CHAPTER VI

QUANTIFIER GUIDED AGGREGATION UNDER NEUTROSOPHIC FUZZY TECHNIQUES

INTRODUCTION:

In fuzzy set (FS) theory, each object is assigned a single real value, called the grade of membership, between zero and one [Zadeh, 1965, 1978, 1983]; Kaufmann & Gupta, 1985]; [Kwang-Lee, 2005]; [Zimmermann, 1991].

Atanassov [1986, 1989, 1999] introduced the concept of intuitionistic fuzzy set (IFS) characterized by a membership function and a non-membership function, which is a generalization of the concept of fuzzy set whose basic component is only a membership function. The intuitionistic fuzzy set has received much attention since its appearance. Gau & Buehrer [1994] introduced the concept of the vague set. But Bustince & Burillo [1996] showed that vague sets are intuitionistic fuzzy sets.

6.1 Previous works:

Gau & Buehrer [1994] pointed out that the drawback of using the single membership value in fuzzy set theory is that the evidence for $u \in U$ and the evidence against $u \in U$ are in fact mixed together (U is the universe of discourse, and u is an element of U). To tackle this problem Gau & Buehrer [1994] proposed the notion of Vague Sets (VSs), which allow using interval-based membership instead of using point-based membership as in FSs.

The interval-based membership generalization in VSs is more expressive in capturing data vagueness. However, VSs are shown to be equivalent to that of intuitionistic fuzzy sets (IFSs) [Bustince & Burillo, 1996]. For this reason, interesting features for handling vague data that are unique to VSs are largely ignored. Lu & Ng [2004, 2005, 2009] pointed out the major differences between IFSs and VSs by the way their interval memberships are defined.

Atanassov [1986, 1989] pointed out that this single value combines the evidence for $x_i \in X$ and the evidence against $x_i \in X$. An IFS A in X is characterised by a membership function $\mu_A(x_i)$ and a non-membership function $\gamma_A(x_i)$. Here, $\mu_A(x_i)$ and $\gamma_A(x_i)$ are associated with each point in X, a real number in $[0,1]$ with the values of $\mu_A(x_i)$ and $\gamma_A(x_i)$ at X representing the grade of membership and non-membership of x_i in A. Thus closeness of the value of $\mu_A(x_i)$ to unity and the value of $\gamma_A(x_i)$ to zero, raise high the grade of membership and lower the grade of non-membership of x_i. An IFS becomes a fuzzy set when $\gamma_A(x_i) = 0$.

6.1.1 The OWA Operator:

An OWA operator of dimension $(n \geq 2)$ is a mapping $F_W : R^n \rightarrow R$ that has an associated weighting vector $W = (w_1 + w_2 + \ldots + w_n)$ having the property: $w_1 + w_2 + \ldots + w_n = 1$ where $0 \leq w_i \leq 1, \quad i = 1, 2, \ldots, n$. In addition, such that $F_W(X) = F_W(x_1, x_2, \ldots + x_n) = \sum_{i=1}^{n} w_i y_i$ with y_i being the i^{th} largest of the x_i.

Definition 6.1.2:

The degree of "orness" associated with this operator is defined as

$$Orness(W) = \sum_{i=1}^{n} \frac{n-i}{n-1} w_i \tag{6.1}$$

Definition 6.1.3:

The max, min and average correspond to W^*, W_* and W_A, respectively, where $W^* = (1, 0, \ldots, 0)$, $W_* = (0, 0, \ldots, 1)$, $W_A = (1/n, 1/n, \ldots, 1/n)$ and $F_{W_*}(X) = min_{1 \leq i \leq n}\{x_i\}$, $F_{W^*}(X) = max_{1 \leq i \leq n}\{x_i\}$, $F_{W_A}(X) = (1/n)\sum_{i=1}^{n} x_i$. Obviously, orness $(W^*) = 1$, orness $(W_*) = 0$ and orness $(W_A) = 1/2$.

