Chapter 2

Fundamental Concepts and Terminology

2.1 Introduction

This chapter introduces the subject matter which are useful for the research work. Basic definitions are explained in this chapter with sufficient illustrations. Figures are also shown in this chapter.

2.2 Fundamental Concepts

Definition 2.2.1: Graph

A graph \( G = (V, E) \) consists of two finite sets: \( V \), the vertex set of the graph which is a nonempty set of element called vertices, and \( E \), the edge set of the graph which is a possibly empty set of element called edges, such that each edge \( e \) in \( E \) is assigned an unordered pair of vertices \( (u, v) \), called the end vertices of \( e \).
Definition 2.2.2: Order of a Graph

The number of vertices in $G$ is called the order of a graph $G$. It is denoted by $|V(G)|$.

Definition 2.2.3: Size of a Graph

The number of edges in $G$ is called the size of a graph $G$. It is denoted by $|E(G)|$.

Definition 2.2.4: Loop

An edge of a graph that joins a vertex to itself is called a loop. A loop at the vertex $v_i$ is an edge $e = (v_i, v_i)$.

Definition 2.2.5: Multiple edges

If two vertices of a graph are joined by more than one edge then these edge are called multiple edges.

Definition 2.2.6: Simple Graph

A graph which has neither loops nor multiple edges is called a simple graph.

Definition 2.2.7: Adjacent Vertices

If two vertices of a graph are joined by an edge then these vertices are called adjacent vertices.

Definition 2.2.8: Degree of Vertices

The number of edges incident on vertex $v$ of any graph $G$ is called degree of $v$. It is denoted by $d_G(v)$ or $d(v)$. 
Definition 2.2.9: Incident edges

Two edges that have an end vertex in common are called *incident edges*.

Definition 2.2.10: Endpoint/Pendent Vertex

A vertex of a graph of degree 1 is called *endpoint* or *pendent vertex*. An edge of the graph $G$ which is incident with a pendent vertex is called a *pendent edge*.

Definition 2.2.11: Connected and Disconnected Graph

A graph is said to be *connected* if there is a path between every pair of vertices of $G$. A graph which is not connected is called a *disconnected graph*.

Definition 2.2.12: Walk

A *walk* is defined as a finite alternating sequence of vertices and edges of the form $v_0e_1v_1e_2\ldots e_nv_n$ which start and end a vertex and each edge in the sequence is incident on the vertex immediately preceding and succeeding it in the sequence.

Definition 2.2.13: Path

A walk in which no vertex is repeated is called a *path*. A path with $n$ vertices is denoted by $P_n$.

Definition 2.2.14: Trail

A walk in which no edge is repeated is called a *trail*.

Definition 2.2.15: Cycle

A closed path in which no vertex is repeated except the terminal vertex is called a *cycle*. A cycle with $n$ vertices is denoted by $C_n$. The number of edges in a walk is called the *length of the walk*.
Definition 2.2.16: Unicyclic Graph

A graph $G$ with exactly one cycle is called a \textit{unicyclic graph}.

Beginning and ending vertices are equal then it is called a \textit{closed walk}.

Illustration 2.2.17 : Let us consider the following graph $G$.

In graph $G$ shown in figure 2.1

- Order of graph $G$ is 8 and size of graph $G$ is 8.
- $e_6$ forms a loop at $v_5$, $e_3$ and $e_4$ are multiple edges.
- $v_1$ and $v_2$ are adjacent vertices by $e_1$ edge.
- $d(v_5) = 3$, $d(v_3) = 2$, $d(v_6) = 1$.
- $e_1$ and $e_2$ are incident edges at $v_2$ vertex.
- $v_1$, $v_6$ and $v_8$ are endpoints.
- $G$ is disconnected graph.
Illustration 2.2.18: Consider the following graph $G$.

For the figure 2.2 we note the following:

- $W = v_2e_2v_3e_3v_4e_4v_5e_5v_6e_6v_2$ is a closed walk.
- $P_4 = v_2e_2v_3e_3v_4e_4v_5$ is a path of length 3.
- $C_5 = v_2e_2v_3e_3v_4e_4v_5e_8v_1e_1v_2$ is a cycle of length 5.

Definition 2.2.19: Euler graph

Let $G = (V, E)$ be a graph. A closed trail in $G$ is called an Euler line if it contains all the edges of the graph $G$. A graph $G$ is called an Euler graph if it admits an Euler line.

