Chapter 2
Nonlinear dynamics of intense EM pulses in plasma

2.1 Introduction

The study of evolution of high intensity lasers as they propagate through the underdense plasmas is an active area of research due to its importance in potential applications such as plasma based accelerators (Sarkisov et al. (1999)), inertial confinement fusion (Regan et al. (1999); Tabak et al. (1994)), and new radiation sources (Benware et al. (1995); Fedotov et al. (2000); Foldes et al. (1999); Suckewer & Skinner (1990, 1995); Yu et al. (1998)). For these applications, a long propagation distance of intense lasers in the medium is desirable. As several effects emerge in such transit of high power laser beam through plasma medium, many instabilities, and nonlinear phenomenon, such as self-phase modulation (SPM), the filamentation instability, group velocity dispersion (GVD), finite pulse effects, relativistic self-focusing effects, plasma waves etc., become important. Therefore, it is important to study analytically, and numerically some of these effects.

Among these, the self-focusing is genuinely nonlinear basic phenomenon which plays important role in the propagation. Self-focusing is due to increase of the on-axis index of refraction relative to edge of the laser beam resulting from averaged quiver motion of the electrons with their subsequent expulsion from the region of high intensity ponderomotive self-focusing. Also the effect of quiver motion is to
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reduce the local plasma frequency which leads to relativistic self-focusing of the laser beam. Experimental observations of relativistic focusing, and ponderomotive channeling have been reported in various context (Borisov et al. (1992, 1998); Chen et al. (1998); Krushelnick et al. (1997); Monot et al. (1995); Wagner et al. (1997)). Most general form of the field envelope in the absence of dispersion, and under slowly varying envelope approximation is:

\[2ik \frac{\partial \psi}{\partial z} + \nabla^2 \psi + Q(|\psi|^2)\psi = 0\]  

(2.1)

This governing equation of field envelope determines the beam evolution, is nonlinear, so conventional separate techniques using Fourier expansions are not appropriate. These higher dimensional nonlinear Schrödinger equation (NLSE) are not integrable so that they do not have special solution such as soliton solutions (spatial or temporal). However, they possess stationary solution which are unstable on propagation. For deeper insight into the physical process, author solves Eq. (2.1) by using some approximate models. Thus, the resulting expressions provide qualitative rather than quantitative estimates.

There are several approximate analytical approaches to analyse the effect of self-focusing namely a systematic perturbation theory (Fibich & Gaeta (2000); Fibich & Papanicolaou (1999)), ray equation approximation (Marburger (1975)), an approach based on Fermat’s principle (Boyd (2003)), variational approach (Firth (1977)), a paraxial ray approximation (PRA)(Akhmanov et al. (1968, 1996); Sodha et al. (1974, 1976)), moment theory approach (Firth (1977); Lam et al. (1975, 1977)), and source-dependent expansion method (Sprangle et al. (2000)). Most commonly used theory being PRA, suffers from a drawback of being local. i.e. it overemphasizes the field closest to beam axis, and lacks global pulse dynamics. Further, it predicts unphysical phase relationship (Karlsson et al. (1991)). With some partial remedies to PRA, moment theory of self-focusing gives results closer to computer simulations. This theory was not actively pursued as it lacks generalization, and phase description. Another global approach which is invogue these days is the variational method, which find diverse applications in a number of fields for the study of nonlinear propagation. Lagrangian for the problem \(L(\psi(r, z))\), chosen to make the corresponding variational function stationary.
2.2 Basic formulation

\[ \delta \int L dr dz = 0 \]  
(2.2)

where \( \psi \) solves the evolution equation. Since ponderomotive and relativistic channeling occur together, author investigates their combined effects on the evolution of intense laser beam in plasma. In this investigation, a model is set up in a weakly relativistic limit starting from Maxwell’s equations, and hydrodynamic equations. These equations under approximate conditions leads to an evolution equation. Lagrangian for the problem is set up, and variational approach is used.

