Chapter 4

Double Path Union and its $\alpha$-labeling

4.1 Introduction

In 1966 Rosa [6] defined $\alpha$-labeling as a graceful labeling with an additional property that there is an integer $k$ ($0 \leq k < |E(G)|$) such that for every $e = (x, y) \in E(G)$, either $f(x) \leq k < f(y)$ or $f(y) \leq k < f(x)$. A graph which admits $\alpha$-labeling is necessarily bipartite graph with partition of $V(G) = V_1 \cup V_2$, where $V_1 = \{v \in V(G)/f(x) \leq k\}$ and $V_2 = \{v \in V(G)/f(x) > k\}$. We call such graph $G$ with $\alpha$-labeling $f$ as an $\alpha$-graceful graph.

In [35] Kaneria, Viradia and Makadia proved that the path union of a semi smooth graceful graph, star of a semi smooth graceful graph and cycle graph of a semi smooth graceful graph are graceful. They also proved step grid graphs $St_n$, Cycle of graphs $C(t \cdot H)$ and $C^m(t \cdot C_n)$ are smooth graceful graphs, where $t \equiv 0 \pmod{4}$, $n \equiv 0 \pmod{4}$, $m \in N$ and $H$ is a semi smooth graceful graph. Some of these results we discuss here and we also discuss equivalency of $\alpha$-labeling and semi smooth graceful labeling. We also derived $\alpha$-labeling for double path union of some graph.
4.2 Graceful Labeling and $\alpha$-labeling

A bipartite graceful graph $G$ with graceful labeling $f$ is said to be *smooth graceful graph* if it admits an injective function $g : V(G) \rightarrow \{0, 1, \ldots, \left\lfloor \frac{q-1}{2} \right\rfloor, \left\lfloor \frac{q+1}{2} \right\rfloor + l, \left\lfloor \frac{q+2}{2} \right\rfloor + l, \ldots, q + l\}$ such that its induced edge labeling map $g^* : E(G) \rightarrow \{1 + l, 2 + l, \ldots, q + l\}$ defined as $g^*(e) = |g(x) - g(y)|$, for every edge $e = (x, y) \in E(G)$, where $l$ be any positive integer. A *semi smooth graceful graph* $G$, we mean it is a bipartite graph with $|E(G)| = q$ and the property that for all $l \in \mathbb{N}$, there is an integer $t$ ($1 \leq t \leq q$) and an injective function $g : V(G) \rightarrow \{0, 1, \ldots, t - 1, t + l, t + l + 1, \ldots, q + l\}$ such that the induced edge labeling function $g^* : E(G) \rightarrow \{1 + l, 2 + l, \ldots, q + l\}$ defined as $g^*(e) = |g(u) - g(v)|$ is a bijection for every edge $e = (u, v) \in E(G)$.

4.2.1 Illustration : $\alpha$-labeling and smooth graceful labeling

for $St_5$ Path union of graph :
4.2.2 Definitions of path union, double path union

Let $G$ be a graph and $G^{(1)}, G^{(2)}, \ldots, G^{(n)}, n \geq 2$ be $n$ copies of $G$. Let $v \in V(G)$. The graph obtained by joining vertex $v$ of $G^{(i)}$ with same vertex of $G^{(i+1)}$ by an edge, $\forall \ i = 1, 2, \ldots, n - 1$ is called path union of graph $G$. We shall denote it by $P(n \cdot G)$. Here $P(n \cdot G)$ can obtain $|V(G)|$ different ways. If $G = K_1$ then $P(n \cdot K_1) = P_n$.

Let $G$ be a graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$. Let $G^{(0)}, G^{(1)}, \ldots, G^{(n)}$ be $n + 1$ copies of $G$. Now each vertex $v_i$ of $G^{(0)}$ join with the corresponding vertex $v_i$ of $G^{(i)}, \forall i = 1, 2, \ldots, n$. Such graph is known as star of $G$ and it is denoted by $G^*$. We call $G^{(0)}$ as central copy of $G^*$. $K^*_1 = K_2$ and $K^*_2 = P_6$.

Let $G$ be a graph and $G^{(1)}, G^{(2)}, \ldots, G^{(n)}, n \geq 2$ be $n$ copies of $G$. Let $v \in V(G)$. The graph obtained by joining vertex $v$ of $G^{(i)}$ with same vertex of $G^{(i+1)}$ by an edge, $\forall \ i = 1, 2, \ldots, n - 1$ and $v$ of $G^{(n)}$ with the same vertex of $G^{(1)}$ by an edge is called cycle of $G$. It is denoted by $C(n \cdot G)$. If we replace $G$ by $C(n \cdot G)$ then such graph $C(n \cdot C(n \cdot G))$, we denote it by $C^2(n \cdot G)$. In general for any $t \geq 2$, $C^t(n \cdot G) = C(n \cdot C^{t-1}(n \cdot G))$. Obviously $C(n \cdot K_1) = C_n$.

The cartesian product of graphs $G$ and $H$ is denoted as $G \times H$, is the graph with vertex set $V(G) \times V(H) = \{(u, v) : u \in V(G) \text{ and } v \in V(H)\}$ and vertex $(u, v)$ adjacent to another vertex $(x, y)$ if and only if either $u = x$ and $(v, y) \in E(H)$ or $v = y$ and $(u, x) \in E(G)$. $(P_n \times P_m)$ is known as grid graph on $mn$ vertices.