6.2 Quantifier guided aggregations with OWA operators

Consensus processes imply that experts achieve an agreement about a problem before taking a decision, thus yielding a solution accepted by the organization, society or themselves. Various consensus approaches were proposed ranging from rigid methods to flexible approaches [Kacprzyk, 1986, 1987]. In these approaches, it is crucial to establish a consensus measure to calculate the level of agreement. Consensus measures are indicators to evaluate how far a group of experts opinions is from unanimity.

Mohanty & Bhasker [2005], Mohanty & Zahir [2007] and Mohanty [2008] have applied the concepts of Linguistic Quantifiers in the product classifications based on customer preference in Internet-Business. In this work, the linguistic quantifiers guided aggregation based on the Ordered Weighted Averaging (OWA) operators are used to derive the weights of the experts.

The problem of determining weights for an OWA operator can be addressed in different ways, for example with the use of the so-called 'Linguistic Quantifiers' introduced by Zadeh [1983]. A relative linguistic quantifier Q, such as 'most', 'few', 'many', and 'all', can be represented as a fuzzy subset of the unit interval, where for a given proportion $r \in [0, 1]$ of the total of the values to aggregate, $Q(r)$ indicates the extent to which this proportion satisfies the semantics defined in Q. For example, given $Q =$ 'most', if $Q(0.7) = 1$ then it would mean that a proportion of 70% totally satisfies the idea conveyed by the quantifier 'most', whereas $Q(0.55) = 0.25$ indicates that the proportion 55% is barely compatible with this concept (i.e., only 25%).

Regular Increasing Monotone (RIM) quantifiers [Liu, 2008; Liu & Han, 2008] are especially interesting for their use in OWA operators. These quantifiers present the following properties:

(a). $Q(0) = 0$; (b). $Q(1) = 1$; (c). If $r_1 < r_2$ then $Q(r_1) \geq Q(r_2)$.

Definition 6.2.1:

Yager [1988] suggested the following method to compute weights w_i, with the use of a RIM quantifier Q:

$$w_i = Q\left(\frac{i}{n}\right) - Q\left(\frac{i-1}{n}\right), \quad i = 1, 2, \ldots, n$$

where the membership function of a linear RIM quantifier $Q(r)$ is defined by two parameters $a, b \in [0, 1]$ as:

$$Q(r) = \begin{cases} 0 & \text{if} \quad r < a \\ \dfrac{r-a}{b-a} & \text{if} \quad a \leq r \leq b \\ 1 & \text{if} \quad r > b \end{cases}$$

Example 6.2.2:

RIM quantifier Q = 'most', with $a = 0.5$ and $b = 0.7$ is given as:

$$Q(r) = \begin{cases} 0 & \text{if} \quad r < 0.5 \\ 5r - 2.5 & \text{if} \quad 0.5 \leq r \leq 0.7 \\ 1 & \text{if} \quad r = 0.7 \end{cases}$$

Since the use of OWA with RIM quantifiers captures the notion of the soft consensus correctly, they can be adopted for the purpose of studying the effect of different aggregation operators on the resolution of a consensus problem with many experts, and expressing a desired group's attitude.

Definition 6.2.3:

A fuzzy subset Q of the real line is called a RIM quantifier if the membership function $Q(x)$ obeys $Q(0) = 0$, $Q(1) = 1$ and $Q(x) \geq Q(y)$ for $x > y$.

Example 6.2.4:

RIM quantifier are "all", "most", "many", "at least".

The quantifier "all" is represented by the fuzzy subset

$$Q_*(x) = \begin{cases} 1 & \text{if} \quad x = 1 \\ 0 & \text{if} \quad x \neq 1 \end{cases}$$

The quantifier "there exists, not none", is defined as

$$Q^*(x) = \begin{cases} 0 & \text{if} \quad x = 0 \\ 1 & \text{if} \quad x \neq 0 \end{cases}$$

With a RIM quantifier Q, Yager, (1988) proposed the *OWA* weighting vector generating rule:

$$w_i = Q\left(\frac{i}{n}\right) - Q\left(\frac{i-1}{n}\right) \tag{6.2}$$

Definition 6.2.5:

