tangent space $T_p M$ and the index of $g_p$ is the same for all $p$.

**Definition 1.6.2** A smooth manifold $M$ equipped with a metric tensor $g$ is a semi-Riemannian (or pseudo-Riemannian) manifold.

**Definition 1.6.3** The common value $\nu$ of index $g_p$ on a semi-Riemannian $M$ is called the index of $M$; $0 \leq \nu \leq m = \dim M$. If $\nu = 0$, $M$ is a Riemannian manifold; each $g_p$ is then a (positive definite) inner product on $T_p M$. If $\nu = 1$ and $m \geq 2$, $M$ is Lorentz manifold.

If $x^1, x^2, \ldots, x^n$ is a coordinate system on a subset $U$ of $M$, the components of the metric tensor $g$ on $U$ are

$$g_{ij} = g(\delta_i, \delta_j), \quad (1 \leq i, j \leq m).$$

For vector fields $X = \sum X^i \delta_i$ and $Y = \sum Y^j \delta_j$

$$g(X, Y) = \sum g_{ij} X^i Y^j.$$

Since $g$ is non-degenerate, at each point $p$ of $U$ the matrix $(g_{ij}(p))$ is invertible, and its inverse matrix is denoted by $(g^{ij}(p))$. The function $g^{ij}$ are smooth on $U$. Since $g$ is symmetric $g_{ij} = g_{ji}$ and hence $g^{ij} = g^{ji}$ for $1 \leq i, j \leq m$. On $U$ the metric tensor can be written as $g = \sum g_{ij} dx^i \otimes dx^j.$
For an integer \( \nu \) with \( 0 \leq \nu \leq m \), changing the first \( \nu \) plus sign to minus gives a metric tensor

\[
g(X, Y) = -\sum_{i=1}^{\nu} X^i Y^i + \sum_{\nu+1}^{m} X^j Y^j,
\]

of index \( \nu \). The resulting semi-Euclidean space \( \mathbb{R}^m_\nu \) reduces to \( \mathbb{R}^m \) if \( \nu = 0 \).

If we fix the notation

\[
\varepsilon_i = \begin{cases} 
-1 & \text{for } 1 \leq i \leq \nu \\
+1 & \text{for } \nu + 1 \leq i \leq m 
\end{cases},
\]

then the metric tensor of \( \mathbb{R}^m_\nu \) can be written as \( g = \sum \varepsilon_i dx^i \otimes dx^i \). The geometric significance of the index of a semi-Riemannian manifold derives from the following trichotomy.

**Definition 1.6.4** A tangent vector \( X \) to \( M \) is

(i) spacelike if \( g(X, X) > 0 \) or \( \nu = 0 \),

(ii) timelike if \( g(X, X) < 0 \),

(iii) null (or lightlike) if \( g(X, X) = 0 \) and \( \nu = 0 \).

The set of all null vectors in \( T_p M \) is called the null-cone at \( p \in M \). The category into which a given tangent vector falls is called its causal character.

A two dimensional subspace \( P \) of the tangent space \( T_p M \) is called tangent plane to \( M \) at \( p \in M \). For tangent vectors \( X, Y \) define \( \|X \wedge Y\|^2 = g(X, X)g(Y, Y) - \|g(X, Y)\|^2 \).

We say a tangent plane \( P \) is non degenerate if and only if \( \|X \wedge Y\|^2 \neq 0 \) for every basis \( X, Y \) for \( P \). Otherwise, it is degenerate.

**Definition 1.6.5** Let \( P \) be a non-degenerate tangent plane to \( M \) at \( P \). The number \( K(X \wedge Y) = \frac{4}{\|X \wedge Y\|^2} \) is independent of the choice of basis \( X, Y \) for \( P \), and is called the sectional curvature \( K(P) \) of \( P \). A semi-Riemannian \( M \) has constant curvature if its sectional curvature is constant. If \( M \) has constant curvature \( c \), then [110]

\[
R(X, Y)Z = c\{g(Z, X)Y - g(Z, Y)X\}.
\]
**Definition 1.6.6** The Ricci curvature tensor $\text{Ric} M$ is symmetric and is given relative to a frame field by

$$
\text{Ric}(X,Y) = \sum_i \varepsilon_i g(R(X,e_i)Y, e_i)
$$

where $\varepsilon_i = g(e_i, e_i)$.