Let $G$ be a graph and $G^{(1)}, G^{(2)}, \ldots, G^{(n)}$ $(n \geq 2)$ be $n$ copies of $G$. Then the graph obtained by joining a pair of distinct vertices $u, v$ of $G^{(i)}$ with same vertices of the graph $G^{(i+1)}$ by two edges, $\forall \ i = 1, 2, \ldots, n - 1$ is called double path union of $n$ copies of the graph $G$, such graph we can obtain $\frac{p(p-1)}{2}$ different ways, where $p = |V(G)|$ and we shall denote such graph by $D(n \cdot G)$. $D(n \cdot K_1)$ is undefined and $D(n \cdot P_2) = P_2 \times P_n$.

Take $P_n, P_n, P_{n-1}, P_{n-2}, \ldots, P_3, P_2$ paths on $n, n, n - 1, n - 2, \ldots, 3, 2$ vertices and arranged them vertically. A graph obtained by joining horizontal vertices of
given successive paths is known as a *step grid graph* of size \( n \) \((n \geq 3)\) and we will denote it by \( St_n \). Obviously \(|V(St_n)| = \frac{1}{2}(n^2 + 3n - 2)\) and \(|E(St_n)| = n^2 + n - 2\).

**4.2.3 Illustration of above definitions:**

![Diagram](image1)

- **figure-4.2** \( P(5 \cdot G) \) at a vertex \( v \in V(G) \)

- **figure-4.3** \( C^*_5(StarofC_5) \)

- **figure-4.4** \( C(6 \cdot G) \) cycle graph of graph \( G \) at a vertex \( v \)

- **figure-4.5** \( P_4 \times W_3 \)
In [36] Kaneria and Meera proved that $B_{m,n}$ is a semi smooth graceful graph. By taking $l = 0$ we get $\alpha$-graceful labeling for $B_{m,n}$.

4.2.4 Illustration : Semi smooth graceful labeling for $B_{4,5}$ and its $\alpha$-graceful labeling.

Here $V(B_{4,5}) = \{v_0, v_1, v_2, v_3, v_4, u_0, u_1, u_2, u_3, u_4, u_5\}$ and $E(B_{4,5}) = \{v_0v_1, v_0v_2, v_0v_3, v_0v_4, v_0u_0, u_0u_1, u_0u_2, u_0u_3, u_0u_5\}$. In [36] they have defined vertex labeling function $f : V(B_{4,5}) \rightarrow \{0, 1, 2, 3, 4, \ldots, 10 + l\}$ as follows.

- $f(v_0) = 10 + l$,
- $f(u_0) = 4$,
- $f(v_i) = i - 1$, $\forall i = 1, 2, \ldots, 4$;
- $f(u_j) = 10 + l - j$, $\forall j = 1, 2, \ldots, 5$.
4.2.5 Theorem: \( B_{m,n} \) is \( \alpha \)-graceful graph.

**Proof:** It is obvious that \( V(B_{m,n}) = \{v_0, v_1, \ldots, v_m, u_0, u_1, \ldots u_n\} \) and \( E(B_{m,n}) = \{v_0v_1, v_0v_2, v_0v_3, \ldots, v_0v_m, v_0u_0, u_0u_1, \ldots, u_0u_n\} \). Define \( f: V(B_{m,n}) \rightarrow \{0, 1, 2, 3, \ldots m + n + 1\} \) as follows.

\[
f(v_i) = \begin{cases} 
q & \text{if } i = 0 \\
q - 1 & \text{if } i = 1, 2, \ldots, m
\end{cases}
\]

Note that \( f(V(B_{m,n})) = \{q, 0, 1, \ldots, m - 1, m + 1, \ldots, m + n\} \)

where \( q = m + n + 1 \). So, \( f \) is a bijective map and so, it is injective. Further we see that

\[
f^*(u_0v_0) = n + 1,
\]

\[
f^*(v_0v_i) = q + 1 - i, \forall i = 1, 2, \ldots m
\]

\[
f^*(u_0v_j) = j, \forall j = 1, 2, \ldots n
\]

Hence, \( f^*(E(B_{m,1})) = \{n + 1, q, q - 1, \ldots, n + 2, 2, \ldots, n\} = \{1, 2, \ldots, m + n + 1\} \) and so, \( f^* : E(B_{m,n}) \rightarrow \{1, 2, \ldots, m + n + 1\} \) is a bijective map. So, \( f \)

is a graceful labeling for \( B_{m,n} \).

Take \( k = m \). Observe that mini \( \{f(u_0), f(v_0)\} = m = k < \max\{f(u_0), f(v_0)\} \)

\[
= q, \min\{f(v_0), f(v_i)\} = i-1 \leq m-1 \leq k < \max\{f(v_0), f(v_i)\}, \forall i = 1, 2, 3, \ldots m.
\]

and mini \( \{f(u_0), f(u_j)\} = m = k < \max\{f(u_0), f(u_j)\} = m + j, \forall j = 1, 2, 3, \ldots, n \).

Therefore, each \( xy \in E(B_{m,n}) \) mini \( \{f(x), f(y)\} \leq k < \max\{f(x), f(y)\} \).

Hence, \( f \) is an \( \alpha \)-graceful labeling for \( B_{m,n} \) and so, it is an \( \alpha \)-graceful graph.

In [35] Kaneria, Makadia and Viradia proved that splitting graph of \( B_{m,n} \) is a semi smooth graceful graph.

