Chapter 2

Literature Survey

Contents

2.1 Overview .............................................. 24
2.2 Linear Time Invariant Systems .............. 25
2.3 Taylor Series ................................. 29
2.4 Discrete Volterra Series ....................... 30
2.5 Two Dimensional Quadratic Systems ... 32
2.6 Isotropy of Quadratic Kernel ............... 33
  2.6.1 Isotropy of 2-D Quadratic Systems ... 34
2.7 Gist of Observations .......................... 35
2.8 Motivation for Research ...................... 36

2.1 Overview

Discrete signals and systems play vital roles in today’s engineering domains such as instrumentation, communication, control systems, medical imaging, radar systems etc. and maintain a steady growth
in cost effective usage and applications. However, most of the theory and research work are focused around linear and time invariant (LTI) systems, owing to the ease of design and simplicity of implementation. But when signals generated by or systems governed by mild polynomial nonlinearities are encountered, one need to consider the usage of polynomial systems which are modeled as extension of the conventional LTI systems. Obviously, one has to resort to mathematical models that link the input and output in the form of a power series that can add quadratic, cubic and higher order components in parallel with the linear term. Although Taylor series is used for such a model, it suffers from lack of memory and huge computational complexity. Sec. 2.2 outlines the LTI systems and their limitations in modeling nonlinearities and Sec. 2.3 discusses Taylor series and its shortcomings. A variant of Taylor series which possesses memory is Volterra series, which is employed to model polynomial relationship between system input and output, the details of which are presented in Sec. 2.4. Most of the nonlinear effects are encompassed by the quadratic term and hence quadratic systems, both one dimensional and two dimensional, are presented in Sec. 2.4, 2.5 and 2.6. The design methodologies for quadratic systems are presented in Sec. 4.2. The computational complexity arising from the use of Volterra series is surmounted by various strategies for realization as presented in detail in Sec. 4.3.

2.2 Linear Time Invariant Systems

Linear time invariant systems are those that obey superposition and shift invariance properties. The output $y[n]$ of a linear and time invariant (LTI) system [Oppenheim and Schafer 1998] driven by input $x[n]$ is given as

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

(2.1)
Eq. 2.1 is denoted as $y[n] = x[n] * h[n]$ where the $*$ sign indicates the linear convolution operation. The parameter $h[n]$ is the kernel or the impulse response of the LTI system that characterizes the system completely. The essential problem in the design of a discrete filter as in Eq. 2.1 is in ascertaining the kernel $h[n]$ for a given input $x[n]$ and for a desired output $y[n]$. This problem is relatively mild when the duration of $h[n]$ is infinite. In this case, discretization of conventional analog filters such as Chebyshev, Bessel filters etc. are possible, resulting in frequency selective infinite impulse response (IIR) filters. When the duration of the kernel $h[n]$ is finite, the design is more involved. Methods based on statistical correlations are proposed\[Hopf 1934\], \[Wiener 1956\], \[Noble 1988\]. Optimization strategies are also proposed for FIR filter design \[McClellan and Parks 2005\], \[Adams 1991\], \[Adams et al. 1993\], \[Kaiser 1983\], \[Hermann 1970\], \[Gold 1975\]. Frequency sampling methods for finding $h[n]$ are available in literature \[Rabiner and Parks 1967\], \[Rader and Gold 1975\].

When it comes to two dimensional signal processing, much of the work is focused around natural images. A natural image is a two dimensional signal rendered as an $M \times N$ array of real numbers by a digital camera \[Ekstrom 1984\], \[Gonzalez and Woods 1992\]. The generation of the image is a nonlinear process and so is the human visual system that perceives the image. Processing the image involves manipulation of the two dimensional array of real numbers from the camera, say $x[n_1, n_2]$, to yield a more useful array of numbers, say $y[n_1, n_2]$, by a two dimensional filter kernel $h[n_1, n_2]$ as given in Eq. 2.2.

$$y[n_1, n_2] = \sum_{k_1=0}^{N-1} \sum_{k_2=0}^{M-1} h[k_1, k_2] x[n_1 - k_1, n_2 - k_2]$$  \hspace{1cm} (2.2)

Eq. 2.2 denoted as $y[n_1, n_2] = x[n_1, n_2] * *h[n_1, n_2]$, is a two dimensional linear convolution of the input with the linear filter kernel. This
The tasks in image processing are segmentation, edge detection, classification, noise removal etc. Extraction of features is a key issue, especially in medical image processing [Jan 2003], [Dougherty 2011]. As the image generation as well as the image perception is nonlinear, it is logical to employ nonlinear filters for processing images. The major nonlinear systems, categorized as in Fig. 2.1, are

- Homomorphic filters
- Order statistic filters
- Morphological filters
- Fuzzy filters

Fig. 2.1. Broad classification of nonlinear systems
• Polynomial filters

Homomorphic Filters are useful when the input signal is a multiplicative combination of two different signals. The two components are processed in different ways using a nonlinear transformation, followed by linear filtering. The filter is popularly used in isolating multiplicative noise and in improving the contrast with normalized brightness.