Since sectional curvature determines the curvature tensor, it also determines $\text{Ric}$. By polarization and scalar multiplication, $\text{Ric}$ can be reconstructed at each point $p$ from its value $\text{Ric}(u,u)$ on the unit vectors at $p$. But if $e_1, e_2, \ldots, e_n$ is a frame at $p$ such that $u = e_i$, then

$$
\text{Ric}(u,u) = \sum_i \varepsilon_i g(R(u,e_i)u, e_i) = g(u,u) \sum K(u\Lambda e_i).
$$

Thus $\text{Ric}(u,u)$ is the sum of sectional curvatures of any $n-1$ orthogonal non degenerate planes through $u$ except for the sign $g(u,u) = \pm 1$.

**Definition 1.6.7** The scalar curvature $\rho$ of $M$ is the contraction of its Ricci tensor. In coordinates

$$
\rho = g^{ij} R_{ij} = g^{ij} R^k_{ijk},
$$

contracting relative to frame field yields:

$$
\rho = \sum_{i \neq j} K(e_i \Lambda e_j) = 2 \sum_{i < j} K(e_i \Lambda e_j).
$$

## 1.7 Complex Geometry

### 1.7.1 Almost Complex Manifold

An almost complex structure on a real differentiable manifold $M$ of dimension $m$ ($m = 2n$, $n$ is a positive integer) is a $(1,1)$ tensor field $F$ which is at every point $p$ of $M$, an endomorphism of the tangent space $T_p M$, such that

$$
F^2 = -I,
$$

(1.7.1)
where \( I \) denotes the identity transformation of \( T_p(M) \).

A manifold \( M \) equipped with an almost complex structure \( F \) is called an almost complex manifold. Every almost complex manifold is always even dimensional.

The Nijenhuis tensor \([F,F]\) with respect to tensor field \( F \) of type \((1,1)\) is a vector valued bilinear tensor field of type \((1,2)\) defined by

\[
[F,F](X,Y) = F^2[X,Y] + \{FX, FY\} - F\{FX, Y\} - F\{X, FY\},
\]

(1.7.2)

for all vector fields \( X, Y \in TM \).

In examining the question of integrability of an almost complex structure, Nijenhuis tensor plays an important role.

### 1.7.2 Complex Manifold

An almost complex manifold \((M, J)\) with vanishing Nijenhuis tensor, that is,

\[
[F,F](X,Y) = 0 \text{ for all } X, Y \in TM,
\]

(1.7.3)

is called complex manifold.

### 1.7.3 Almost Hermite Manifold

On a differentiable manifold, we can always introduce a Riemannian metric \( g \). A metric \( g \) is called Hermite metric on an almost complex manifold, if

\[
g(FX, FY) = g(X, Y).
\]

(1.7.4)

"An almost complex manifold with an almost complex structure \( F \) and a Hermite metric \( g \) is called almost Hermite manifold."

A Hermite metric thus defines a Hermite inner product on each tangent space \( T_pM \) with respect to the complex structure defined by \( F \).
Let \( M \) be an almost Hermite manifold with an almost complex structure \( F \) and a Hermite metric \( g \). The fundamental 2-form \( \Omega \) of \( M \) is defined by

\[
\Omega(X, Y) = g(X, FY),
\]

for all vector fields \( X \) and \( Y \) on \( M \). Then, we have

\[
\Omega(X, Y) = \Omega(FX, FY) \quad \text{and} \quad \Omega(X, Y) = -\Omega(Y, X).
\]

The fundamental 2-form \( \Omega \) on \( M \) is closed, if

\[
d\Omega = 0,
\]

or equivalently

\[
(\tilde{\nabla}_X \Omega)(Y, Z) + (\tilde{\nabla}_Y \Omega)(Z, X) + (\tilde{\nabla}_Z \Omega)(X, Y) = 0,
\]

for all vector fields \( X, Y, Z \) on \( M \); where \( \tilde{\nabla} \) is Riemannian connection on \( M \) and \( d \) is the exterior derivative.

### 1.7.4 Kähler Manifold

A Hermite metric \( g \) on an almost complex manifold \( M \) is called a Kähler metric if the fundamental 2-form \( \Omega \) is closed. An almost complex manifold \( M \) with a Kähler metric is called an almost Kähler manifold. A complex manifold with a Kähler metric is called a Kähler manifold.

An almost Hermite manifold for which

\[
(\tilde{\nabla}_X F)(Y) = 0,
\]

is called a Kähler manifold [104], [81].
1.8 Contact Geometry

The word contact has been borrowed from a “contact transformation” which is natural in view of the example of Huygen’s Principle. Consider the propagation of light (or various other disturbances) in a medium (for the moment we do not specify the properties of this medium). The medium will be given by a 3-dimensional manifold \( M \). Given a point \( p \) in \( M \) and \( t > 0 \), let \( I_p(t) \) be the set of all points to which light can travel in time less than or equal to \( t \). The wave front of \( p \) at time \( t \) is the boundary of this set and is denoted by \( \Phi_p(t) = \partial I_p(t) \). Huygen’s principle state that “\( \Phi_p(t + \tau) \) is the envelope of the wave fronts \( \Phi_p(\tau) \) for all \( \tau \in \Phi_p(t) \)”.

