Chapter 6

On an R-Randers $m^{th}$-root Space

6.1 Introduction

The theory of an $m^{th}$-root metric was introduced by H. Shimada [35] in 1979. By introducing the regularity of the metric, various fundamental quantities of a Finsler metric could be found. In particular, the Cartan connection of a Finsler space with $m^{th}$-root metric was introduced from the theoretical standpoint. M. Matsumoto and K. Okubo [98] studied Berwald connection of a Finsler space with $m^{th}$-root metric and gave main scalars in two dimensional case and defined higher order Christoffel symbols. The $m^{th}$-root metric is used in many problems of theoretical physics [109]. T. N. Pandey, et. al. [199] studied three dimensional Finsler space with $m^{th}$-root metric. To discuss general relativity with the electromagnetic field, G. Randers [33] introduced a metric of the form $L(x, y) = \alpha(x, y) + \beta(x, y)$, where $\alpha$ is a square root metric and $\beta$ is a differential one form. In his honor, this metric is called Randers metric, and it has been extensively studied by several geometers and physicists [13, 43, 207].

In 2010, Otilia Lungu and Valer Niminet [106] studied a special Finsler space with the metric $L$ of the form $L(x, y) = F(x, y) + \alpha(x, y)$, $\forall y \in TM$, where $F$ is a quartic root metric and $\alpha$ is a square root metric.
They regarded this space as an R-Randers quartic space and obtained many results related to it. The aim of the present chapter is to study a more general space with the metric \( L(x, y) = F(x, y) + \alpha(x, y) \), where \( F \) is an \( m^{th} \)-root metric and \( \alpha \) is a Riemannian metric. We call the space endowed with this metric as an R-Randers \( m^{th} \)-root space. We obtain the expressions for the fundamental metric tensor, Cartan tensor, geodesic spray coefficients and the coefficients of nonlinear connection in an R-Randers \( m^{th} \)-root space. Some other properties of such space have also been discussed.

### 6.2 R-Randers \( m^{th} \)-root space

Let \( F^n = (M, L(x, y)) \) \((n > 2)\) be an \( n \)-dimensional Finsler space. The \( m^{th} \)-root metric on \( M \) is defined as \( L^m = a_{i_1i_2...i_m}(x)y^{i_1}y^{i_2}...y^{i_m} \), where \( a_{i_1i_2...i_m}(x) \) are components of an \( m^{th} \) order covariant symmetric tensor.

In case of \( m = 2 \), the metric \( L \) is Riemannian and in the case \( m = 3 \) and \( m = 4 \) these metrics are cubic and quartic respectively.

The geodesic of a Finsler space is given by

\[
\frac{d^2x^i}{dt^2} + 2G^i = 0, \tag{6.2.1}
\]

where \( G^i \) are the geodesic spray coefficients given by

\[
2G^i = \frac{1}{2}g^{ij}\left\{y^k\partial_j\partial_kL^2 - \partial_jL^2\right\}, \quad \partial_j \equiv \frac{\partial}{\partial x^j}. \tag{6.2.2}
\]

The nonlinear connection of a Finsler space is defined as

\[
N^i_j = \dot{\partial}_j G^i. \tag{6.2.3}
\]

In this chapter, we study the space whose fundamental function is given by

\[
L(x, y) = F(x, y) + \alpha(x, y), \tag{6.2.4}
\]
where

\[ F^m = a_{i_1i_2...i_m}(x)y^{i_1}y^{i_2}...y^{i_m} \]

is an \( m^{th} \)-root metric and

\[ \alpha^2 = b_{i_1i_2}(x)y^{i_1}y^{i_2}, \]

is a Riemannian metric. We call this space as an R-Randers \( m^{th} \)-root space.