Definition 2.2.20: Tree

A graph $G$ is called a acyclic graph if it contains no cycle. A graph $T$ is called a tree if it contains a acyclic graph.

Definition 2.2.21: Caterpillar

A caterpillar is a tree with the property that the removal of its pendant vertices leaves a path. This path is known as spine of the caterpillar.
2.3 Some Classes of Graph

Definition 2.3.1: Complete graph

A complete graph is a simple graph such that every pair of vertices is joined by an edge. A complete graph on \( n \) vertices is denoted by \( K_n \).

Definition 2.3.2: Bipartite graph

A graph \( G \) is said to be bipartite if the vertices can be partitioned into two disjoint subsets \( V_1 \) and \( V_2 \) such that for every edge \( e_i = (v_i, v_j) \in E(G) \), \( v_i \in V_1 \) and \( v_j \in V_2 \).

Definition 2.3.3: Complete bipartite graph

A simple bipartite graph, whose two vertices are adjacent if and only if they are in different partite sets is called a complete bipartite graph. If partite sets \( V_1 \) and \( V_2 \) are having \( m \) and \( n \) vertices respectively then the related complete bipartite graph is denoted by \( K_{m,n} \) and \( V_1 \) is called \( m \)-vertices part and \( V_2 \) is called \( n \)-vertices part of \( K_{m,n} \).

Definition 2.3.4: Regular graph

A regular graph is a graph if degree of each vertices are same.

Definition 2.3.5: \( k \)-regular graph

A regular graph with vertices of degree \( k \) is called a \( k \)-regular graph.

Definition 2.3.6: Star graph

A complete bipartite graph \( K_{1,n} \) is known as star graph.
Definition 2.3.7: Banana tree

A banana tree is a tree which is obtained from a family of stars by joining one end vertex of each star to a new vertex.

Definition 2.3.8: Super subdivision

Let $G = (V, E)$ be a graph. A graph $H$ is called a super subdivision of $G$ if $H$ is obtained from $G$ by replacing every edge $e_i$ of $G$ by a complete bipartite graph $K_{2,m_i}$ for some $m_i$, $1 \leq i \leq q$ in such a way that the ends of each $e_i$ are merged with the two vertices part of $K_{2,m_i}$ after removing the edge $e_i$ from the graph $G$.

Definition 2.3.9: Grid graph

The cartesian product of two paths $P_n, P_m$ is known as grid graph, which is denoted by $P_n \times P_m$. We redefine this in 4.2.2 in detail.

Definition 2.3.10: Petersen graph

The Petersen graph is an undirected graph with 10 vertices and 15 edges, as shown in figure 2.3

![Petersen graph](figure-2.3)

Definition 2.3.11: Bistar

Bistar $B_{n,n}$ is the graph obtained by joining the apex vertices of two copies of star $K_{1,n}$ by an edge.
Definition 2.3.12: The square of bistar

$B_{n,n}^2$ is the square of $B_{n,n}$ obtained by joining each pair of two vertices of $B_{n,n}$ at distance 2 by a new edge.

Definition 2.3.13: Path union of graphs with arbitrary length

Let $G_1, G_2, \ldots, G_t$ be connected graphs. Consider $P_{n_1}, P_{n_2}, \ldots, P_{n_{t-1}}$ paths on vertices $n_1, n_2, \ldots, n_{t-1}$ respectively. Then path union of graphs by path of arbitrary length is denoted by $\langle G_1, P_{n_1}, G_2, P_{n_2}, \ldots, G_{t-1}, P_{n_{t-1}}, G_t \rangle$ and such graph obtained by joining two graphs $G_i, G_{i+1}$ by a path $P_n(1 \leq i \leq t-1)$.

If we replace each paths $P_{n_1}, P_{n_2}, \ldots, P_{n_{t-1}}$ by a path $P_n$, i.e. $P_{n_1} = P_n = P_{n_2} = \ldots = P_{n_{t-1}}$, such path union of graphs $G_1, G_2, \ldots, G_t$ we shall denote it by $P_n(G_1, G_2, \ldots, G_t)$.