In terms of contact geometry this can be best understood. Let \( \pi |_S : (TM \setminus \{0\}) \to PM \) be the natural projection \([76]\) and let \( S \) be any smooth subbundle of \( TM \setminus \{0\} \) that is transverse to the radial vector field in each fiber and for which \( \pi : \to PM \) is a diffeomorphism; where the set \( PM = \text{non zero covectors in } TM \) is the projectivized cotangent space of a manifold \( M \) and defined by \( PM = (TM \setminus \{0\})/R_+ \). The restriction of the Liouville form to \( S \) gives a contact form \( \eta \) and a corresponding Reeb vector field \( v \). Given a subset \( F \) of \( M \) with a well defined tangent space at every point set
\[
L_F = \{ p \in S : \pi(p) \in F \text{ and } p(w) = 0 \text{ for all } w \in T_p(L) \},
\]
the set \( L_F \) is a Legendrian submanifold of \( S \) and is called the Legendrian lift of \( F \). If \( L \) is a generic Legendrian submanifold in \( S \) then \( \pi(L) \) is called the front projection of \( L \) and \( L_{\pi}(L) = L \). Given a Legendrian submanifold \( L \), let \( \Psi_t(L) \) be the Legendrian submanifold obtained from \( L \) by flowing along \( v \) for time \( t \).

1.8.1 Contact Metric Manifold

A contact manifold is a \((2n + 1)\)-dimensional manifold \( M \) equipped with a global 1-form \( \eta \), called a contact form of \( M \) such that
\[
\eta \wedge (d\eta)^n \neq 0, \tag{1.8.1}
\]
everywhere on the manifold. In particular \( \eta \wedge (d\eta)^n \neq 0 \) is a volume element on \( M \), so that a contact manifold is orientable.

A Riemannian metric \( g \) is said to be an associated metric if there exists a tensor field of type \((1,1)\) such that

\[
d\eta(X,Y) = g(X,\varphi Y),
\]

\( \eta(X) = g(X,\xi), \)  

(1.8.2)

and

\[\varphi^2 = -I + \eta \otimes \xi,\]

(1.8.4)

for any vector field \( X,Y \) on \( M \), where \( I \) denote the identity map of the tangent space \( T_p M \) for \( p \in M \) and the symbol \( \otimes \) is the tensor product. From these equations one can easily obtain

\[\varphi \xi = 0, \ \eta \circ \varphi = 0,\]

(1.8.5)

and

\[g(\varphi X,\varphi Y) = g(X,Y) - \eta(X)\eta(Y).\]

(1.8.6)

The structure \((\varphi,\xi,\eta,g)\) on \( M \) is called a contact metric structure or contact Riemannian structure and the manifold \( M \) equipped with a contact metric structure is said to be a contact metric manifold on contact Riemannian manifold.

1.8.2 Almost Contact Metric Manifold

The \((\varphi,\xi,\eta,g)\)-structure satisfying the conditions:

\[\varphi \xi = 0, \ \eta(\varphi X) = 0, \ \eta(\xi) = 1, \]

(1.8.7)

\[\varphi^2 X = -X + \eta(X)\xi \ \text{and} \ g(X,\xi) = \eta(X),\]

(1.8.8)

\[g(\varphi X,\varphi Y) = g(X,Y) - \eta(X)\eta(Y),\]

(1.8.9)
for any vector fields $X$ and $Y$ on $M$, is called an almost contact metric structure, so the manifolds $M$ with structure $(\varphi, \xi, \eta, g)$, is now known as almost contact metric manifold.