### 4.2.6 Illustration: Semi smooth graceful labeling and its \( \alpha \)-graceful labeling for \( B_{3,5} \).  

Here \( V(S'(B_{3,5})) = \{v_0, v_1, v_2, v_3, u_0, u_1, \ldots, v_5, v'_0, v'_1, v'_2, v'_3, u'_0, u'_1, \ldots u'_3\} \) and \( E(S'(B_{3,5})) = \{v_0v_1, v_0v_2, v_0v_3, v_0u_0, u_0u_1, u_0u_2, \ldots, u_0u_5, v_0v'_0, v_0v'_1, v_0v'_2, v'_3, v'_0v_1, v'_0v_2, v'_0v_3, \ldots, u'_0u_5, u_0u'_1, \ldots u_0u'_5\} \). In [35] Kaneria, Makadia and Viradia have defined
$f(V(S'(B_{3,5}))) \rightarrow \{0, 1, 2, \ldots, 7+l, 8+l, \ldots, 27+l\}$ as follows.

$f(v_0) = 27+l$, $f(u_0) = 6$, $f(v'_0) = 22+l$, $f(u'_0) = 7$; $f(v_i) = 2+i$, $\forall i = 1, 2, 3$; $f(v'_i) = 3-i$, $i = 1, 2, 3$; $f(u_j) = 6+2j+l$, $\forall j = 1, 2, 3, 4, 5$ and $f(u'_j) = 16+j+l$, $\forall j = 1, 2, 3, 4, 5$.

figure 4.8  Semi smooth graceful labeling of $S'(B_{3,5})$

figure 4.9  $\alpha$-graceful labeling of $S'(B_{3,5})$
4.2.7 Theorem: $S'(B_{m,n})$ is a $\alpha$-graceful graph.

Proof: Let $G = S'(B_{m,n})$, $V(G) = \{v_0, v_1, v_2, \ldots, v_m, u_0, u_1, \ldots, u_n, u'_0, u'_1, \ldots, u'_n, v'_0, v'_1, v'_2, \ldots, v'_m\}$ and $E(G) = E(B_{m,n}) \cup \{v_0v'_1, v_0v'_2, \ldots, v_0v'_m, v'_0v_1, v'_0v_2, \ldots, v'_0v_m, u_0u'_1, u_0u'_2, \ldots, u_0u'_n, u'0u_1, u'0u_2, \ldots, u'0u_n, u'_0v_0, u'_0v'_0\}$. i.e. $|V(G)| = 2(m + n + 1)$ and $|E(G)| = 3(m + n + 1)$. Define a vertex labeling function $f : V(G) \rightarrow \{0, 1, 2, \ldots, 3(m + n + 1)\}$ as follows.

$$f(v_0) = 3(m + m + 1), f(u_0) = 2m, f(v'_0) = 2m + 3n + 1, f(u'_0) = 2m + 1, f(v_i) = m + (i - 1), \forall i = 1, 2, 3, \ldots, m; f(v'_i) = m - i, \forall i = 1, 2, \ldots, m; f(u_j) = 2(m + j);$$

$$\forall j = 1, 2, 3, \ldots, n \text{ and } f(u'_j) = 2(m + n) + j, \forall j = 1, 2, \ldots, n.$$

Note that $f(V(G)) = \{0, 1, \ldots, m - 1, m, m + 1, \ldots, 2m - 1, 2m, 2m + 1, 2m + 2, 2m + 4, \ldots, 2m + 2n, 2m + 2n + 1, \ldots, 2m + 3n, 2m + 3n + 1, 3(m + n + 1)\}$ $\subset \{0, 1, \ldots, 3(m + n + 1)\}$ and $f$ is an injective map, as there is no $x, y \in V(G)$ such that $f(x) = f(y)$. Let $q = 3(m + n + 1)$. Observe that

$$f^*(v_0v'_i) = q - m + i, \forall i = 1, 2, \ldots, m;$$

$$f^*(v_0v_i) = q - m - i + 1, \forall i = 1, 2, \ldots, m;$$

$$f^*(v_0u_0) = m + 3n + 3;$$

$$f^*(v_0u'_0) = m + 3n + 2;$$

$$f^*(v'_0v_i) = m + 3n + 2 - i, \forall i = 1, 2, \ldots, m;$$

$$f^*(v'_0u_0) = 3n + 1;$$

$$f^*(u_0u'_j) = 2n + j, \forall j = 1, 2, \ldots, n$$

$$f^*(u_0u_j) = 2j, \forall j = 1, 2, \ldots, n;$$

$$f^*(u'_0u_j) = 2j - 1, \forall j = 1, 2, \ldots, n;$$

Hence, $f^*(E(G)) = \{1, 2, \ldots, q\}$ and $f^*$ is an injective map as there is no $e_1, e_2 \in E(G)$ such that $f^*(e_1) = f^*(e_2)$. Therefore, $f^*$ is a bijective map. So, $f$ is a graceful labeling for $G$. Take $k = F^*(u'_0 = 2m + 1$. Further we see that

$$\min\{f(v_0), f(v'_i)\} < m < k < \max\{f(v_0), f(v'_i)\} = q,$$

$$\min\{f(v_0), f(v_i)\} < 2m < k < \max\{f(v_0), f(v_i)\} = q,$$

$$\min\{f(v_0), f(u_0)\} = 2m < k < \max\{f(v_0), f(u_0)\} = q,$$
\[ \min\{f(v_0), f(u'_0)\} = 2m + 1 \leq k < \max\{f(v_0), f(u'_0)\} = q, \]
\[ \min\{f(v'_0), f(u_i)\} = 2m < k < \max\{f(v'_0), f(u_i)\} = 2m + 3n + 1, \]
\[ \min\{f(v_i), f(u_j)\} = 2m < k < 2m + 2 \leq \max\{f(u_i), f(u_j)\}, \]
\[ \min\{f(u'_0), f(u_j)\} = 2m + 1 \leq k < 2m + 2 \leq \max\{f(u'_0), f(u_j)\}, \]
\[ \forall i = 1, 2, \ldots, m \text{ and } \forall j = 1, 2, \ldots, n. \]

Therefore, each \( xy \in E(G) \), \( \min\{f(x), f(y)\} \leq k < \max\{f(x), f(y)\} \) and so, \( f \) is an \( \alpha \)-graceful labeling for \( G \). Hence, \( G = S'(B_{m,n}) \) is a \( \alpha \)-graceful graph.