Therefore, the quantifier guided aggregation with *OWA* operator is

$$F_Q(X) = F_W(X) = \sum_{i=1}^{n} \left(Q\left(\frac{i}{n}\right) - Q\left(\frac{i-1}{n}\right) \right) y_i \tag{6.3}$$

Definition (orness) 6.2.6:

Yager[1988] also extended the ornesss measure of *OWA* operator, and defined the orness of a RIM quantifier:

$$\begin{aligned} orness(Q) &= lim_{n\to\infty} \textstyle\sum_{i=1}^{n} \frac{n-i}{n-1} \left(Q\left(\frac{i}{n}\right) - Q\left(\frac{i-1}{n}\right) \right) \\ &= lim_{n\to\infty} \frac{1}{n-1} \textstyle\sum_{i=1}^{n-1} Q\left(\frac{i}{n}\right) \\ &= \int_0^1 Q(x)dx \end{aligned} \tag{6.4}$$

6.3 The analytical solutions of the maximum entropy problems:

Maximum Entropy-OWA (ME-OWA) operator problem was first proposed by O'Hagan, (1988), as a constrained non-linear programming problem, given the level of orness. Filev & Yager, (1998) further analyzed the properties of ME-OWA operators, and proposed a method for obtaining weights as a function of one parameter.

Definition 6.3.1:

The ME-OWA operator problem can be formulated as:

$$max - \sum_{i=1}^{n} w_i ln w_i \tag{6.5}$$

such that

$$\frac{1}{n-1}\sum_{i=1}^{n}(n-i)w_i = \alpha, \quad 0 < \alpha < 1, \tag{6.6}$$

$$\sum_{i=1}^{n} w_i = 1, \quad 0 \leq w_i \leq 1, \quad i = 1, 2, \ldots, n. \tag{6.7}$$

The solution of problem (7), $W^{ent} = (w_1^{ent}, w_2^{ent}, \ldots, w_n^{ent})$ can be expressed analytically as a geometric series, which is determined by

$$w_{i+1}^{ent} = w_i^{ent} q, \quad i = 1, 2, \ldots, n-1, \quad \text{where} \quad q \in (0, +\infty) \tag{6.8}$$

Therefore,

$$w_i^{ent} = \frac{q^{i-1}}{\sum_{j=0}^{n-1} q^j} \tag{6.9}$$

where q is the unique positive real root of

$$(n-1)\,\alpha q^{n-1} + \sum_{i=2}^{n} \left((n-1)\,\alpha - i + 1\right) q^{n-i} = 0 \tag{6.10}$$

Remark 6.3.2:

For simplification, in problem (7) we do not consider the two extreme cases with $\alpha = 1$ and 0, which correspond to the unique solution W^* and W_* of $q = 0$ and $+\infty$, respectively.

As the generating function $f(t)$ in RIM quantifier $Q(x)$ plays the role of w_i in OWA operator, a similar maximum entropy problem can be defined.

The (Shannon) entropy of the generating function $f(t)$ is

$$H_f = -\int_0^1 f(t)\ ln\ f(t)dt \tag{6.11}$$

with $\int_0^1 f(t)dt = 1$.

6.4: The Fuzzy Linguistic quantifier for MAGDM

The aggregation weighted vector W is a mapping to membership function $Q(r)$ guided by a monotonically non-decreasing fuzzy linguistic quantifier, Q represented as equation (6.11) and (6.12). The membership function $Q(r)$ represents the membership grade on r that belongs to Q.

Definition 6.4.1:

The membership function also differs from Q (Herrera et al., 2000).

$$w_k = Q\left(\frac{k}{n}\right) - Q\left(\frac{k-1}{n}\right), \quad k = 1, 2, \ldots, n \tag{6.12}$$

$$Q(r) = \begin{cases} 0 & \text{if} \quad r < a \\ \dfrac{r-a}{b-a} & \text{if} \quad a \leq r \leq b \quad a, b, r \in [0, 1] \\ 1 & \text{if} \quad r > b \end{cases} \tag{6.13}$$

To enable the use of different fuzzy linguistic quantifiers to aggregate behaviour among attributes to produce the fuzzy majority rule, the following perspectives are considered. “Critical” factor is used for fuzzy linguistic quantifier “At least half” to emphasize the strong influence of aggregating on results.