Define 2-form on $M$ by $\Phi(X, Y) = g(X, \varphi Y)$. We call $\Phi$ the fundamental 2-form of the almost contact metric structure $(\varphi, \xi, \eta, g)$. Since $\varphi$ has rank $2n$, we have

$$\eta \wedge \Phi^\alpha \neq 0.$$ 

An almost contact structure $(\varphi, \xi, \eta)$ on $M$ is normal if and only if

$$N(X, Y) = [\varphi, \varphi](X, Y) + 2d\eta(X, Y)\xi.$$ 

An almost contact metric structure $(\varphi, \xi, \eta, g)$ on $M$ is said to be ( [89], [96])

(i) quasi Sasakian if $\Phi$ is closed and $(\varphi, \xi, \eta)$ is normal,
(ii) cosymplectic if $\Phi$ and $\eta$ are closed and $(\varphi, \xi, \eta)$ is normal,
(iii) Sasakian if $\Phi = d\eta$ and $(\varphi, \xi, \eta)$ is normal,
(iv) contact metric manifold if $\Phi = d\eta(X, Y)$ for all $X, Y \in TM$;
(v ) $K$-contact manifold if $\tilde{\nabla}\xi = -\varphi$, where $\tilde{\nabla}$ is Levi-Civita connection.

### 1.8.3 Sasakian Manifold

A Riemannian manifold $(M, g)$ is called a Sasakian manifold if there exists a killing vector field $\xi$ of unit length on $M$ so that the tensor field $\varphi$ of type $(1, 1)$, defined by $\varphi(X) = -\nabla_X \xi$, satisfies the condition:

$$(\tilde{\nabla}_X \varphi)(Y) = g(X, Y)\xi - g(\xi, Y)X,$$  \hspace{1cm} (1.8.10)

for any pair of vector fields $X$ and $Y$ on $M$. This is a curvature condition which can be easily expressed in terms of the Riemannian curvature tensor as

$$R(X, \xi)Y = g(\xi, Y)X - g(X, Y)\xi.$$  \hspace{1cm} (1.8.11)
Sasakian geometry is a special kind of contact metric geometry such that the structure transverse to the Reeb vector field \( \xi \) is Kähler and invariant under the flow of \( \xi \). In fact, \( \eta \) is the contact 1-form, and \( \varphi \) is a \((1,1)\) tensor field which defines a complex structure on the contact subbundle \( \ker \eta \) which annihilates \( \xi \).

1.8.4 \((k,\mu)\)-Contact Metric Manifold

The \((k,\mu)\)-nullity distribution of a contact metric manifold \( M \) \((\varphi, \xi, \eta, g)\) is a distribution \([24]\).

\[
N(k, \mu) : p \to N_p(k, \mu) = \{Z \in T_p M : R(X,Y)Z = k[g(Y,Z)X - g(X,Z)Y] + \mu[g(Y,Z)hX - g(X,Z)hY]\} \quad (1.8.12)
\]

where \(k, \mu\) are real constants. Hence, if the characteristic vector field \( \xi \) belongs to the \((k,\mu)\)-nullity distribution.

A contact metric manifold satisfying the relation:

\[
R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY],
\]

is called a \((k,\mu)\)-contact metric manifold. The class of \((k,\mu)\)-contact metric manifolds contains both the class of Sasakian and non-Sasakian manifolds. In the case of Sasakian manifolds, \(k = 1\) (and hence \(h = 0\)). But for non-Sasakian case \(k < 1\).

1.8.5 \(\eta\)-Einstein Manifold

An almost contact metric structure \((\varphi, \xi, \eta, g)\) on \( M \) is called an \(\eta\)-Einstein, if its Ricci-tensor \(S\) is of the form

\[
S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \quad (1.8.13)
\]

where \(a\) and \(b\) are smooth functions on \( M \).

An \(\eta\)-Einstein manifold becomes an Einstein, if \(b = 0\).
1.8.6 Trans-Sasakian Manifold

Trans-Sasakian manifold is given by Oubina [111]. In 1990 Blair and Oubina [23] expressed it by the condition:

\[(\tilde{\nabla}_X \varphi)Y = \alpha(g(X,Y)\xi - \eta(Y)X) + \beta(g(\varphi X,Y)\xi - \eta(Y)\varphi X),\]  \hspace{1cm} (1.8.14)  

for some smooth functions \(\alpha, \beta\) on \(M\) and we say that trans-Sasakian structure is of type \((\alpha, \beta)\). It is known that a trans-Sasakian structure is always normal.

Particularly, trans-Sasakian structures of type \((0,0)\), \((0,\beta)\) and \((\alpha,0)\) are called cosymplectic, \(\beta\)-Kenmotsu and \(\alpha\)-Sasakian, respectively.