4.3 Semi Smooth Graceful Labeling and \( \alpha \)-labeling

In [35] Kaneria, Makadia and Viradia proved that every smooth graceful graph \( G \) is also an \( \alpha \)-graceful graph and every \( \alpha \)-graceful graph \( G \) admits a semi smooth graceful labeling. In fact by definitions of smooth graceful and semi smooth graceful labelings, every smooth graceful graph is a semi smooth graceful graph, as by taking \( \left \lceil \frac{q+1}{2} \right \rceil = t \), smooth gracefulfulness is a particular case of semi smooth gracefulfulness. In this section we shall prove that every semi smooth graceful graph \( G \) admits an \( \alpha \)-graceful labeling. We also discuss about universal graceful labeling and universal \( \alpha \)-graceful labeling.

4.3.1 Theorem : Every semi smooth graceful graph \( G \) is also an \( \alpha \)-graceful graph.

**Proof** : Let \( G \) be a semi smooth graceful graph with a semi smooth graceful labeling, \( g : V(G) \to \{0, 1, \ldots, t-1, t+l, t+l+1, \ldots, q+l\} \) for some \( t \in N \) and for every \( l \in N \), its edge induced labeling function \( g^* : E(G) \to \{1+l, 2+l, \ldots, q+l\} \) defined by \( g^*(xy) = |g(x) + g(y)| \) is a bijective map.
By taking \( l = 0 \), \( g \) becomes a graceful labeling for \( G \) and so, \( G \) is graceful labeling, i.e. \( g \) is an injective map and \( g^* : E(G) \rightarrow \{1, 2, \ldots, q\} \) is bijective map.

Take \( k = t - 1 \). It can be observe that \( G \) is a bipartite graph with \( V(G) \) has a partition \( v_1 = \{v \in V(G)/g(v) \leq t - 1\} \) and \( v_2 = \{v \in V(G)/g(v) \geq t\} \). i.e. \( v_2 = \{v \in V(G)/g(v) > t\} \). Thus, for any \( xy \in E(G) \) we get \( \min\{g(x), g(y)\} \leq k = t - 1 < \max\{g(x), g(y)\} \) and so, \( g \) is an \( \alpha \)-graceful labeling \( g \) by taking \( l = 0 \) and so, \( G \) is an \( \alpha \)-graceful graph.

Let \( G \) be a graceful graph with graceful labeling \( f : V(G) \rightarrow \{0, 1, \ldots, E(G)\} \).
A vertex \( v \in V(G) \) is called a graceful center w.r.t \( f \) if \( f(v) = 0 \). A graph \( G \) is said to be universal graceful graph if for any \( v \in V(G) \), \( v \) is a graceful center for \( G \) w.r.t. some graceful labeling \( g \) of \( G \). Also we call \( G \) is an universal \( \alpha \)-graceful graph if for every \( v \in V(G) \) there exits an \( \alpha \)-graceful labeling \( g \) on \( G \) and \( g(v) = 0 \).

As \( q - f \) is also a graceful labeling for \( G \) \( (q - f)(w) = 0 \), where \( w \in V(G) \) and \( f(w) = q \), \( w \) is also a graceful center w.r.t \( q - f \) for \( G \), where \( q = |E(G)| \). Thus, any graceful graph \( G \) has at least two graceful center \( v \) and \( w \), when \( f(v) = 0 \) and \( f(w) = 0 \).

In [37] Makadia, Karvadiya and Kaneria proved that any \( \alpha \)-graceful graph \( G \) has at least four \( \alpha \)-graceful center, as \( f, q - f, h, q - h \) four \( \alpha \)-labelings for \( G \), where \( h : V(G) \rightarrow \{0, 1, \ldots, q\} \) defined by \( h|v_1 = k - f \) and \( h|v_2 = q + k + 1 - f \), where \( k = \min\{f(x), f(y)\} \) and \( xy \in E(G) \) such that \( f^*(xy) = 1 \), \( v_1 = \{v \in V(G)/f(v) \leq k\} \) and \( V_2 = V(G) - V_1 \). In some paper they have proved that the following tree \( T \) is an \( \alpha \)-graceful graph, but it is not a universal graph, as the vertex \( v \) can not be a graceful center for \( T \) w.r.t any graceful labeling \( T \).
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Every cycle $C_n (n \equiv 0 \pmod{4})$, star $K_{1,n}$ are universal $\alpha$-graceful graphs. While $C_n (n \equiv 3 \pmod{4})$ and $W_n$ are universal graceful graphs but they are not universal $\alpha$-graceful graphs.

4.3.2 Theorem : The graph obtained by joining two copies of a universal $\alpha$-graceful graph $G$ say $G^{(1)}$ and $G^{(2)}$ by an edge with any two corresponding vertices $v^{(1)} \in V(G^{(1)})$ and $v^{(2)} \in V(G^{(2)})$, for some $v \in V(G)$ is also a universal $\alpha$-graceful graph.