“Major” factor is used for fuzzy linguistic quantifier “Most” to emphasize the medium influence of aggregation on the results. Finally, “Fundamental” factor is used for fuzzy linguistic quantifier “As many as possible” to represent the degree to which essential requests are satisfied.

6.4.2 Optimizing the aggregation weighting vector:

Optimizing the aggregation weighted vector requires calculating the degree of "Orness" and "Entropy" (Dispersion). The calculation is based on the aggregation weighted vector W, displayed in equations (6.13) and (6.14). Orness, which lies in the unit interval, is a good measurement for characterizing the degree to which the aggregation is an Or-like (Max-like) or And-like (Min-like) operation. When Orness equals 1, the aggregation equals the maximum operation; when equals 0, the aggregation equals the minimum operation; and when Orness equals 0.5, the aggregation equals the arithmetic mean operation. Simultaneously, Entropy represents the measurement for characterizing the degree to which information on the individual behaviors in the aggregation process is used (Yager, 1988).

$$Orness\,(W) = \frac{1}{n-1}\sum_{k=1}^{n}(n-k)\,w_k \tag{6.14}$$

$$Entropy\,(W) = -\sum_{k=1}^{n} w_k \; ln \; w_k \tag{6.15}$$

6.4.3 Fuzzy linguistic quantifier:

The concept and purpose of optimization is based on the premise that the current Orness should be kept constant to implement an amendment process for maximizing the Entropy. Equation (6.16) illustrates the approach to proceed.

6.4.4 Optimization Problem :

$$Max \quad -\sum_{k=1}^{n} w_k \; ln \; w_k \tag{6.16}$$

Subject to:

$$Orness(W) = \frac{1}{n-1}\sum_{k=1}^{n}(n-k)\,w_k \tag{6.16a}$$

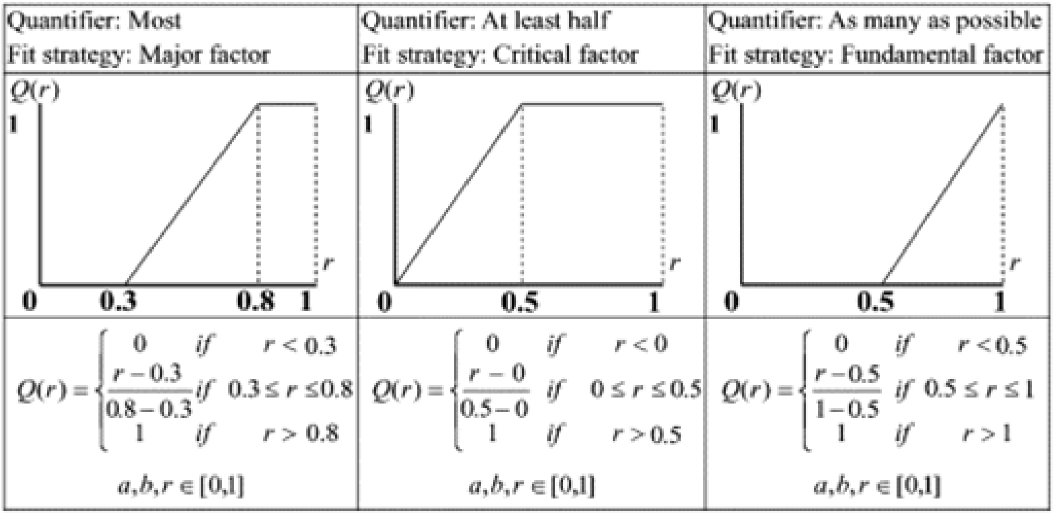

Figure 1. Monotonically non-decreasing fuzzy linguistic quantifier

$$\sum_{k=1}^{n} w_k = 1, \quad w_k \in [0,1], \quad k = 1,2,\ldots,n \tag{6.16b}$$