1.8.7 Nearly Trans-Sasakian Manifold

An almost contact metric structure \((\varphi, \xi, \eta, g)\) on \(M\) is called nearly trans-Sasakian structure [70], if

\[(\tilde{\nabla}_X \varphi)Y + (\tilde{\nabla}_Y \varphi)X = \alpha(2g(X,Y)\xi - \eta(X)Y - \eta(X)Y) - \beta(\eta(X)X + \eta(X)\varphi Y).\]  \hspace{1cm} (1.8.15)  

A trans-Sasakian structure is always nearly trans-Sasakian. Also, nearly trans-Sasakian structures of type \((0,0)\), \((0,\beta)\) and \((\alpha,0)\) are called nearly cosymplectic, nearly \(\beta\)-Kenmotsu and nearly \(\alpha\)-Sasakian, respectively.

1.8.8 Generalized Ricci-recurrent manifold

A non-flat Riemannian manifold \(M\) is called a generalized Ricci-recurrent manifold [58], if its Ricci tensor \(S\) satisfies the condition:

\[(\nabla_X S)(Y,Z) = A(X)S(Y,Z) + B(X)g(Y,Z),\]  \hspace{1cm} (1.8.16)  

where \(\nabla\) is Levi-Civita connection of the Riemannian metric \(g\), and \(A, B\) are 1-forms on \(M\). In particular, if the 1-form \(B\) vanishes identically, then \(M\) reduces to the well known Ricci-recurrent manifold [116].
A Riemannian manifold is said to admit cyclic Ricci tensor, if

\[(\nabla_X S)(Y, Z) + (\nabla_Y S)(Z, X) + (\nabla_Z S)(X, Y) = 0.\] (1.8.17)

### 1.8.9 Kenmotsu Manifold

At first, in 1972, a class of contact Riemannian manifolds satisfying some special conditions was studied by Kenmotsu [91] and it was termed as Kenmotsu manifold.

Let \((M, g)\) be an almost contact Riemannian manifold with a contact form \(\eta\) the associated vector field \(\xi\), a \((1,1)\)-tensor \(\varphi\) and the associated Riemannian metric \(g\) satisfying the following condition:

\[ (\tilde{\nabla}_X \varphi)Y = -g(X, \varphi Y)\xi - \eta(Y)\varphi X, \] (1.8.18)

is called Kenmotsu manifold.

### 1.8.10 K-Contact Manifold

Let \(M\) be a \((2n + 1)\)-dimensional contact metric manifold with a contact metric structure \((\varphi, \xi, \eta, g)\), we define a \((1,1)\)-tensor field \(h\) by

\[ h = \frac{1}{2} \mathcal{L}_\xi \varphi, \] (1.8.19)

where \(\mathcal{L}_\xi\) denotes the Lie differentiation. In a contact metric manifold the symmetric operator \(h\) anti-commutes with \(\varphi\), i.e.,

\[ h\varphi + \varphi h = 0. \] (1.8.20)

Thus, if \(\lambda\) is an eigenvalue of \(h\) with eigenvector \(X\), \(-\lambda\) is also an eigenvalue of \(h\) with eigenvector \(X\). We also have

\[ h\xi = 0, \] (1.8.21)

\[ \text{trace } h = \text{trace } \varphi h = 0. \] (1.8.22)
Moreover, if $\nabla$ denotes the Riemannian connection of $g$, then we have
\begin{align*}
\nabla_X \xi &= \varphi X - \varphi h X, \\
\nabla_\xi \xi &= 0.
\end{align*} (1.8.23) (1.8.24)

A vector field $X$ on a Riemannian manifold $M$ is called a Killing vector field if and only if $\mathcal{L}_X g = 0$, where $\mathcal{L}$ denotes the operator of Lie differentiation.

A $(2n+1)$-dimensional contact metric manifold $M$ with a contact metric structure $(\varphi, \xi, \eta, g)$ for which the structural vector field $\xi$ is Killing vector field, is called a $K$-contact metric manifold. It is well known that a contact metric manifold is $K$-contact if and only if $h = 0$, i.e.,
\begin{equation}
\nabla_X \xi = -\varphi X,
\end{equation} (1.8.25)
for all $X \in \chi(M)$.

### 1.8.11 Lorentzian Manifold

An $n$-dimensional differentiable manifold $M$ is a smooth connected para compact Hausdorff manifold with a Lorentzian metric $g$, that is, $M$ admits a smooth symmetric tensor field $g$ of type $(0, 2)$ such that for each point $p \in M$, the tensor $g_p : T_p M \times T_p M \rightarrow \mathbb{R}$ is a non-degenerate inner product of signature $(-, +, ..., +)$, where $T_p M$ denotes the tangent vector space of $M$ at $p$ and $\mathbb{R}$ is the real number space. A non-zero vector $v \in T_p M$ is said to be timelike (resp., non-spacelike, null, spacelike) if it satisfies $g_p(v, v) < 0$ (resp., $\leq 0$, $= 0$, $> 0$) [110].