2.2 Basic formulation

The present model is set up in a weakly relativistic limit starting from Maxwell’s equations, and hydrodynamic equations. Two coupled equations for density perturbation, and laser beam vector potential in a preformed plasma channel are given as follows:

\[
\left( \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \psi = \frac{\omega_p^2}{c^2} \left( 1 + \frac{r^2}{r_{ch}^2} + \frac{\delta n}{n_0} - \frac{|\psi|^2}{2} \right) \psi
\]  
(2.3)

\[
\left( \frac{\partial^2}{\partial t^2} + \omega_p^2 \right) \frac{\delta n}{n_0} = c^2 \nabla^2 \frac{|\psi|^2}{2}
\]  
(2.4)

where \( \psi \) is the vector potential normalized by \( \frac{m_0 c^2}{e} \), \( m_0 \) and \( e \) are the rest mass and charge of electron, \( c \) is the velocity of light in vacuum. \( n_0 \) is the initial axial electron density and \( \delta n \) is the perturbed electron density respectively. \( r_{ch} \) is the effective channel radius. Assuming parabolic density profile and long pulse limit approximation and in slowly varying envelope approximation, author obtains the following evolution equation governing the electric field envelope in collisionless plasmas as follows:

\[
\left( 2 \frac{2k}{r_{ch}} \frac{\partial}{\partial z} + \nabla^2_\perp - k_p^2 \frac{r_{ch}^2}{r_{ch}^2} - \nabla^2_\perp \frac{|\psi|^2}{2} + k_p^2 \frac{|\psi|^2}{2} \right) \psi(r, z) = 0
\]  
(2.5)

Eq. (2.5) which is a special case of Eq. (2.1), is a nonlinear parabolic partial differential equation in which second term has its origin in diffractional divergence, third and fourth corresponds to channel and ponderomotive self-focusing (PSF).
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while the last one has its origin to relativistic self-focusing (RSF) respectively. This can be solved by a number of approximate techniques. It may be mentioned that another technique in vogue these days, is source-dependent technique. This has frequently been used in free electron laser theory, where nonlinear term is expanded in terms of associated Laguerre polynomials. However, it amounts to solve nonlinear problems in terms of linear theory as nonlinear source term is expanded by superposition principle. *i.e.* in terms of Laguerre polynomials lacks/needs justification. Variational approach which have rigorous basis, as applied in other fields, is used here to nonlinear wave propagation. We can reformulate Eq. (2.5) into a variational problem corresponding to a Lagrangian $L$, so as to make $\frac{\delta L}{\delta z} = 0$, is equivalent to Eq. (2.5). Thus Lagrangian $L$, is given by:

$$L = i k \left( \psi \frac{\partial \psi}{\partial z} - \psi^* \frac{\partial \psi^*}{\partial z} \right) - \left| \frac{\partial \psi}{\partial x} \right|^2 - \left| \frac{\partial \psi}{\partial y} \right|^2 + \frac{1}{4} \left( \left( \frac{\partial |\psi|^2}{\partial x} \right)^2 + \left( \frac{\partial |\psi|^2}{\partial y} \right)^2 \right)$$

$$- \frac{x^2}{r_{ch}^2} k_p^2 \left| \psi \right|^2 - \frac{y^2}{r_{ch}^2} k_p^2 \left| \psi \right|^2 + k_p^2 \left| \psi \right|^4$$

(2.6)

Thus, the solution to the variational problem

$$\delta \int \int \int L dxdydz = 0$$

(2.7)

also solves the nonlinear Schrödinger Eq. (2.5). Using the trial function as elliptic Gaussian beam of the form as follows:

$$\psi(r, z) = \psi_0(z) \exp \left[ -\frac{x^2}{2X^2(z)} - \frac{y^2}{2Y^2(z)} + i \left( x^2b_x(z) + y^2b_y(z) + \phi(z) \right) \right]$$

(2.8)

where $X, Y$ are the beam width parameters in the $x$ and $y$ direction respectively. It may be further noted that $X$ and $Y$ are real functions of $z$. As most of the laser systems usually generate a beam which is more nearly elliptical than circular in cross-section. This makes our choice of trial function of above form as more realistic one. Using the ansatz, with expression for $\psi$ as trial function and substituted in $L$, we can perform the integration to write:

$$\langle L \rangle = i k \pi XY \left( \psi_0 \frac{\partial \psi_0}{\partial z} - \psi_0^* \frac{\partial \psi_0^*}{\partial z} \right) - \left| \psi_0 \right|^2 \frac{X^3 Y}{4} \left[ \frac{2k \frac{db_x}{dz}}{dz} + 4b_x^2 + \frac{1}{X^4} \right] -$$

$$\left| \psi_0 \right|^2 \frac{Y^3}{4} \left[ \frac{2k \frac{db_y}{dz}}{dz} + 4b_y^2 + \frac{1}{Y^4} \right] - 2k \pi XY \left| \psi_0 \right|^2 \frac{d\phi}{dz} -$$

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for an initially parallel beam, initial conditions are:

\[
\frac{dX_0}{dz} = \frac{dY_0}{dz} = 0
\]
2. NONLINEAR DYNAMICS OF INTENSE EM PULSES IN PLASMA

Figure 2.1: Variation of normalized beam width parameters $X_n$, $Y_n$ and $\rho_n^2$ with dimensionless distance of propagation $\eta$ for the following set of parameters: $X_0 = 0.002 cm$, $Y_0 = 0.0016 cm$, $k = 0.53 \times 10^4 cm^{-1}$. Channel term is assumed to be zero.