Furthermore, the Langrange multiplier method can be used to obtain the maximal Entropy aggregation weighted vector W^*, which can aggregate the maximum information from behaviors. Filev & Yager, (1998) presented a detailed information in this regard. Equation (6.16) can be further simplified as equations (6.17) and (6.18). Moreover, the numerical analysis approach can be used to obtain b from equation (6.17), and can be substituted into equation (6.18) to obtain W^* . The initial vector of W thus is replaced by the new W^* , thus optimizing the aggregation weighted vector.

$$\sum_{k=1}^{n} \left(\frac{n-k}{n-1} - Orness(W) \right) b^{n-k} = 0 \tag{6.17}$$

$$w_k^* = \frac{b^{n-k}}{\sum_{k=1}^{n} b^{n-k}} \tag{6.18}$$

6.5 The Algorithm for getting weights by Quantifier (RIM) guided entropy method

PROPOSED MODEL OF DECISION MAKING

Let $A = \{A_1, A_2, \ldots, A_n\}$ be a set of alternatives, $G = \{G_1, G_2, \ldots, G_n\}$ be the set of attributes, $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ is the weighting vector of the attribute G_j , $j = 1, 2, \ldots, n$, where $\omega_j \in [0, 1]$, such that $\sum_{j=1}^{n} \omega_j = 1$. Let $D = \{D_1, D_2, \ldots, D_k\}$ be the set of decision makers, $V = (V_1, V_2, \ldots, V_n)^T$ be the weighting vector of the decision makers, with $V_k \in [0, 1]$, $\quad \sum_{k=1}^{t} V_k = 1$.

6.5.1 The developed model of MAGDM is given as follows:

Step 1:

Let $R_i = (r)_{ij}^{(k)} = \left(T_{ij}^{(k)} I_{ij}^{(k)}, F_{ij}^{(k)}\right)$ be the neutrosophic fuzzy decision matrix, where, $(T)_{ij}^{(k)}$ is the degree of the truth membership value that the alternative satisfies the attribute G_j given by the decision maker D_k, $(I)_{ij}^{(k)}$ is the degree of the indeterministic membership value that the alternative A_i satisfies the attribute G_j given by the decision maker D_k, and $(F)_{ij}^{(k)}$ is the degree of false membership value for the alternative A_i, where $T_{ij}^{(k)}, I_{ij}^{(k)}, F_{ij}^{(k)}$ are in [0, 1], $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, $k = 1, 2, \ldots, t$.

Step 2:

Fix $i = 1, 2, \ldots, m$, and $j = 1, 2, \ldots, n$

$\mu_{ij}^{(k)} = max_k \left\{T_i j^((k)) :: k = 1, 2, \ldots, t.\right\}$

$\gamma_{ij}^{(k)} = min \left\{I_{ij}^{(k)} :: k = 1, 2, \ldots, t.\right\}$.

$\lambda_{ij}^{(k)} = min \left\{F_{ij}^{(k)} :: k = 1, 2, \ldots, t.\right\}$.

Step 3:

To derive the weights by Quantifier (RIM) guided entropy method with Orness -

for weights by using

$$w_k = Q\left(\frac{k}{n}\right) - Q\left(\frac{k-1}{n}\right)\frac{1}{2}, \quad k = 1, 2, \ldots, n$$

$$Q(r) = \begin{cases} 0 & \text{if} \quad r < a \\ \\ \dfrac{r-a}{b-a} & \text{if} \quad a \leq r \leq b; \quad a, b, r \in [0,1] \\ \\ 1 & \text{if} \quad r > b \end{cases}$$

$$Orness\,(W) = \frac{1}{n-1}\sum_{k=1}^{n}(n-k)\,w_k, \quad Entropy\,(W) = -\sum_{k=1}^{n} w_k \; ln \; w_k$$