### 6.3 Fundamental Metric Tensor and Cartan Tensor

In this section, we find the fundamental metric tensor \( g_{ij} \), its inverse \( g^{ij} \), angular metric tensor \( h_{ij} \) and the Cartan tensor \( C_{ijk} \) for an R-Randers \( m^{th} \)-root space. Differentiating (6.2.5) partially with respect to \( y^i \), we get

\[ F^{m-1}(\dot{\partial}_i F) = a_i, \]

where \( a_i(x,y) = a_{ii_2...i_m}(x)y^{i_2}y^{i_3}...y^{i_m}. \)

Differentiating (6.3.1) partially with respect to \( y^j \), we find

\[ (m-1)F^{m-2}(\dot{\partial}_j F)(\dot{\partial}_i F) + F^{m-1}(\dot{\partial}_j \dot{\partial}_i F) = (m-1)a_{ij}, \]

where \( a_{ij}(x,y) = a_{iji_3...i_m}(x)y^{i_3}y^{i_4}...y^{i_m}. \)

Take \( \rho_{-(m-1)} = (m-1)F^{-(m-1)} \) and \( \rho_{-(2m-1)} = (m-1)F^{-(2m-1)} \), where the subscripts denote the degree of homogeneity of the corresponding entities with respect to \( y^i \). In view of this, (6.3.1) and (6.3.2) give

\[ \dot{\partial}_i F = \frac{\rho_{-(m-1)}}{(m-1)a_i} \]

and

\[ \dot{\partial}_j \dot{\partial}_i F = \rho_{-(m-1)}a_{ij} - \rho_{-(2m-1)}a_ia_j, \]
where

\[(6.3.5) \quad \dot{\partial}_{j}\rho_{-(m-1)} = -(m - 1)\rho_{-(2m-1)}a_j.\]

Now, differentiating (6.2.6) partially with respect to \(y^j\), we get

\[(6.3.6) \quad \dot{\partial}_i\alpha = \mu_{-1}b_i,\]

where \(b_i(x, y) = b_{ii}(x)y^2\) and \(\mu_{-1} = \alpha^{-1}\).

Further differentiating (6.3.6) partially with respect to \(y^j\), we find

\[(6.3.7) \quad \dot{\partial}_j\dot{\partial}_i\alpha = \mu_{-1}b_{ij} + \mu_{-3}b_i b_j,\]

where \(\mu_{-3} = -\alpha^{-3}\).

Differentiating \(\mu_{-1} = \alpha^{-1}\) partially with respect to \(y^j\), we get

\[(6.3.8) \quad \dot{\partial}_j\mu_{-1} = \mu_{-3}b_j.\]

Thus, we have

**Proposition 6.3.1:** In an R-Randers \(m^{th}\)-root space, the following identities hold

\[
\dot{\partial}_i F = \frac{\rho_{-(m-1)}}{(m - 1)}a_i, \quad \dot{\partial}_j\dot{\partial}_i F = \rho_{-(m-1)}a_{ij} - \rho_{-(2m-1)}a_i a_j, \quad \dot{\partial}_i\alpha = \mu_{-1}b_i,
\]

\[
\dot{\partial}_j\dot{\partial}_i\alpha = \mu_{-1}b_{ij} + \mu_{-3}b_i b_j, \quad \dot{\partial}_j \rho_{-(m-1)} = -(m - 1)\rho_{-(2m-1)}a_j \quad \text{and} \quad \dot{\partial}_j\mu_{-1} = \mu_{-3}b_j.
\]

Differentiating (6.2.4) partially with respect to \(y^i\) and using (6.3.3) and (6.3.6), we have

\[(6.3.9) \quad \dot{\partial}_i L = \frac{\rho_{-(m-1)}}{(m - 1)}a_i + \mu_{-1}b_i.\]

Further, differentiating (6.3.9) partially with respect to \(y^i\) and using (6.3.4) and (6.3.7), we get

\[(6.3.10) \quad \dot{\partial}_j\dot{\partial}_i L = \rho_{-(m-1)}a_{ij} - \rho_{-(2m-1)}a_i a_j + \mu_{-1}b_{ij} + \mu_{-3}b_i b_j.\]