2.3 Discussion

Eqs. (2.13) and (2.14) describes the beam dynamics in plasma with relativistic self-focusing, ponderomotive self-channeling and channel focusing. The Eqs. (2.13) and (2.14) are nonlinearly coupled equations which govern the beam width parameters $X$ and $Y$ in the $x$ and $y$ directions. Further Eq. (2.15) describes the longitudinal phase. It is observed that R.H.S. of these equations contain various terms, each representing some physical mechanism responsible for focusing/defocusing of laser beam. e.g. the first term on R.H.S of Eq. (2.13) is responsible for diffractional divergence of the laser beam. It has its origin in the Laplacian appearing in the evolution Eq. (2.5). The other terms represent the combined effects of relativistic self-focusing, PSF and self-channeling. In the absence of these terms, beam diverges due to diffraction.

However, as mentioned in introduction, long distance of several Rayleigh lengths are required for novel applications of lasers. Earlier research work e.g. relativistic and ponderomotive self-focusing using different approaches have been reported in various context. However, results of variational approach are more appropriate and we here study the effects of various physical mechanisms in the evolution of beam. We have chosen following set of parameters for numerical computation:
2.3 Discussion

Figure 2.2: Variation of normalized beam width parameter $X_n$ with $\eta$ for the same set of parameters as in the caption of Figure 2.1 except channel radius, $r_{ch} = 0.0017\, cm$.

Figure 2.3: Variation of normalized beam width parameter $Y_n$ with dimensionless distance of propagation $\eta$ for the same set of parameters as in the caption of Figure 2.1 except $r_{ch} = 0.0017\, cm$.

\begin{align*}
X_0 &= 0.002\, cm, \quad Y_0 = 0.0016\, cm, \quad k = 0.53 \times 10^4\, cm^{-1}, \quad |\psi_0|^2 = 0.1, \quad k_p = \frac{\omega_p}{c}, \\
r_{ch} &\sim \frac{c}{\omega_p}.
\end{align*}

In the absence of plasma channel and for the above mentioned parameters, the diffraction is the dominant mechanism leading to monotonically increase in normalized beam width parameters $X_n$ and $Y_n$. This is in contrast to earlier observations where nonlinearity due to ponderomotive and relativistic mechanism are sufficient to result in self-focusing phenomenon. The results are investigated and succinctly shown in Figure 2.1 where $X_n$, $Y_n$ and $\rho_n^2$ have been plotted for various values of intensity parameter, $\Pi(=|\psi_0|^2)$.

However, situation changes drastically as finite channel is introduced. The
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\[ \{X_n, Y_n, \rho_n^2\} \]

\[ \eta \]

\[ \Pi \]

\[ 0.1 \]

\[ 0.13 \]

\[ 0.19 \]

\[ \Phi \]

\[ \phi \]

\[ 0.2 \]

\[ 0.1 \]

\[ 0.0 \]

\[ -0.1 \]

\[ -0.2 \]

\[ -0.3 \]

\[ -0.4 \]

\[ 5 \]

\[ 10 \]

\[ 15 \]

\[ 20 \]

Figure 2.4: Variation of beam width parameter \( X_n, Y_n \) and \( \rho_n^2 \) with \( \eta \) for the same set of parameters as in the caption of Figure 2.1 with \( r_{ch} = 0.0017cm \).

Figure 2.5: Plot of longitudinal phase \( \phi \) versus \( \eta \) for different values of \( \Pi \) and the other parameters are same as in the caption of Figure 2.1.

Nonlinear ordinary differential equations start the occurrence of individual self-focusing of \( X_n \) and \( Y_n \) as well as phenomenon of cross-focusing is observed. In this process, focusing of \( X_n \) leads to defocusing of \( Y_n \) and vice-versa. Figure 2.2 and Figure 2.3 exhibit the oscillatory self-focusing of \( X_n \) and \( Y_n \) with the distance of propagation.