Order Statistic Filters are used for removing impulsive noise from the input image without blurring its edges. Median, weighted median and mean filters are popular members in this category.

Morphological Filters employ geometric or morphological transformations on input image to extract the skeletal sketches of objects in the image. These morphological operations rely only on the relative ordering of pixel values and not on their numerical values, and therefore are especially suited to the processing of binary images.

Fuzzy Filters apply fuzzy reasoning to model the uncertainty in the image and are employed in filtering, interpolation and morphology.

While all the above methods are used in various image processing applications, polynomial systems have the distinctive advantage that they can model a larger class of nonlinear systems with a smaller number of coefficients. Besides they are constructed as extensions of a linear system by suitable power series expansion. The popular Taylor series is deselected based on reasons that will be elaborated in Sec. 2.3. In its stead, Volterra power series that can add quadratic, cubic and higher order systems in parallel with the linear system is selected. The work focuses on one dimensional and two dimensional quadratic systems based on Volterra series for practical applications.
The details of Volterra systems and the work done for realization for various applications are detailed from chapters 2 to 8 as outlined in the next section. Now that the idea of systems following a polynomial relationship between input and output is conceived, it is imperative to look for a general mathematical framework for polynomial systems. It is intuitive to consider Taylor series [Struik 1969] for this purpose.

2.3 Taylor Series

The power series is proposed by Brook Taylor in 1715 for expressing the function $y$ as

$$y = f(x) = \sum_{i=0}^{\infty} \frac{f^{(n)}(a)}{i!} (x - a)^i$$

where $f^{(n)}(a)$ denotes the $n^{th}$ derivative of $f$ at the point $a$. When $a = 0$, the series is called Maclaurin series. Even though Taylor series provides a polynomial relationship between $x$ and $y$, it cannot easily be employed for modeling polynomial systems due to the following shortcomings.

- Taylor series, being a memoryless series, [Sicuranza 1992] only the present value of the input $x$ but not its history is incorporated in the model.

- The convergence being slow, more coefficients and powers are needed for the approximation of $y$, resulting in huge complexity.

- Practical implementations of polynomial systems are much more convenient if square, cubic and higher powers are added with Eq. 2.1 as this enables the designer to add quadratic, cubic or higher order systems in parallel with the existing LTI system. Such a representation is not possible with Taylor series.
The above points warrant another series representation for polynomial systems that can augment Eq. 2.1 with quadratic, cubic and higher order systems. Such a power series was proposed by Vito Volterra, the discrete version of which is discussed in the next section.

### 2.4 Discrete Volterra Series

Vito Volterra proposed a power series for describing the input-output relation of an $N$th order nonlinear system, the discrete version of which is given as

$$y[n] = h_0 + \sum_{r=1}^{\infty} \sum_{n_1=1}^{N} \sum_{n_2=1}^{N} \ldots \sum_{n_r=1}^{N} h_r[n_1, n_2, \ldots n_r] \cdot x[n - n_1]x[n - n_2] \ldots x[n - n_r]. \quad (2.4)$$

The term $h_0$ denotes the output offset when no input is present and the term $h_r[n_1, n_2, \ldots n_r]$ denotes the $r$th order Volterra kernel. Identification of this kernel for a nonlinear system is the chief issue in designing polynomial systems. By employing Volterra series, one can surmount many disadvantages with Taylor series for modeling polynomial systems. With order $r = 1$, Eq. 2.4 defaults to a convolution between input and output, representing a linear system. For $r = 2$, it becomes a quadratic system given as

$$y[n] = h_0 + \sum_{m_1=1}^{N} h_1[m_1]x[n - m_1] + \sum_{n_1=1}^{N} \sum_{n_2=1}^{N} h_2[n_1, n_2]x[n - n_1]x[n - n_2] \quad (2.5)$$