From (1.3.2), we have

\[(6.3.11) \quad g_{ij} = \left(\dot{\partial}_i L\right) \left(\dot{\partial}_j L\right) + L \left(\dot{\partial}_i \dot{\partial}_j L\right).\]
Using (6.3.9) and (6.3.10), (6.3.11) yields
\begin{equation}
  g_{ij} = \left( \frac{\rho-(m-1)}{(m-1)} a_i + \mu_{-1} b_i \right) \left( \frac{\rho-(m-1)}{(m-1)} a_j + \mu_{-1} b_j \right) \\
  + L \left( \rho-(m-1) a_i a_j - \rho-(2m-1) a_i a_j + \mu_{-1} b_i + \mu_{-3} b_i b_j \right).
\end{equation}

Taking \( d_{ij} = \rho-(m-1) a_{ij} + \mu_{-1} b_{ij} \) and \( c_i = q_0 a_i + q_{-1} b_i \),

where \( q_0 \) and \( q_{-1} \) satisfy
\begin{align*}
  q_0^2 &= \left( \frac{\rho-(m-1)}{(m-1)} \right)^2 - L \rho-(2m-1), \quad
  q_{-1}^2 = (\mu_{-1}^2 + L \mu_{-3}) \quad \text{and} \\
  q_0 q_{-1} &= \frac{\rho-(m-1)}{(m-1)} \mu_{-1}.
\end{align*}

Then (6.3.12) takes the form
\begin{equation}
  g_{ij} = L d_{ij} + c_i c_j.
\end{equation}

Thus, we have

**Theorem 6.3.1:** The fundamental metric tensor of an \( R \)-Randers \( m \)-th-root space is given by (6.3.13).

**Theorem 6.3.2:** In an \( R \)-Randers \( m \)-th-root space, the inverse \( g^{ij} \) of the fundamental metric tensor \( g_{ij} \), is given by
\begin{equation}
  g^{ij} = \frac{1}{L} \left( d^{ij} - \frac{1}{L + c^2} c^i c^j \right),
\end{equation}

where \( c^i = d^{ij} c_j \) and \( c^2 = c_i c^i \).

*Proof.* Utilizing the Lemma 6.1.2.1 of [108] for the nonsingular matrix \( g_{ij} \) given by (6.3.13), we have the result. \( \square \)

Using (6.3.9) and (6.3.12) in (1.6.4), we get the angular metric tensor of an \( R \)-Randers \( m \)-th-root space:
\begin{equation}
  h_{ij} = L \left( \rho-(m-1) a_{ij} - \rho-(2m-1) a_i a_j + \mu_{-1} b_{ij} + \mu_{-3} b_i b_j \right).
\end{equation}

Thus, we have
**Theorem 6.3.3:** In an R-Randers $m^{th}$-root space, the angular metric tensor $h_{ij}$ is given by (6.3.15).

Differentiating $\rho_{-(2m-1)} = (m - 1)F^{-(2m-1)}$ partially with respect to $y^k$, we get

\[
\dot{k}\rho_{-(2m-1)} = -(2m - 1)\rho_{-(3m-1)}a_k,
\]

where $\rho_{-(3m-1)} = (m - 1)F^{-(3m-1)}$.

Also, we have

\[
\dot{k}\mu_{-3} = 3\mu_{-5}b_k,
\]

where $\mu_{-5} = \alpha^{-5}$.