Proof : Let $H$ is a graph obtained by two copies of a universal $\alpha$-graceful graph $G$ say $G^{(1)}$ and $G^{(2)}$ by an edge with any two corresponding vertices $v^{(1)} \in V(G^{(1)})$ and $v^{(2)} \in V(G^{(2)})$, for some $v \in V(G)$.

Let $u \in V(H)$. Implies either $u \in V(G^{(1)})$ or $u \in V(G^{(2)})$. We may assume that $u \in V(G^{(1)})$, call $u$ by $u^1$, as $u^1 \in V(G^{(1)})$. Now $u \in V(G)$ and since, $G$ is a universal $\alpha$-graceful graph, there exit an $\alpha$-labeling $f : V(G) \rightarrow \{0, 1, \ldots, \left|E(G)\right|\}$ such that $f(u) = 0$, since $f$ is an $\alpha$-graceful labeling for $G$, there exit $k \ (0 \leq k < \left|E(G)\right|)$ such that for each $xy \in E(G)$, $\min\{f(x), f(y)\} \leq k < \max\{f(x), f(y)\}$. Take $v_1 = \{w \in V(G)/f(w) \geq k\}$ and $V_2 = V(G) - V_1$.

Here $H$ is the graph obtained by joining $G^{(1)}$ and $G^{(2)}$, two copies of $G$ by an edge between $v^{(1)}$ and $v^{(2)}$, where $v^{(1)} \in V(G^{(1)})$ and $v^{(2)} \in V(G^{(2)})$ for some $v \in V(G)$. It is obvious that $V(H) = V(G^{(1)}) \cup V(G^{(2)})$ and $E(H) = E(G^{(1)}) \cup E(G^{(2)}) \cup \{(v^{(1)}), (v^{(2)})\}$, $\left|V(H)\right| = 2\left|V(G)\right|$ and $\left|E(H)\right| = 2q + 1$, where $q = \left|E(G)\right|$.
Define \( g : V(H) \rightarrow \{0, 1, \ldots, 2q + 1\} \) as follows. \( g|V^{(1)}_1 = f|V^{(1)}_1 \), \( g|V^{(2)}_1 = f|V^{(2)}_1 \), \( g|V^{(2)}_2 = f|V^{(2)}_2 + q + 1 \) and \( g|V^{(1)}_2 = f|V^{(1)}_2 + (q + 1) \). Since \( f \) is \( 1 - 1 \), \( g \) is also a \( 1 - 1 \) map. Take \((w_1, w_2) \in E(G)\) be any edge for any \( i = 1, 2 \).

\[
g^*(w^i_1, w^i_2) = |g(w^i_1) - g(w^i_2)|.
\]

\[
g^*(w^i_1, w^i_2) = \begin{cases} 
(q + 1) + f(w^i_1) - f(w^i_2), & \text{if } w^i_1 \in V^{(1)}_2 \cup V^{(2)}_1 \\
(q + 1) + f(w^i_1) - f(w^i_2), & \text{if } w^i_1 \in V^{(2)}_2 \cup V^{(1)}_1 \\
(q + 1) + f^*((w^i_1, w^i_2)), & \text{if } w^i_1 \in V_1 \quad \& \quad i = 2 \quad \text{or} \\
(q + 1) - f^*((w^i_1, w^i_2)), & \text{otherwise}.
\end{cases}
\]

Since range of \( f^* = \{1, 2, 3, \ldots, q\} \), \( g^*((v^{(1)}, v^{(2)})) = q + 1 \), range of \( g^* = \{1, 2, \ldots, 2q + 1\} \). Thus, \( g^* : E(H) \rightarrow \{1, 2, \ldots, 2q + 1\} \) is a bijective map,\( g^* \) is \( 1 - 1 \) and range of \( g^* = \text{codomain of } g^* \). Hence, \( g^* \) is an graceful labeling for \( H \). Take \( k = q \). Now for each \( u \in V(G) \), \( f(u) \leq q \). Therefore, \( \min\{g(u^{(1)}), g(u^{(2)})\} \leq q \) and \( \max\{g(u^{(1)}), g(u^{(2)})\} \geq q + 1 \).

Therefore, \( \min\{g(u^{(1)}), g(u^{(2)})\} \leq k < \max\{g(v^{(1)}), g(v^{(2)})\} \). For any \((w^i_1, w^i_2) \in E(H)\), we have \((w_1, w_2) \in E(G)\) and so, one of them lies in \( V_1 \) and another of them lies in \( V_2 \), as \( G \) is a bipartite graph.

For any \((w^i_1, w^i_2) \in E(H)\), \( \min\{g(w^i_1), g(w^i_2)\} \leq k < \max\{g(w^i_1), g(w^i_2)\} \) for any \( i = 1, 2 \). Hence, for any \((w_1, w_2) \in E(H)\) and \( \min\{g(w_1, g(w_2)) \leq k < \max\{g(w_1, g(w_2)) \}. Therefore, \( g \) is an \( \alpha \)-graceful labeling for \( H \).

Moreover \( g(u^{(1)}) = f(u) = 0 \), as \( u \in V_1 \). So, \( u \) is also an \( \alpha \)-graceful center for \( H \) w.r.t. \( \alpha \)-graceful labeling \( g \) of \( H \). Since \( u \in V(H) \) be the arbitrary vertex of \( H \), \( H \) must be universal \( \alpha \)-graceful graph.