It has been observed that majority of the effects due to polynomial nonlinearities are confined to the quadratic term and so the interest
2.4. DISCRETE VOLTERRA SERIES

in quadratic systems. Eq. 2.5 is expressed as a matrix equation

\[ y[n] = h_0 + X_1^T[n]H_1 + X_1^T[n]H_2X_1[n] \]  

(2.6)

where

\[ X_1 = [x[n], x[n-1], \ldots, x[n-N+1]]^T \]  

(2.7)

and

\[ H_1 = [h_1[0], h_1[1], \ldots, h_1[N-1]]^T \]  

(2.8)

The term \( H_2 \) is expressed as

\[
H_2 = \begin{bmatrix}
h_2[0,0] & h_2[0,1] & \cdots & h_2[0,N-1] \\
h_2[1,0] & h_2[1,1] & \cdots & h_2[1,N-1] \\
h_2[2,0] & h_2[2,1] & \cdots & h_2[2,N-1] \\
\vdots & \vdots & \ddots & \vdots \\
h_2[N-1,0] & h_2[N-1,1] & \cdots & h_2[N-1,N-1]
\end{bmatrix}
\]  

(2.9)

This system is realized as a parallel combination of three components as in Fig. 2.2. The quadratic system is defined by the third term in Eq. 2.6 is

\[ y_q[n] = X_1^T[n]H_2X_1[n] \]  

(2.10)

**Fig. 2.2.** Quadratic filter as a parallel combination of 3 components

Eq. 2.10 is understood as the output of linear filter of kernel \( H_2 \) operating on

\[ X_2[n] = X_1[n] \otimes X_1[n] \]  

(2.11)
where $\otimes$ represents Kronecker product. Quadratic nonlinearities are predominant in image generation and processing mechanisms. Employing two dimensional quadratic filters can lead to improved performance. Sec. 2.5 details the theory of 2-D quadratic filters.

### 2.5 Two Dimensional Quadratic Systems

The two dimensional quadratic filter is governed by the equation

$$y[n_1, n_2] = \sum_{m_{11}=0}^{N_1-1} \sum_{m_{12}=0}^{N_2-1} \sum_{m_{21}=0}^{N_1-1} \sum_{m_{22}=0}^{N_2-1} h_2[m_{11}, m_{12}, m_{21}, m_{22}] \times$$

$$x[n_1 - m_{11}, n_2 - m_{12}] x[n_1 - m_{21}, n_2 - m_{22}] \quad (2.12)$$

Out of the four indices $m_{11}, m_{12}, m_{21}, m_{22}$ of the kernel $h_2$, two stem from the quadratic nature of the kernel and the remaining two denote the two dimensions of the signal processed. Eq. 2.12 is represented in the matrix form as

$$y[n_1, n_2] = X^T[n_1, n_2] H_2 X[n_1, n_2] \quad (2.13)$$

where

$$X[n_1, n_2] = \begin{bmatrix}
  x[n_1, n_2] \\
  x[n_1 - 1, n_2] \\
  \vdots \\
  x[n_1 - N_1 + 1, n_2] \\
  x[n_1, n_2 - N_2 + 1] \\
  x[n_1 - 1, n_2 - N_2 + 1] \\
  \vdots \\
  x[n_1 - N_1 + 1, n_2 - N_2 + 1]
\end{bmatrix} \quad (2.14)$$
The quadratic kernel $H_2$ has $N_1N_2 \times N_1N_2$ elements and each element consists of $N_2^2$ sub-matrices $H(i,j)$ with $N_1 \times N_2$ elements given as

$$H_2 = \begin{bmatrix}
H[0,0] & H[0,1] & \cdots & H[0,N_2-1] \\
H[1,0] & H[1,1] & \cdots & H[1,N_2-1] \\
\vdots & \vdots & \ddots & \vdots \\
H[N_2-1,0] & H[N_2-1,1] & \cdots & H[N_2-1,N_2-1]
\end{bmatrix}$$ (2.15)

where each sub-matrix $H[i,j]$ is given by

$$H(i,j) = \begin{bmatrix}
h[0,i,0,j] & h[0,i,1,j] & \cdots & h[0,i,N_1-1,j] \\
h[1,i,0,j] & h[1,i,1,j] & \cdots & h[1,i,N_1-1,j] \\
\vdots & \vdots & \ddots & \vdots \\
h[N_1-1,i,0,j] & h[N_1-1,i,1,j] & \cdots & h[N_1-1,i,N_1-1,j]
\end{bmatrix}$$ (2.16)

The design of the quadratic filter invariably means the determination of the $N^2$ elements of $H_2$. This task is tedious as $N^2$ is large enough, but not all the coefficients are independent. The number of independent coefficients is considerably reduced if the isotropy of the kernel matrix is considered as in the following section.