Thus, we have the following

**Proposition 6.3.2:** In an R-Randers $m^{th}$-root space, the following hold good

\[
\dot{k}\rho_{-(2m-1)} = -(2m - 1)\rho_{-(3m-1)}a_k \text{ and } \dot{k}\mu_{-3} = 3\mu_{-5}b_k.
\]

Differentiating (6.3.12) partially with respect to $y^k$, we get

\[
\dot{k}g_{ij} = 2\rho_{-(m-1)} \left( \dot{k}\rho_{-(m-1)} \right) a_i a_j + (\rho_{-(m-1)})^2 \left( a_j \dot{k}a_i + a_i \dot{k}a_j \right)
\]

\[+ \left[ \left( \dot{k}\rho_{-(m-1)} \right) \mu_{-1} + \rho_{-(m-1)} \left( \dot{k}\mu_{-1} \right) \right] (b_i a_j + b_j a_i)
\]

\[+ \rho_{-(m-1)} \mu_{-1} \left[ a_j \dot{k}b_i + b_i \dot{k}a_j + a_i \dot{k}b_j + b_j \dot{k}a_i \right] + 2\mu_{-1}(\dot{k}\mu_{-1})b_i b_j
\]

\[+ \mu_{-1}^2 \left[ b_j \dot{k}b_i + b_i \dot{k}b_j \right] + (\dot{k}L) \left[ \rho_{-(m-1)} a_{ij} - \rho_{-(2m-1)} a_i a_j \right]
\]

\[+ \mu_{-1}b_{ij} + \mu_{-3}b_i b_j \right]
\]

\[- \left( \dot{k}\rho_{-(2m-1)} \right) a_i a_j - \rho_{-(2m-1)} \left( \left( \dot{k}a_i \right) a_j + \left( \dot{k}a_j \right) a_i \right) + \left( \dot{k}\mu_{-1} \right) b_{ij}
\]

\[+ \left( \dot{k}b_{ij} \right) \mu_{-1} \left( \dot{k}\mu_{-3} \right) b_i b_j + \mu_{-3} \left( \dot{k}b_i \right) b_j + \left( \dot{k}b_j \right) b_i \right].
\]

Partial differentiation of $a_i$ and $b_i$, with respect to $y^k$, yields

\[
\dot{k}a_i = (m - 1)a_{ik} \text{ and } \dot{k}b_i = b_{ik}.
\]
Further, differentiation of $a_{ij}$ and $b_{ij}$ with respect to $y^k$ gives

\begin{equation}
\dot{a}_{ij} = (m - 2)a_{ijk} \quad \text{and} \quad \dot{b}_{ij} = 0,
\end{equation}

where $a_{ijk}(x, y) = a_{ijki \ldots i_m(x)}y^{i_1} \ldots y^{i_m}$.

Thus, we have

**Proposition 6.3.3:** In an R-Randers $m^{th}$-root space the following hold good

\begin{align*}
\dot{a}_i &= (m - 1)a_{ik}, \\
\dot{a}_{ij} &= (m - 2)a_{ijk}, \\
\dot{b}_i &= b_{ik} \quad \text{and} \quad \dot{b}_{ij} = 0.
\end{align*}

If we use (1.6.1), (6.3.5), (6.3.8), (6.3.9), (6.3.16), (6.3.17), (6.3.19) and (6.3.20) in (6.3.18), on simplification it follows that

\begin{equation}
2C_{ijk} = \rho_{-(2m-2)}a_{k}a_{ij} + \xi_{-(2m-2)}(a_ja_{ik} + a_ia_{jk}) + \mu_{-2}(b_ia_{jk} + b_ja_{ik} + b_kb_{ij}) \\
+ \mu_{-4}b_ib_jb_k + \rho_{-(3m-2)}a_ia_ja_k + \rho_{-(m-2)}a_{ijk} \\
+ \rho_{-(m-1)}\mu_{-1}\left\{ \frac{1}{m-1}a_kb_{ij} + b_ka_{ij} + a_jb_{ik} \right\} \\
+ (m - 1)(b_ia_{jk} + b_ja_{ik} + a_kb_{jk}) \\
- (m - 1)\rho_{-(2m-1)}\mu_{-1}(b_ia_j + b_ia_i)a_k + \rho_{-(m-1)}\mu_{-3}(b_ia_j + b_ia_i)b_k,
\end{equation}