### 1.8.12 Almost Para Contact Manifold

A $(2n+1)$-dimensional differentiable manifold $M$ is said to be an almost paracontact structure $(\varphi, \xi, \eta)$ if it admits a tensor field $\varphi$, a unit timelike contravariant vector field $\xi$ and a 1-form $\eta$ which satisfy the relations:
\begin{equation}
\eta(\xi) = -1,
\end{equation} (1.8.26)
\[ \varphi^2 X = X + \eta(X)\xi, \]

(1.8.27)

for any vector field \( X \) on \( M \).

In a \( (2n+1) \)-dimensional almost paracontact manifold with structure \( (\varphi, \xi, \eta) \), the following conditions hold:

\[ \varphi \circ \xi = 0, \]

(1.8.28)

\[ \eta \circ \varphi = 0, \]

(1.8.29)

\[ \text{rank } \varphi = 2n. \]

(1.8.30)

In this definition, if we put \( \xi = -\xi \), then \( (\varphi, \xi, \eta) \) is an almost paracontact structure defined by I. Sato [122].

Every almost paracontact manifold satisfying a certain condition admits a Lorentzian metric [99] which stands analogous situation to the almost paracontact Riemannian metric for any almost paracontact manifold [123].

### 1.9 Space-Forms

In differential geometry, the curvature of a Riemannian manifold \( (M, g) \) plays a fundamental role and as it is well known, the sectional curvatures of a manifold determine the curvature tensor \( R \) completely. For any point \( p \in M \) and any plane section \( \pi \subseteq T_p M \), the sectional curvature \( K(\pi) \) is defined by

\[ K(\pi) = g(R(X,Y)Y, X), \]

(1.9.1)

where \( X, Y \) are orthonormal vector fields in \( \pi \). In such a case, we also denote \( K(\pi) \) by \( K(X \wedge Y) \). A Riemannian manifold with constant sectional curvature \( c \) is called real-space-form, and its curvature tensor satisfies the equation:

\[ R(X, Y)Z = c\{g(Y, Z)X - g(X, Z)Y\}. \]

(1.9.2)

Models for these spaces are the Euclidean spaces \( (c = 0) \), the spheres \( (c > 0) \) and the hyperbolic spaces \( (c < 0) \).
1.9.1 Complex Space Forms

A similar situation as above can be found in the study of complex manifolds from a Riemannian point of view. If $(M, F, g)$ is a Kählerian manifold with constant holomorphic sectional curvatures

$$K(X \wedge FY) = c,$$

then it is said to be a complex space form and it is well known that its curvature tensor is given by

$$R(X, Y)Z = \frac{c}{4}\{g(Y, Z)X - g(X, Z)Y + g(X, FZ)FY - g(Y, FZ)FX + 2g(X, FY)FZ\}. \quad (1.9.3)$$

The models now are $C^n$, $CP^n$ and $CH^n$, depending on $c = 0$, $c > 0$ or $c < 0$. More generally, if the curvature tensor of an almost Hermitian manifold $\tilde{M}$ satisfies:

$$R(X, Y)Z = F_1\{g(Y, Z)X - g(X, Z)Y\} + F_2\{g(X, FZ)FY - g(Y, FZ)FX + 2g(X, FY)FZ\}. \quad (1.9.4)$$

$F_1$ and $F_2$ being differentiable functions on $M$, then $M$ is said to be a generalized complex space form (see [130, 135]). In [130], an important obstruction for such a space was presented by F. Tricerri and L. Vanhecke: If $M$ is connected, $\dim(M) > 6$, and $F_2$ is not identically zero, then $M$ is a complex space form (in particular, $F_1$ and $F_2$ must be constant). Nevertheless, there are examples of 4-dimensional generalized complex space forms with non-constant functions, such as those given by Z. Olszak in [109]. Many other authors have studied these manifolds and their submanifolds.
1.9.2 Sasakian Space Forms

Sasakian space forms play a similar role in contact metric geometry as that of complex space forms.