2.4 Some More Definitions

Definition 2.4.1: One point union

A graph $G$ obtained by replacing each edge of $K_{1,t}$ by a path $P_n$ on $n+1$ vertices is called one point union for $t$ copies of path $P_n$. We denote such graph $G$ by $P^t_n$.

Definition 2.4.2: Super graph

A subgraph of a graph $G$ is a graph $H$ such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. In such a case, $G$ is known as a super graph of $H$.

Definition 2.4.3: Index of cordiality

The index of cordiality for $G$ is $n$ if union of $n$ copies of $G$ is cordial, but union of less than $n$ copies of $G$ do not have cordial labeling.
**Definition 2.4.4: Step grid graph**

Take $P_n, P_n, P_{n-1}, \ldots, P_2$ paths on $n, n, n-1, n-2, \ldots, 3, 2$ vertices and arrange them vertically. A graph obtained by joining horizontal vertices by edges of given successive paths is known as a *step grid graph* of size $n$, where $n \geq 3$. It is denoted by $St_n$. It is obvious that $|V(St_n)| = \frac{1}{2}(n^2 + 3n - 2)$ and $|E(St_n)| = n^2 + n - 2$.

Above definition introduced by Kaneria and Makadia [1].

**Definition 2.4.5: Open star of graph**

A graph obtained by replacing each vertex of $K_{1,n}$ except the apex vertex by the connected graphs $G_1, G_2, \ldots, G_n$ is known as *open star of graphs*. We denote such graph by $S(G_1, G_2, \ldots, G_n)$.

If we replace each vertices of $K_{1,n}$ except the apex vertex by a connected graph $G$. i.e. $G_1 = G, G_2 = G, \ldots, G_n = G$, such open star of graph is denoted by $S(n \cdot G)$.

**Definition 2.4.6: One point union for path of graphs**

A graph $G$ obtained by replacing each vertices of $P_n^t$ except the central vertex by the connected graphs $G_1, G_2, \ldots, G_{tn}$ is known as *one point union for path of graphs*. We shall denote such graph $G$ by $P_{n}^{t}(G_1,G_2,\ldots,G_{tn})$, where $P_{n}^{t}$ is the one point union of $t$ copies of path $P_n$. If we replace each vertices of $P_n^t$ except the central vertex by a connected graph $H$, i.e. $G_1 = G_2 = \ldots = G_{tn} = H$, such one point union of path graphs, we shall denote it by $P_{n}^{t}(tn \cdot H)$.

**Definition 2.4.7: Join sum of graph**

Consider $t$ copies of a graph $G_0$. Then graph $G = \langle G^{(1)}_0; G^{(2)}_0; \ldots; G^{(t)}_0 \rangle$ obtained by joining two copies of the graph $G^{(i)}_0$ and $G^{(i+1)}_0$ by a vertex $1 \leq i \leq t-1$ is called *join sum of graphs*.
Definition 2.4.8: $t$ copies of stars

Consider $t$ copies of stars namely $K_{1,n_1}, K_{1,n_2}, \ldots, K_{1,n_t}$. Then the $G = \langle K_{1,n_1}, K_{1,n_2}, \ldots, K_{1,n_t} \rangle$ is the graph obtained by joining apex vertices of $K_{1,n_i}$ and $K_{1,n_{i+1}}$ to a vertex $u_i$, where $1 \leq i \leq t - 1$.

A subgraph of a graph $G$ is a graph $H$ such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. In such a case, $G$ is known as a super graph of $H$.

Definition 2.4.9: Splitting graph

Let $G = (V, E)$ be a graph. Then the splitting graph of $G$ is denoted by $S'(G)$, which obtained by adding to each $v \in V$, by a new vertex $v'$ such that $v'$ is adjacent to those vertex of $G$, which are adjacent to $v$ in $G$. i.e. If $V = \{v_1, v_2, \ldots, v_n\}$, then take $V' = \{v'_1, v'_2, \ldots, v'_n\}$ and $S'(G) = (V \cup V', E \cup \{(u, v'), (u', v)/(u, v) \in E\})$.

Observe that $|V(S'(G))| = 2|V(G)|$ and $|E(S'(G))| = 3|E(G)|$.

2.5 Conclusive Remarks

This chapter provides a brief account of basic concepts, definitions and notations which are prerequisites for the advancement of the remaining chapters. Other standard terminology and notations we refer to Harary [2], West [3], Gross and Yellen [4], Clark and Holton [5].