### 2.6 Isotropy of Quadratic Kernel

Isotropy of a one dimensional linear system means the invariance of the filter output with respect of 180° rotation of the filter input. The input to a linear system is

$$X_1[n] = [x[n] \ x[n-1] \ \cdots \ x[n-N+1]]^T$$ (2.17)

and its image after 180° rotation is

$$X'_1[n] = [x[n-N+1] \ \cdots \ x[n-1] \ x[n]]^T$$ (2.18)
The output of the isotropic system will be the same for both inputs $X_1[n]$ and $X_2[n]$. A class of linear phase FIR filters with impulse response $h_1[n]$ with

$$h_1[n] = h_1[N - 1 - n]$$

(2.19)
can achieve this condition. For a one dimensional quadratic system, the input vector and its image after 180° rotation are

$$X_2[n] = X_1[n] \otimes X_1[n]$$

(2.20)

$$X'_2[n] = X'_1[n] \otimes X'_1[n]$$

(2.21)
The condition under which the outputs of a quadratic system for the inputs $X_2[n]$ and $X'_2[n]$ are the same is

$$h_2[n_1, n_2] = h_2[N - 1 - n_1, N - 1 - n_2]$$

(2.22)

### 2.6.1 Isotropy of 2-D Quadratic Systems

The input-output relation for a two dimensional quadratic system is

$$y[n_1, n_2] = Tr[H_2X_T^2[n_1, n_2]]$$

(2.23)

where

$$X_2[n_1, n_2] = X_1[n_1, n_2] \oplus X_1[n_1, n_2]$$

(2.24)

Isotropy requires that the output is the same for 90°, 180° and 270° rotations of $X_2[n_1, n_2]$. Isotropy leads to less number of independent coefficients and ease of design [Mathews and Sicuranza 2000].

Two dimensional filters with a small plane of support is ideal for spatial filtering. A filter of $3 \times 3$ mask is considered, with $N = 9$. The number of independent coefficients are reduced from 81 to 45, on application of the symmetry of the $H_2$ kernel. On application of the isotropy constraints, it reduces to 11 [Ramponi 1990].
2.7 Gist of Observations

This chapter presents the limitations of LTI systems and the signal processing needs that necessitate the use of quadratic systems. Volterra series, an extension of Taylor series, is presented as a suitable power series expansion to model quadratic systems. Two dimensional quadratic systems and the properties of Volterra quadratic kernels such as symmetry and isotropy, which will be later used in subsequent designs, are reviewed. The principal distinction when working with quadratic filters is that in the case of 1-D and 2-D linear time invariant systems, the linear convolutions as in Eq. 2.1 and Eq. 2.2 translate into multiplication of Fourier transforms of the input and impulse response functions in the frequency domain i.e. \( Y(\omega) = H(\omega)X(\omega) \). The latter statement implies that the frequencies at the output of LTI systems contain the input frequencies or a subset therein, depending on the shape of \( H(\omega) \) where \( H(\omega) \) is the Fourier transform of \( h[n] \), but never anything outside the input frequency band. Such a convenient situation does not exist when polynomial systems such as \( y[n] = x^2[n] \) is considered. With such a system, if \( x[n] \) is band limited in \( [-\omega_c, \omega_c] \), then the spectrum of \( y[n] \) can spread from \( [-2\omega_c, 2\omega_c] \). Not only that the convenient relationship as proposed by Eq. 2.1 or by its frequency domain counterpart does not exist in the case of quadratic systems. Besides, one has to worry about spectral aliasing not only at the input side but at the output side also. If one takes a closer look at systems whose input and output are related by polynomial equations such as the case just discussed, one can observe that there are equivalences with linear systems in the time domain but there is no exact equivalent of the frequency domain. This imposes a great challenge in the implementation of polynomial systems compared with linear systems that are implemented using frequency domain methods that rely on computationally efficient fast Fourier transform algorithms.
2.8 Motivation for Research

The research in quadratic systems is motivated by the need to model the effects of polynomial components present in speech and image signals. It is estimated that quadratic systems can outperform LTI systems and other nonlinear filters in the tasks of edge detection, impulsive noise removal and statistical prediction. The design methodology and scheme of implementation need to be selected for each application. Based on the review of literature and the requirements in polynomial signal processing in three application areas, the scheme of work and methodology have been established in tune with the objectives. Chapter 3 is devoted for explaining the scheme of work and methodology.