where

\begin{align*}
\rho_{-(2m-2)} &= \frac{\rho_{-(m-1)}^2}{m-1} - L(m-1)\rho_{-(2m-1)}, \\
\xi_{-(2m-2)} &= (m - 1)\left( \rho_{-(m-1)}^2 - L\rho_{-(2m-1)} \right), \\
\mu_{-2} &= (\mu_{-1}^2 + L\mu_{-3}), \\
\mu_{-4} &= (3L\mu_{-5} + 2\mu_{-1}\mu_{-3}), \\
\rho_{-(m-2)} &= L(m - 2)\rho_{-(m-1)}, \\
\rho_{-(3m-2)} &= (2m - 1)L\rho_{-(3m-1)} - 2(m - 1)\rho_{-(m-1)}\rho_{-(2m-1)}.
\end{align*}

Thus, we have

**Theorem 6.3.4:** In an R-Randers $m^{th}$-root space, the Cartan tensor $C_{ijk}$ is given by (6.3.21).
6.4 Spray and Equation of Geodesics

In this section, we discuss about the spray of an R-Randers $m^{th}$-root Space and obtain its local coefficients. We also obtain the equation of geodesics in such space.

If we differentiate (6.2.5) partially with respect to $x^j$, we get

$$\partial_j F = \frac{\rho^{-(m-1)}}{m(m-1)} A_j,$$

where

$$A_j = (\partial_j a_{i_1...i_m}) y^{i_1}...y^{i_m}.$$  

Further, differentiating (6.3.3) partially with respect to $x^k$ and utilizing $\partial_k \rho^{-(m-1)} = \frac{-(m-1)}{m} \rho^{-(2m-1)} A_k$, we have

$$\partial_k \dot{\partial}_i F = -\frac{1}{m} \rho^{-(2m-1)} A_k a_i + \frac{\rho^{-(m-1)}}{(m-1)} \partial_k a_i.$$  

Differentiating (6.2.6) partially with respect to $x^j$, we get

$$\partial_j \alpha = \frac{1}{2} \mu_{-1} B_j,$$

where

$$B_j = (\partial_j b_{i_1i_2}) y^{i_1}y^{i_2}.$$  

Also, we have

$$\partial_k \mu_{-1} = \frac{1}{2} \mu_{-3} B_k.$$  

Next, differentiating (6.3.6) partially with respect to $x^k$ and using (6.4.6), we obtain

$$\partial_k \dot{\partial}_i \alpha = \frac{1}{2} \mu_{-3} b_i B_k + \mu_{-1} \partial_k b_i.$$  

Thus, we have
Proposition 6.4.1: In an $R$-Randers $m^{th}$-root space, the following hold good

\[
\partial_j F = \frac{\rho^{-(m-1)}}{m(m-1)} A_j, \quad \partial_j \alpha = \frac{1}{2} \mu^{-1} B_j, \\
\partial_k \dot{\partial}_i F = -\frac{1}{m} \rho^{-(2m-1)} A_k a_i + \frac{\rho^{-(m-1)}}{(m-1)} \partial_k a_i, \quad \partial_k \dot{\partial}_i \alpha = \frac{1}{2} \mu^{-3} b_i B_k + \mu^{-1} \partial_k b_i.
\]

If we differentiate (6.2.4) partially with respect to $x^k$ and use (6.4.1) and (6.4.4), it follows that

\[\partial_k L = \frac{\rho^{-(m-1)}}{m(m-1)} A_k + \frac{1}{2} \mu^{-1} B_k.\]  

(6.4.8)

Next, differentiating (6.3.9) partially with respect to $x^k$ and using (6.4.3) and (6.4.7), we get

\[\partial_k \dot{\partial}_i L = -\frac{1}{m} \rho^{-(2m-1)} A_k a_i + \frac{\rho^{-(m-1)}}{(m-1)} \partial_k a_i + \frac{1}{2} \mu^{-3} b_i B_k + \mu^{-1} \partial_k b_i.\]  

(6.4.9)