Figure 2.4 exhibits the graphs of \( X_n, Y_n \), and \( \rho_n^2 \) for all other parameters mentioned above along with channel. Further, the phenomenon of waveguide is not observed as \( X_n \) and \( Y_n \) keeps on oscillatory self-focusing. From variational approach, it is observed that propagation extends to several Rayleigh lengths. This confirms the experimental observations of (Fauser & Langhoff (2000)).
2.4 Stability Criterion

In Figure 2.5, author has shown longitudinal phase, which may be positive or negative with distance of propagation. However, regularised phase [not shown] is always negative with the distance of propagation.

2.4 Stability Criterion

To study the stability properties of the system (Lakshman & Rajasekar (2003)), the following Jacobi determinant is constructed from derivatives with respect to amplitude, width and curvature in terms of $S$, $F$, and $G$ where

\[
S = \frac{d\psi_0}{dz} = \frac{2b_x |\psi_0|}{k} \tag{2.20}
\]

\[
F = \frac{dX}{dz} = \frac{4Xb_x}{k} \tag{2.21}
\]

\[
G = \frac{db_x}{dz} = -\frac{2b_x^2}{k} + \frac{1}{2kX^4} + \frac{3A_0^2X_0Y_0}{4kX^5Y} + \frac{A_0^2X_0Y_0}{16kX^3Y^3} - \frac{k^2A_0^2X_0Y_0}{8kX^3Y} - \frac{k^2}{2kY^2_{ch}} \tag{2.22}
\]

\[
\text{det} \left| J - \lambda I \right| = \left| \begin{array}{ccc} \frac{\partial S}{\partial \psi_0} - \lambda & \frac{\partial S}{\partial X} & \frac{\partial S}{\partial b_x} \\ \frac{\partial F}{\partial \psi_0} & \frac{\partial F}{\partial X} - \lambda & \frac{\partial F}{\partial b_x} \\ \frac{\partial G}{\partial \psi_0} & \frac{\partial G}{\partial X} & \frac{\partial G}{\partial b_x} - \lambda \end{array} \right| = 0 \tag{2.23}
\]

This leads to the following characteristic equation cubic in $\lambda$:

\[
\lambda^3 + \alpha_1 \lambda^2 + \alpha_2 \lambda + \alpha_3 = 0 \tag{2.24}
\]

where,

\[
\alpha_1 = -\frac{2b_x}{k} \tag{2.25}
\]

\[
\alpha_2 = \frac{8}{kX^4} + 15 \frac{|\psi_0|^2}{k^2X^4} + \frac{3}{4} \frac{|\psi_0|^2}{k^2X^2Y^2} - 16 \frac{b_x^2}{k^2} \tag{2.26}
\]

\[
\alpha_3 = \frac{32b_x^3}{k^3} - \frac{16b_x}{k^2X^4} - 30 \frac{|\psi_0|^2 b_x}{k^3X^4} - \frac{3}{2} \frac{|\psi_0|^2 b_x}{k^3X^2Y^2} + \frac{3k_{ch}^2b_x}{kX^2} \frac{|\psi_0|^2}{k^3X^2} \tag{2.27}
\]

In order to have Lyapunov’s stability, Hurwitz conditions must be fulfilled. i.e. $
\alpha_1\alpha_2 - \alpha_3$ must be positive. According to the Routh-Hurwitz criterion, a necessary and sufficient condition for the stationary solutions to be stable is:

\[
\alpha_1\alpha_2 - \alpha_3 > 0 \tag{2.28}
\]
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Equation (2.24) has a pair of purely imaginary roots at a critical point:

\[ \lambda = \pm \iota v, \; v > 0 \quad (2.29) \]

We may substitute (2.29) into (2.24) and we get:

\[ v^2 - \alpha_2 = 0 \quad (2.30) \]

and

\[ \alpha_1 v^2 - \alpha_3 = 0 \quad (2.31) \]

The critical condition of the Hopf-bifurcation is:

\[ f = \alpha_1 \alpha_2 - \alpha_3 = 0 \quad (2.32) \]

\( f > 0 \) is a necessary condition for the stationary solution to be stable, \( f < 0 \) is a necessary condition for the Hopf-bifurcation to emerge. It is observed that the condition \( f > 0 \) is satisfied for chosen set of parameters and therefore Hopf-bifurcation, resulting from the unstable fixed point, does not come into play, leading to overall stability of the beam dynamics.