Hence,

$$Max \quad -\sum_{k=1}^{n} w_k \; ln \; w_k \tag{6.19}$$

Subject to:

$$Orness(W) = \frac{1}{n-1}\sum_{k=1}^{n}(n-k)\,w_k$$

$$\sum_{k=1}^{n} w_k = 1, \quad w_k \in [0,1], \quad k = 1, 2, \ldots, n$$

$$\sum_{k=1}^{n}\left(\frac{n-k}{n-1} - Orness(W)\right)b^{n-k} = 0, \quad w_k^* = \frac{b^{n-k}}{\sum_{k=1}^{n} b^{n-k}}$$

Step 4:

Find reduced matrix $R = (R_{ij}) = (T_{ij}, I_{ij}, F_{ij})$ from the given k decision making matrices as follows:

$T_{ij} = \sum_{k=1}^{t} T_{ij}^{(k)}\mu_{ij}^{(k)}; I_{ij} = \sum_{k=1}^{t} I_{ij}^{(k)}\gamma_{ij}^{(k)}; F_{ij} = \sum_{k=1}^{t} F_{ij}^{(k)}\lambda_{ij}^{(k)}$.

Step 5:

Taking $r = (1, 0, 0)$ as decision row is fixed.

Step 6:

The distance $d(r, r_i) = \sqrt{\frac{1}{2}\left[\prod_{j=1}^{n} T_{ij} + \prod_{j=1}^{n}(1 - I_{ij}) + \prod_{j=1}^{n}(1 - F_{ij})\right]}$ where i varies from 1 to m.

Step 7:

Select the alternative $max_i d(r, r_i)$, say s **Step 8:**

Identity the best alternative is A_s.

6.5.2 Numerical Illustration:

STEP 1:

As same as Step-1 of the chapter V.

$$R^1 = \begin{bmatrix} <0.25, 0.54, 0.8> & <0.3, 0.4, 0.9> & <0.7, 0.35, 0.5> & <0.9, 0.2, 0.8> \\ <0.6, 0.5, 0.5> & <0.6, 0.2, 0.3> & <0.2, 0.4, 0.9> & <0.6, 0.23, 0.7> \\ <0.3, 0.45, 0.9> & <0.7, 0.1, 0.4> & <0.6, 0.5, 0.5> & <0.4, 0.2, 0.9> \\ <0.45, 0.38, 0.27> & <0.37, 0.68, 0.16> & <0.6, 0.25, 0.3> & <0.1, 0.4, 0.8> \end{bmatrix}$$

$$R^2 = \begin{bmatrix} <0.1, 0.3, 0.7> & <0.6, 0.6, 0.5> & <0.4, 0.2, 0.1> & <0.3, 0.7, 0.6> \\ <0.3, 0.55, 0.37> & <0.75, 0.42, 0.1> & <0.32, 0.67, 0.56> & <0.35, 0.56, 0.72> \\ <0.5, 0.4, 0.32> & <0.65, 0.25, 0.32> & <0.6, 0.3, 0.1> & <0.75, 0.25, 0.55> \\ <0.27, 0.9, 0.81> & <0.31, 0.4, 0.6> & <0.75, 0.65, 0.55> & <0.3, 0.7, 0.9> \end{bmatrix}$$

$$R^3 = \begin{bmatrix} < 0.32, 0.47, 0.6 > & < 0.9, 0.1, 0.3 > & < 0.6, 0.4, 0.5 > & < 0.3, 0.5, 0.7 > \\ < 0.12, 0.32, 0.52 > & < 0.17, 0.81, 0.9 > & < 0.5, 0.3, 0.1 > & < 0.45, 0.65, 0.27 > \\ < 0.50, 0.6, 0.23 > & < 0.56, 0.52, 0.23 > & < 0.3, 0.6, 0.1 > & < 0.57, 0.52, 0.55 > \\ < 0.54, 0.83, 0.72 > & < 0.73, 0.86, 0.61 > & < 0.5, 0.52, 0.4 > & < 0.6, 0.4, 0.2 > \end{bmatrix}$$