Given an almost contact metric manifold \((M, \varphi, \xi, \eta, g)\), a \(\varphi\)-section of \(M\) at \(p \in M\) is a section \(\pi \subseteq T_pM\) spanned by a unit vector \(X_p\) orthogonal to \(\xi_p\) and \(\varphi X_p\). The \(\varphi\)-sectional curvature of \(\pi\) is defined by

\[
K(X \wedge \varphi X) = R(X, \varphi X; \varphi X, X). \tag{1.9.5}
\]

A Sasakian manifold with constant \(\varphi\)-sectional curvature \(c\) is called a Sasakian space form. In such a case, its Riemann curvature tensor is given by

\[
R(X, Y)Z = \frac{c + 3}{4} \{g(Y, Z)X - g(X, Z)Y\} + \frac{c - 1}{4} \{g(X, \varphi Z)\varphi Y - g(Y, \varphi Z)\varphi X + 2g(X, \varphi Y)\varphi Z\} + \frac{c - 1}{4} \{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X\} + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi. \tag{1.9.6}
\]

These spaces can also be modeled, depending on \(c > -3\), \(c = -3\) or \(c < -3\).

Further, given an almost contact metric manifold \((M, \varphi, \xi, \eta, g)\), we say that \(M\) is a generalized Sasakian space form if there exist three functions \(f_1, f_2\) and \(f_3\) on \(M\) such that

\[
R(X, Y)Z = f_1 \{g(Y, Z)X - g(X, Z)Y\} + f_2 \{g(X, \varphi Z)\varphi Y - g(Y, \varphi Z)\varphi X + 2g(X, \varphi Y)\varphi Z\} + f_3 \{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X\} + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi, \tag{1.9.7}
\]

for any vector fields \(X, Y, Z\) on \(M\), where \(R\) denotes the curvature tensor of \(M\). In such a case, we will write \(M(f_1, f_2, f_3)\).
1.10 Submanifolds

Let $M$ and $\tilde{M}$ be two $C^\infty$ manifolds with dimensions $m$ and $n$ respectively such that $m \leq n$. A $C^\infty$-function $f : M \rightarrow \tilde{M}$ is called immersion, if its Jacobian map i.e. $f_* : M_p \rightarrow M_{f(p)}$, for all $p \in M$ is injective and the difference $(n - m)$ is called the codimension of the immersion. And $f$ is said to be submersion if $f_*$ is surjective for all $p \in M$. If $f$ is injective and immersion, then $f$ is called embedding. $(M, f)$ is called a submanifold of $\tilde{M}$ if $f$ is embedding.

Let $M$ be a submanifold of a Riemannian manifold $\tilde{M}$ equipped with a Riemannian metric $g$. Then the Gauss and Weingarten formulae are given by

$$\tilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y), \quad (1.10.1)$$

$$\tilde{\nabla}_X V = -A_V X + \nabla^\perp_X V, \quad (1.10.2)$$

for all $X, Y \in TM$ and $V \in T^\perp M$, where $\tilde{\nabla}$, $\nabla$ and $\nabla^\perp$ are called Riemannian, induced Riemannian and induced normal connections in $\tilde{M}$, $M$ and normal bundle $T^\perp M$ of $M$, respectively. And second fundamental form $\sigma$ of $M$ is related to $A$ by

$$g(\sigma(X, Y), V) = g(A_V X, Y). \quad (1.10.3)$$

1.11 Semi-Symmetric Metric Connection

Let $M$ be an $n$-dimensional Riemannian manifold of class $C^\infty$ endowed with the Riemannian metric $g$ and $\nabla$ be the Levi-Civita connection on $(M^n, g)$.

A linear connection $\tilde{\nabla}$ defined on $(M, g)$ is said to semi-symmetric [74], if its torsion tensor $T$ is if the form

$$T(X, Y) = \eta(Y)X - \eta(X)Y, \quad (1.11.1)$$

where $\eta$ is a 1-form and $\xi$ is a vector field given by

$$\eta(X) = g(X, \xi),$$
for all vector fields $X \in \chi(M^n)$, $\chi(M^n)$ is the set of all differentiable vector fields on $M^n$.

A semi-symmetric connection $\tilde{\nabla}$ is called a semi-symmetric metric connection [79], if it further satisfies:

$$\tilde{\nabla} g = 0. \quad (1.11.2)$$

A relation between the semi-symmetric metric connection $\tilde{\nabla}$ and the Levi-Civita connection $\nabla$ on $(M^n, g)$ has been obtained by K. Yano [141] which is given by

$$\tilde{\nabla}_X Y = \nabla_X Y + \eta(Y)X + \eta(\xi)g(X,Y). \quad (1.11.3)$$

Further, a relation between the curvature tensor $\tilde{R}$ of the semi-symmetric metric connection $\tilde{\nabla}$ and the curvature tensor $R$ of the Levi-Civita connection $\nabla$ is given by

$$\tilde{R}(X,Y)Z = R(X,Y)Z + A(X,Z)Y - A(Y,Z)X$$

$$+ g(X,Z)QY - g(Y,Z)QX, \quad (1.11.4)$$

where $A$ is a tensor field of type $(0,2)$ and $Q$ is a tensor field of type $(1,1)$ which is given by

$$A(Y,Z) = g(QY,Z) = (\nabla_Y \eta)(Z) - \eta(Y)\eta(Z) + \frac{1}{2} \eta(\xi)g(Y,Z), \quad (1.11.5)$$

for all vector fields $X, Y$ on $M$.