### 4.4 Double Path Union of Some Graphs

#### 4.4.1 Theorem

Let \( G = C_m, m \equiv 0 \pmod{4} \). Any double path union of \( G \) obtained by \( u, v \in V(G) \) is \( \alpha \)-graceful, where \( d_G(u, v) \) is odd.
Proof: Let $v_1, v_2, \ldots, v_m$ be vertices of $C_m$. We know that $f : V(C_m) \rightarrow \{0, 1, \ldots, m\}$ defined by

$$f(v_i) = m - \left(\frac{i-1}{2}\right), \quad \forall i = 1, 3, \ldots, m - 1$$

$$= \left(\frac{m-2}{2}\right), \quad \forall i = 2, 4, \ldots, \frac{m}{2}$$

$$= \left(\frac{1}{2}\right), \quad \forall i = \frac{m}{2} + 2, \frac{m}{2} + 4, \ldots, m.$$ 

is $\alpha$-graceful labeling for $C_m$, where $m \equiv 0 \pmod{4}$. Let $G = D(n \cdot C_m)$, double path union of $n$ copies of the cycle $C_m$ ($m \equiv 0 \pmod{4}$). Let $u_{i,j}$ ($1 \leq j \leq m$) be vertices for $i^{th}$ copy $C_m^{(i)}$ in $G$, $\forall i = 1, 2, \ldots, n$ with $u_{1,j} = v_j$ ($1 \leq j \leq m$).

Obviously $P = |V(G)| = mn$ and $Q = |E(G)| = (m+2)n - 2$. Now join vertices $u_{i, \frac{m}{2} + 1}$ with $u_{i+1, \frac{m}{2} + 1}$ and $u_{i,m}$ with $u_{i+1,m}$ by two edges, $\forall i = 1, 2, \ldots, n - 1$ to form the graph $G$. We define labeling function $g : V(G) \rightarrow \{0, 1, \ldots, Q\}$ as follows

$$g(u_{1,j}) = f(v_j), \quad \text{when } f(v_j) \leq \frac{m}{2}$$

$$= f(v_j) + (Q - m), \quad \text{when } f(v_j) > \frac{m}{2}, \forall j = 1, 2, \ldots, m;$$

$$g(u_{2,j}) = g(u_{1,j}) + (Q - m), \quad \text{when } g(u_{1,j}) < \frac{Q}{2}$$

$$= g(u_{1,j}) - (Q - m - 1), \quad \text{when } g(u_{1,j}) > \frac{Q}{2}, \forall j = 1, 2, \ldots, m;$$

$$g(u_{i,j}) = g(u_{i-2,j}) + (m + 2), \quad \text{when } g(u_{i-2,j}) < \frac{Q}{2}$$

$$= g(u_{i-2,j}) - (m + 2), \quad \text{when } g(u_{i-2,j}) > \frac{Q}{2}, \forall j = 1, 2, \ldots, m, \forall i = 3, 4, \ldots, n.$$ 

Above defined labeling pattern give rise edge labels $Q, Q - 1, \ldots, Q - m + 1$ for first copy $C_m^{(1)}$ in $G$, edge labels $Q - m - 2, Q - m - 3, \ldots, Q - 2m + 1$ for second copy $C_m^{(2)}$ in $G$ and $Q - (i - 1)m - 2i + 2, \ldots, Q - im - 2i + 3$ for $i^{th}$ copy $C_m^{(i)}$ in $G$, $\forall i = 3, 4, \ldots, n$. i.e. $g^*$ give rise edge labels in $n^{th}$ copy $C_m^{(n)}$ in $G$. In order to make $G$ as a graceful graph, we require edge labels $Q - m, Q - m - 1, Q - 2m - 2, Q - 2m - 3, \ldots, Q - im - 2i + 2, Q - im - 2i + 1, \ldots, m + 2, m + 1$.

We see that, absolute difference of $g(u_{1,m})$ and $g(u_{2,m})$ is $Q - m$. Similarly absolute difference of $g(u_{1,\frac{m}{2} + 1})$ and $g(u_{2,\frac{m}{2} + 1})$ is $Q - m - 1$, absolute difference of $g(u_{i,m})$ and $g(u_{i+1,m})$ is $Q - im - 2i + 2$, absolute difference of $g(u_{i,\frac{m}{2} + 1})$ and $g(u_{i+1,\frac{m}{2} + 1})$ is $Q - im - 2i + 1, \forall i = 2, 3, \ldots, n - 1$, which fulfill the require edge
labels in the graph $G$. Thus, $G$ is a graceful graph.

Moreover, we see that $1 = g^*((u_{n,1}, u_{n,2}))$, when $n$ is even and $1 = g^*((u_{n,m-1}, u_{n,m}))$, when $n$ is odd. Take $k = g(u_{n,1})$, when $n$ is even and $k = g(u_{n,m})$, when $n$ is odd.

Obviously for any edge $e = (u, v) \in E(G)$, either $g(u) \leq k < g(v)$ or $g(v) \leq k < g(u)$. Therefore, $D(n \cdot C_m)$ is $\alpha$-graceful graph.

4.4.2 Illustration: $D(5 \cdot C_8)$ and its $\alpha$-labeling

![Diagram of $D(5 \cdot C_8)$ and its $\alpha$-labeling](image)

Figure 4.10 $D(5 \cdot C_8)$ and its $\alpha$-labeling [Here $k = 24$]

4.4.3 Theorem: Let $G = K_{r,s}$. Any double path union of $G$ obtained by $u, v \in V(G)$, is $\alpha$-graceful, where $d_G(u, v)$ is 1.