$$R^4 = \begin{bmatrix} < 0.7, 0.3, 0.1 > & < 0.5, 0.4, 0.4 > & < 0.2, 0.1, 0.6 > & < 0.7, 0.9, 0.6 > \\ < 0.3, 0.56, 0.73 > & < 0.57, 0.24, 0.1 > & < 0.23, 0.76, 0.65 > & < 0.53, 0.65, 0.27 > \\ < 0.32, 0.32, 0.6 > & < 0.56, 0.52, 0.32 > & < 0.1, 0.3, 0.9 > & < 0.57, 0.52, 0.55 > \\ < 0.72, 0.5, 0.18 > & < 0.13, 0.6, 0.4 > & < 0.55, 0.56, 0.78 > & < 0.7, 0.1, 0.6 > \end{bmatrix}$$

$$R^5 = \begin{bmatrix} < 0.52, 0.45, 0.1 > & < 0.57, 0.37, 0.1 > & < 0.76, 0.65, 0.23 > & < 0.57, 0.52, 0.55 > \\ < 0.3, 0.6, 0.7 > & < 0.7, 0.4, 0.1 > & < 0.3, 0.7, 0.6 > & < 0.5, 0.4, 0.6 > \\ < 0.2, 0.3, 0.2 > & < 0.6, 0.2, 0.5 > & < 0.1, 0.6, 0.65 > & < 0.3, 0.9, 0.7 > \\ < 0.27, 0.5, 0.81 > & < 0.75, 0.25, 0.32 > & < 0.32, 0.67, 0.56 > & < 0.35, 0.56, 0.72 > \end{bmatrix}$$

STEP 2:

To derive a weight vector w by using Quantifier (RIM) guided entropy method with orness for weight.

Most Quatifier:

$$Q(r) = \begin{cases} 0 & \text{if} \quad r < 0.3 \\ \dfrac{r - 0.3}{0.8 - 0.3} & \text{if} \quad 0.3 \leq r \leq 0.8 \quad 0.3, 0.8, r \in [0, 1] \\ 1 & \text{if} \quad r > 0.8 \end{cases}$$

$$Max \quad -\sum_{k=1}^{n} w_k \; ln \; w_k \tag{6.20}$$

Subject to:

$$Orness(W) = \frac{1}{n-1}\sum_{k=1}^{n}(n-k)\, w_k$$

$$\sum_{k=1}^{n} w_k = 1, \quad w_k \in [0,1], \quad k = 1,2,\ldots,n$$

$$\sum_{k=1}^{n}\left(\frac{n-k}{n-1} - Orness(W)\right) b^{n-k} = 0, \quad w_k^* = \frac{b^{n-k}}{\sum_{k=1}^{n} b^{n-k}}$$

$$w_1^* = \frac{b^3}{\sum_{j=1}^{4} b^{4-j}} = \frac{b^3}{b^3+b^2+b+1}$$

$$w_2^* = \frac{b^2}{\sum_{j=1}^{4} b^{4-j}} = \frac{b^2}{b^3+b^2+b+1}$$

$$w_3^* = \frac{b}{\sum_{j=1}^{4} b^{4-j}} = \frac{b}{b^3+b^2+b+1}$$

$$w_4^* = \frac{1}{\sum_{j=1}^{4} b^{4-j}} = \frac{1}{b^3+b^2+b+1}$$

$$\begin{aligned} w_1 &= Q\,(1/5) - Q\,(0/5) \\ &= Q(0.2) - Q(0) \\ &= 0 - 0 \\ w_1 &= 0 \end{aligned} \qquad \begin{aligned} w_2 &= Q\,(2/5) - Q\,(1/5) \\ &= Q(0.4) - Q(0) \\ &= 0.2 - 0 \\ w_2 &= 0.2 \end{aligned}$$

$$\begin{aligned} w_3 &= Q\,(3/5) - Q\,(2/5) \\ &= Q(0.6) - Q(0.4) \\ &= 0.6 - 0.2 \\ w_3 &= 0.4 \end{aligned} \qquad \begin{aligned} w_4 &= Q\,(4/5) - Q\,(3/5) \\ &= Q(0.8) - Q(0.6) \\ &= 1 - 0.6 \\ w_4 &= 0.4 \end{aligned}$$