In view of (6.4.8) and (6.4.9), (6.2.2) gives

\[G^i = \frac{1}{4} g^{ij} \left[ \frac{2}{m} \left( \frac{\rho^{2-(m-1)}}{(m-1)^2} - L \rho^{-(2m-1)} \right) a_j A_k y^k + \frac{\rho^{-(m-1)}}{(m-1)^2} \mu^{-1} (2b_j A_k + a_j B_k) y^k + 2L \rho^{-(m-1)} \left( \partial_k a_j y^k \right) \\
+ 2L \mu^{-1} \left( \partial_k b_j y^k - \mu^{-1} (\partial_k a_j)^2 \right) \right],\]

that is

\[G^i = \frac{1}{4} g^{ij} \left[ \frac{2}{m} \rho^{-(m-2)} a_j A_0 + \mu^{-2} b_j B_0 \\
+ \frac{1}{m-1} \rho^{-(m-1)} \mu^{-1} (2b_j A_0 + a_j B_0) \\
+ \frac{2}{(m-1)(m-2)} \rho^{-(m-2)} \left( (\partial_k a_j y^k - A_j) + \eta^{-2} (2(\partial_k b_j y^k - B_j) \right) \right],\]

where

\[A_0 = A_k y^k, \quad B_0 = B_k y^k, \quad \eta^{-2} = \mu^{-1}\]

and $g^{ij}$ is given by (6.3.14). Thus, we have

\[(6.4.10) \quad A_0 = A_k y^k, \quad B_0 = B_k y^k, \quad \eta^{-2} = \mu^{-1}\]
Theorem 6.4.1: In an R-Randers $m^{th}$-root space, the spray coefficients are given by (6.4.10).

In view of (6.2.1) and Theorem 6.4.1, we have

**Corollary 6.4.1:** In an R-Randers $m^{th}$-root space, the equation of geodesics is given by

$$\frac{d^2x^i}{dt^2} + 2G^i = 0,$$

where the spray coefficients $G^i$ are given by (6.4.10).

### 6.5 Nonlinear Connection

In this section, we obtain the coefficients of nonlinear connection of the space under consideration.

Differentiating (6.4.10) partially with respect to $y^k$, we have

\[
\dot{\partial}_k G^i = \frac{1}{4} \left( \dot{\partial}_k g^{ij} \right) \left[ \frac{2}{m(m-1)} \rho_-(2m-2)a_j A_0 + \mu_- 2 b_j B_0 \right.
\]
\[
+ \frac{1}{m-1} \rho_-(m-1) \mu_-(2b_j A_0 + a_j B_0)
\]
\[
+ \frac{2}{(m-1)(m-2)} \left( \left( \dot{\partial}_k a_j y^j - A_j \right) + \eta_- 2 \left( \partial_k b_j y^k - B_j \right) \right]
\]
\[
+ \frac{1}{4} g^{ij} \left[ \frac{2}{m(m-1)} \left( \dot{\partial}_k \rho_-(2m-2)a_j A_0 + \rho_-(2m-2)(A_0 \dot{\partial}_k a_j + a_j \dot{\partial}_k A_0) \right) \right.
\]
\[
\left. + \left( \dot{\partial}_k \mu_- 2 b_j B_0 + \mu_- 2 (b_j \dot{\partial}_k B_0 + B_0 \dot{\partial}_k b_j) \right) \right]
\]
\[
+ \frac{1}{m-1} \left\{ \left( \dot{\partial}_k \rho_-(m-1) \mu_- 1 + \rho_-(m-1)(\dot{\partial}_k \mu_- 1) \right) \right\} (2b_j A_0 + a_j B_0)
\]
\[
+ \frac{1}{m-1} \rho_-(m-1) \mu_- 1 \left\{ 2(\dot{\partial}_k b_j) A_0 + 2b_j (\dot{\partial}_k A_0) + (\dot{\partial}_k a_j) B_0 + a_j (\dot{\partial}_k B_0) \right\}
\]
\[
+ \frac{2}{(m-1)(m-2)} \left( \dot{\partial}_k \rho_-(m-2) \right) \left( (\partial_l a_j) y^l - A_j \right)
\]
\[
+ \rho_- (m-2) \left( \dot{\partial}_k (\partial_l a_j) y^l + (\partial_l a_j)(\dot{\partial}_k y^l) - \dot{\partial}_k A_j \right)
\]
\[
+ (\dot{\partial}_k \eta_- 2) \left( 2(\dot{\partial}_k b_j) y^k - B_j \right)
\]
\[
+ \eta_- 2 \left[ 2(\dot{\partial}_k \partial_l b_j) y^k + 2(\partial_l b_j)(\dot{\partial}_k y^k) - \dot{\partial}_k B_j \right].
\]
Differentiation of $g^i_j g_{jm} = \delta^i_m$ partially with respect to $y^k$, yields