$$\tilde{R}(X,Y,Z,W) = R(X,Y,Z,W) + A(X,Z)g(Y,W)$$

$$- A(Y,Z)g(X,W) + g(X,Z)A(Y,W)$$

$$- g(Y,Z)A(X,W), \quad (1.11.6)$$

$$\tilde{R}(X,Y,Z,W) = g(R(X,Y)Z,W) \text{ and } R(X,Y,Z,W) = g(R(X,Y)Z,W). \quad (1.11.7)$$
1.12 Quarter-Symmetric Metric Connection

A linear connection $\tilde{\nabla}$ in a Riemannian manifold $M$ is said to be a quarter-symmetric connection [73] if the torsion tensor $T$ of the connection $\tilde{\nabla}$

$$T(X, Y) = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X, Y],$$

satisfies

$$T(X, Y) = \eta(Y)\varphi X - \eta(X)\varphi Y,$$

where $\eta$ is a 1-form and $\varphi$ is a $(1, 1)$-tensor field. If moreover, a quarter-symmetric connection $\tilde{\nabla}$ satisfies the condition:

$$(\tilde{\nabla}_X g)(Y, Z) = 0,$$

where $X, Y, Z \in \chi(M)$, then $\tilde{\nabla}$ is said to be a quarter-symmetric metric connection, otherwise it is said to be a quarter-symmetric non-metric connection. If we replace $\varphi X$ by $X$, then the quarter-symmetric metric connection reduces to a semi-symmetric metric connection [146]. Thus the notion of quarter-symmetric connection generalizes the idea of the semi-symmetric connection.

1.13 Ricci Soliton

The notion of Ricci soliton which is a natural generalization of an Einstein metric (i.e. the Ricci tensor $S$ is a constant multiple of $g$) was introduced by Hamilton [77] in 1982. A pseudo-Riemannian manifold $(M, g)$ is called a Ricci soliton if it admits a smooth vector field $V$ (potential vector field) on $M$ such that

$$\frac{1}{2}(\mathcal{L}_V g)(X, Y) + S(X, Y) + \lambda g(X, Y) = 0, \quad (1.13.1)$$

where $\mathcal{L}_V$ denotes the Lie-derivative in the direction $V$, $\lambda$ is a constant and $X, Y$ are arbitrary vector fields on $M$. A Ricci soliton is said to be shrinking, steady or
expanding according to $\lambda$ being negative, zero or positive, respectively. It is obvious that a trivial Ricci soliton is an Einstein manifold with $V$ zero or Killing vector field. Since Ricci solitons are the fixed points of the Ricci flow, they are important in understanding Hamilton’s Ricci flow [78] $\frac{\partial}{\partial t} g_{ij} = -2S_{ij}$, viewed as a dynamical system, on the space of Riemannian metrics modulo diffeomorphisms and scalings. In differential geometry, the Ricci flow is an intrinsic geometric flow. It can be viewed as a process that deforms the metric of a Riemannian manifold in a way formally analogous to the diffusion of heat, smoothing out the irregularities in the metric.

Geometric flows, especially Ricci flows, have become important tools in theoretical physics. Ricci soliton is known as quasi Einstein metric in physics literature [68] and the solutions of the Einstein field equations correspond to Ricci solitons [4]. Relation with the string theory and the fact that (1.13.1) is a particular case of Einstein field equation makes the equation of Ricci soliton interesting in theoretical physics.

### 1.14 $\eta$-Ricci Soliton

The concept of $\eta$-Ricci soliton was initiated by Cho and Kimura [39].

An $\eta$-Ricci soliton $(M, g, \lambda, \mu)$ is the generalization of Ricci solitons $(M, g, \lambda)$ on a pseudo-Riemannian manifold, which is defined as

$$L_\xi g + 2S + 2\lambda g + 2\mu \eta \otimes \eta = 0,$$

(1.14.1)

where $L_\xi$ is the Lie derivative operator along the vector field $\xi$, $S$ is the Ricci curvature tensor field of the metric $g$, $\lambda$ and $\mu$ are real constants.

In particular, if $\mu = 0$, $(M, g, \lambda, \mu)$ is a Ricci soliton [78].