$$\begin{aligned} w_5 &= Q\left(5/5\right) - Q\left(4/5\right) \\ &= Q(1) - Q(0.8) \\ &= 1 - 1 \\ w_5 &= 0 \end{aligned}$$

$$\begin{aligned} Orness(W) &= \tfrac{1}{5-1}\textstyle\sum_{j=1}^{5}\left(5-j\right)w_j \\ &= \tfrac{1}{4}\left(4w_1 + 3w_2 + 2w_3 + w_4\right) \\ &= \tfrac{1}{4}\left(4(0) + 3(0.2) + 2(0.4) + 0.4\right) \\ &= 0.45 \end{aligned}$$

$$\sum_{j=1}^{5}\left(\frac{5-j}{5-1} - 0.45\right)b^{5-j} = 0$$

$$\Rightarrow \left(\frac{5-1}{5-1} - 0.45\right)b^4 + \left(\frac{5-2}{5-1} - 0.45\right)b^3 + \left(\frac{5-3}{5-1} - 0.45\right)b^2 + \left(\frac{5-4}{5-1} - 0.45\right)b + \left(\frac{5-5}{5-1} - 0.45\right) = 0$$

$$\Rightarrow \left(\frac{4}{4} - 0.45\right)b^4 + \left(\frac{3}{4} - 0.45\right)b^3 + \left(\frac{2}{4} - 0.45\right)b^2 + \left(\frac{1}{4} - 0.45\right)b + \left(\frac{0}{4} - 0.45\right) = 0$$

$$\Rightarrow 0.55b^4 + 0.3b^3 + 0.05b^2 - 0.2b - 0.45 = 0$$

$$w_1^* = \frac{b^4}{\sum_{j=1}^{5} b^{5-j}} = \frac{b^4}{b^4 + b^3 + b^2 + b + 1}$$

$$w_2^* = \frac{b^3}{\sum_{j=1}^{5} b^{5-j}} = \frac{b^3}{b^4 + b^3 + b^2 + b + 1}$$

$$w_3^* = \frac{b^2}{\sum_{j=1}^{5} b^{5-j}} = \frac{b^2}{b^4 + b^3 + b^2 + b + 1}$$

$$w_4^* = \frac{b}{\sum_{j=1}^{5} b^{5-j}} = \frac{b}{b^4 + b^3 + b^2 + b + 1}$$

$$w_5^* = \frac{1}{\sum_{j=1}^{5} b^{5-j}} = \frac{1}{b^4 + b^3 + b^2 + b + 1}$$

Hence it finds that

$w_1^* = 0.1619,\ w_2^* = 0.1791,\ w_3^* = 0.1979,\ w_4^* = 0.2189,\ w_5^* = 0.2421$

NEW REDUCED MATRIX

$$R = \begin{bmatrix} <0.3334, 0.6021, 0.7166> & <0.5518, 0.6789, 0.6791> & <0.4769, 0.7124, 0.6766> & <0.5042, 0.4742, 0.3651> \\ <0.2803, 0.5009, 0.4371> & <0.4997, 0.6287, 0.8151> & <0.2971, 0.4537, 0.5481> & <0.4795, 0.5242, 0.5439> \\ <0.3346, 0.6072, 0.6369> & <0.6067, 0.7227, 0.6534> & <0.2294, 0.5574, 0.6696> & <0.4842, 0.55358, 0.3682> \\ <0.4171, 0.4122, 0.5228> & <0.3877, 0.5051, 0.6178> & <0.5078, 0.4803, 0.4923> & <0.3601, 0.6452, 0.4313> \end{bmatrix}$$

$d(r, r_1) = 0.2027 = A_1$

$d(r, r_2) = 0.2856 = A_2$

$d(r, r_3) = 0.1874 = A_3$

$d(r, r_4) = 0.4022 = A_4$

$A_4 > A_2 > A_1 > A_3$

A_4 is best alternative.