Proof: Let $v_1, v_2, \ldots, v_r, u_1, u_2, \ldots, u_s$ be vertices of $K_{r,s}$. We know that $f : V(K_{r,s}) \rightarrow \{0, 1, \ldots, rs\}$ defined by

$$f(v_i) = i - 1, \quad \forall \ i = 1, 2, \ldots, r$$

$$f(u_j) = (s - j + 1)r, \quad \forall \ j = 1, 2, \ldots, s$$

is $\alpha$-graceful labeling for $K_{r,s}$.

Let $G = D(n \cdot K_{r,s})$ be the double path union of $n$ copies of the complete bipartite graph $K_{r,s}$. Let $w_{l,k}$ ($1 \leq k \leq r + s$) be vertices for $l^{th}$ copy $K_{r,s}^{(l)}$ in $G$, $\forall \ l = 1, 2, \ldots, n$ with $w_{1,i} = v_i$ ($1 \leq i \leq r$), $w_{1,j} = u_{j-r}$ ($r + 1 \leq j \leq r + s$). Obviously $P = |V(G)| = n(r + s)$ and $Q = |E(G)| = n(rs + 2) - 2$. 
Now join vertices \( w_{l,r} \) with \( w_{l+1,r} \) and \( w_{l,r+s} \) with \( w_{l+1,r+s} \) by two edges, \( \forall \ l = 1, 2, \ldots, n - 1 \) to form the graph \( G \).

We define labeling function \( g : V(G) \rightarrow \{0, 1, \ldots, Q\} \) as follows

\[
g(w_{1,k}) = f(w_{1,k}), \quad \text{if} \ f(w_{1,k}) \leq r - 1; \quad \text{[i.e.} \ g(w_{1,k}) = f(v_k), \quad (1 \leq k \leq r);
\]

\[
g(w_{1,k}) = f(w_{1,k}) + (Q - rs), \quad \text{if} \ f(w_{1,k}) \geq r; \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \text{[i.e.} \ g(w_{1,k}) = f(u_{k-r}),(r + 1 \leq k \leq r + s)];
\]

\[
g(w_{2,k}) = g(w_{1,k}) + (Q - rs), \quad \text{if} \ g(w_{1,k}) < \frac{Q}{2}
\]

\[
= g(w_{1,k}) - (Q - rs - 1), \quad \text{if} \ g(w_{1,k}) > \frac{Q}{2}, \quad \forall \ k = 1, 2, \ldots, r + s;
\]

\[
g(w_{1,k}) = g(w_{l-2,k}) + (rs + 2), \quad \text{if} \ g(w_{l-2,k}) < \frac{Q}{2}
\]

\[
= g(w_{l-2,k}) - (rs + 2), \quad \text{if} \ g(w_{l-2,k}) > \frac{Q}{2}, \quad \forall \ k = 1, 2, \ldots, r + s, \forall
\]

\( l = 3, 4, \ldots, n. \)

Above defined labeling pattern give rise edge labels \( Q, Q - 1, \ldots, Q - rs + 1 \) for first copy \( K_{r,s}^{(1)} \) in \( G \), edge labels \( Q - rs - 2, Q - rs - 3, \ldots, Q - 2rs - 1 \) for second copy \( K_{r,s}^{(2)} \) in \( G \) and \( Q - (l - 1)rs - 2l + 2, \ldots, Q - lrs - 2l + 3 \) for \( l \)th copy \( K_{r,s}^{(l)} \) in \( G \), \( \forall \ l = 3, 4, \ldots, n. \)

Remaining edge labels \( Q - rs, Q - rs - 1, Q - 2rs - 2, Q - 2rs - 3, \ldots, Q - lrs - 2l + 2, Q - lrs - 2l + 1, \ldots, rs + 2, rs + 1 \) are produced by added edges as

\[
|g(w_{1,r}) - g(w_{2,r})| = Q - rs, \quad |g(w_{1,r+s}) - g(w_{2,r+s})| = Q - rs - 1, \quad |g(w_{2,r}) - g(w_{3,r})| = Q - 2rs - 2,
\]

\[
|g(w_{2,r+s}) - g(w_{3,r+s})| = Q - 2rs - 3, \quad |g(w_{1,r}) - g(w_{l+1,r})| = Q - lrs - 2l + 2 \quad (3 \leq l \leq n - 2),
\]

\[
|g(w_{l,r+s}) - g(w_{l+1,r+s})| = Q - lrs - 2l + 1 \quad (3 \leq l \leq n - 2), \quad |g(w_{n-1,r}) - g(w_{n,r})| = rs + 2 \quad \text{and} \quad |g(w_{n-1,r+s}) - g(w_{n,r+s})| = rs + 1.
\]

Thus, \( G \) is a graceful graph.

Take \( K = g(w_{n,1}) \), when \( n \) is even and \( K = g(w_{n,r}) \), when \( n \) is odd. Obviously, for any edge \( e = (x, y) \in E(G) \), either \( g(x) \leq K < g(y) \) or \( g(y) \leq K < g(x) \). Therefore, \( D(n \cdot K_{r,s}) \) is \( \alpha \)-graceful graph.
4.4.4 Illustration: \( D(3 \cdot K_{5,4}) \) and its \( \alpha \)-labeling

Illustration - 4.6.2: \( D(3 \cdot K_{5,4}) \) and its \( \alpha \)-labeling shown in figure - 4.11.

![Graph](image)

Edge labels 45 to 64 23 to 42 1 to 20

Figure - 4.11. \( D(3 \cdot K_{5,4}) \) and its \( \alpha \)-labeling. [Here \( k = 26 \)]

4.4.5 Theorem: Let \( G = P_m \), where \( m \) even. Any double path union of path \( G \) is obtained by \( u, v \in V(G) \), is \( \alpha \)-graceful, where \( d_G(u, v) \) is odd. Double Path Union of a Graph \( G = P_m \) and its \( \alpha \)-labeling.