\[(6.5.2) \quad \dot{\partial}_k g^i_l = -2C^i_{km} g^{ml},\]

where $C^i_{km} = C^{jkm} g^i_j$.

Next, differentiating $\rho_{-(2m-2)} = \rho_{-(m-1)}^2 - L(m-1)\rho_{-(2m-1)}$ partially with respect to $y^k$ and using (6.3.5), (6.3.9) and (6.3.16), we get

\[(6.5.3) \quad \dot{\partial}_k \rho_{-(2m-2)} = (m-1)(-3\rho_{-(3m-2)} a_k - \rho_{-(2m-1)} \mu_1 b_k)
+ (2m - 1) L\rho_{-(3m-1)} a_k),\]

where $\rho_{-(3m-2)} = (m - 1) F^{-(3m-2)}$.

Differentiating $A_0 = A_l y^l$ and $B_0 = B_l y^l$ partially with respect to $y^k$, we respectively have

\[(6.5.4) \quad \dot{\partial}_k A_0 = m(\partial_l a_k) y^l + A_k \text{ and}\]

\[(6.5.5) \quad \dot{\partial}_k B_0 = 2(\partial_l b_k) y^l + B_k.\]

Also, we have

\[(6.5.6) \quad \dot{\partial}_k (\partial_l a_j) = (m - 1) \partial_l a_{jk}, \quad \dot{\partial}_k (\partial_l b_j) = \partial_l b_{jk}, \quad \dot{\partial}_k A_j = m \partial_j a_k \]

and $\dot{\partial}_k B_j = 2 \partial_j b_k$.

Using (6.2.3), (6.3.5), (6.3.8), (6.3.16), (6.3.17), (6.3.19) and (6.5.2)-(6.5.6) in (6.5.1), we have

\[(6.5.7) \quad N^i_k = -2C^i_{km} G^m + \frac{1}{4} g^{ij} \left[ \frac{2}{m} (-3\rho_{-(3m-2)} - (2m - 1)\rho_{-(3m-1)}) a_k a_j A_0 
- \rho_{-(2m-1)} \mu_1 (b_k a_j A_0 + 2a_k b_j A_0 + a_k a_j B_0) 
+ \frac{2\rho_{-(2m-2)}}{m(m-1)} (m(\partial_l a_k) y^l a_j + A_k a_j + (m - 1) A_{jk} A_0 + (\partial_l a_j) y^l a_k - A_j a_k)
+ (3\mu_4 - \mu_1 \mu_3) b_k b_j B_0
+ \frac{1}{m-1} \rho_{-(m-1)} \mu_3 (a_k b_j B_0 + (2b_j A_0 + a_j B_0) b_k)
+ \mu_2 (b_{jk} B_0 + 2(\partial_l b_k) y^l b_j + B_k b_j + b_k (2(\partial_l b_j) y^l - B_j))\right].\]
Thus, we have

**Theorem 6.5.1:** The local coefficients of the nonlinear connection of an R-Randers $m^{th}$-root space are given by (6.5.7).