Proof: Let \( v_1, v_2, \ldots, v_m \) (\( m \) is even) be vertices of \( P_m \). We know that \( f : V(P_m) \rightarrow \{0, 1, \ldots, m - 1\} \) defined by

\[
\begin{align*}
    f(v_i) &= m - \left(\frac{i+1}{2}\right), \quad \forall i = 1, 3, \ldots, m - 1 \\
    &= \left(\frac{i-2}{2}\right), \quad \forall i = 2, 4, \ldots, m
\end{align*}
\]

is \( \alpha \)-graceful labeling for \( P_m \).

Let \( G = D(n \cdot P_m) \) be the double path union of \( n \) copies of path \( P_m \) (\( m \) is even). Let \( u_{i,j} \) (\( 1 \leq j \leq m \)) be vertices for \( i^{th} \) copy \( P_m^{(i)} \) in \( G \), \( \forall i = 1, 2, \ldots, n \) with \( u_{1,j} = v_j \) (\( 1 \leq j \leq m \)).

Obviously, \( P = |V(G)| = mn \) and \( Q = |E(G)| = n(m + 1) - 2 \). Now join vertices \( u_{i,1} \) with \( u_{i+1,1} \) and \( u_{i+1,m} \) with \( w_{i+1,m} \) by two edges, \( \forall i = 1, 2, \ldots, n - 1 \) to form the graph \( G \). We define labeling function \( g : V(G) \rightarrow \{0, 1, \ldots, Q\} \) as follows
\[ g(u_{1,j}) = f(v_j), \quad \text{when } f(v_j) < \frac{m}{2}; \]
\[ = f(v_j) + (Q - m + 1), \quad \text{when } f(v_j) \geq \frac{m}{2}, \forall j = 1, 2, \ldots, m; \]
\[ g(u_{2,j}) = g(u_{1,j}) + (Q - m + 1), \quad \text{when } g(u_{1,j}) < \frac{Q}{2} \]
\[ = g(u_{1,j}) - (Q - m), \quad \text{when } g(u_{1,j}) > \frac{Q}{2}, \forall j = 1, 2, \ldots, m; \]
\[ g(u_{i,j}) = g(u_{i-2,j}) + (m + 1), \quad \text{when } g(u_{i-2,j}) < \frac{Q}{2} \]
\[ = g(u_{i-2,j}) - (m + 1), \quad \text{when } g(u_{i-2,j}) > \frac{Q}{2}, \forall j = 1, 2, \ldots, m, \forall i = 3, 4, \ldots, n. \]

Above defined labeling pattern gives rise edge labels \( Q, Q - 1, \ldots, Q - m + 2 \) for first copy \( P_m^{(1)} \) in \( G \), edge labels \( Q - m - 1, Q - m - 2, \ldots, Q - 2m + 1 \) for second copy \( P_m^{(2)} \) in \( G \) and \( Q - (i - 1)(m + 1), \ldots, Q - i(m + 1) + 3 \) for \( i^{th} \) copy \( P_m^{(i)} \) in \( G \), \( \forall i = 3, 4, \ldots, n. \) Remaining edge labels \( Q - m + 1, Q - m, \]
\( Q - 2m, Q - 2m - 1, \ldots, m + 1, m \) are produced by added edges as \( |g(u_{1,1}) - g(u_{2,1})| = Q - m, |g(u_{1,m}) - g(u_{2,m})| = Q - m + 1, |g(u_{2,m}) - g(u_{3,m})| = Q - 2m, |g(u_{2,1}) - g(u_{3,1})| = Q - 2m - 1, \]
\( |g(u_{n-1,m}) - g(u_{n,m})| = m + 1 \) and \( |g(u_{n-1,1}) - g(u_{n,1})| = m. \) Thus, \( G = D(n \cdot P_m) \) is a graceful graph.

Take \( k = g(u_{n,1}) \), when \( n \) is even and \( k = g(u_{n,m}) \), when \( n \) is odd. i.e. \( k = \left\lceil \frac{Q}{2} \right\rceil \). Obviously for any edge \( e = (x, y) \in E(G) \), either \( g(x) \leq K < g(y) \) or \( g(y) \leq K < g(x) \). Therefore, \( D(n \cdot P_m) \) is \( \alpha \)-graceful graph.

### 4.4.6 Illustration : \( D(4 \cdot P_6) \) and its \( \alpha \)-labeling

![Figure 4.12](image-url)\( D(4 \cdot P_6) \) and its \( \alpha \)-labeling [Here \( k = 13 \)]
4.4.7 Corollary: Lader graph is $\alpha$-graceful graph

Proof: Obviously $P_2 \times P_n = D(n \cdot P_2)$ and so, by Theorem 4.5.5, it is $\alpha$-graceful graph.

4.5 Some Open Problems

1. Does any path $P_n$’s are universal graceful graph?
2. Does every grid graph $P_n \times P_m$ are universal graceful graph?

4.6 Conclusive Remarks

This chapters gives brief idea about double path union of graphs with its $\alpha$-labeling and will provide ready reference for any researcher. It is generalized concept of graceful labeling. In this chapter we have discussed graceful labeling, $\alpha$-labeling, smooth graceful graphs, Semi smooth graceful graphs and applied $\alpha$-labeling on some known graphs in detail. The discussion includes definitions and known results for labeling. We proved three new results based on $\alpha$-labeling on some known graphs. The next chapter is focused on odd-even sum graphs and Regular trees.