CHAPTER 4

THE CHARACTERIZATION PROPERTIES AND BASIC HYPERGEOMETRIC FUNCTIONS OF 
(P, Q)-ANALOGUE

4.1 Introduction

In this chapter, some new results on (P, Q)-analogue are acquainted by giving definition of (P, Q)-shifted factorials, and some new relations on negative shifted factorial. Some new relations for (P, Q)-shifted factorials and (P, Q)-hypergeometric functions are also proved.

The (P, Q)-analysis or post quantum calculus was discovered in the last decade. Many mathematicians and physicists have widely developed the theory of (P, Q)-numbers, along the traditional lines of classical and quantum calculus. Burban and Klimyk et al. [12] presented the (P, Q)-Derivative, (P, Q)-anti-Derivative and (P, Q) hypergeometric functions related to quantum groups. Sadjang et al. [87–89] produced the fundamental theorem of (P, Q)-calculus, the (P, Q)-Gamma and the (P, Q)-Beta functions. Duran at al. [26,28,29] establishment a new class of Bernoulli, Euler and Genocchi polynomials using (P, Q)-calculus and checked some of their properties. The (P, Q)-anti-Derivative $[n]_{(P,Q)}$ are specified as (see [42,87])

$$[n]_{P,Q} = P^{n-1} + P^{n-2}Q + P^{n-3}Q^2 + \cdots + P \cdot Q^{n-2} + Q^{n-1}$$
The \((P, Q)\)-basic number is a generalization of the \(Q\)-number, that is
\[
\lim_{P\to 1} [n]_{P, Q} = [n]_Q.
\]
The definition for \((P, Q)\)-factorial is explained in [43, 87]
\[
[n]_{P, Q}! = \prod_{m=1}^{n} [m]_{P, Q}!, \quad n \geq 1, \quad [0]_{P, Q}! = 1.
\]
The \((P, Q)\)-analogue of the binomial coefficients are given as
\[
\binom{n}{m}_{P, Q} = \frac{[n]_{P, Q}!}{[m]_{P, Q}![n-m]_{P, Q}!}, \quad 0 \leq m \leq n.
\]
Note that, as \(P \to 1\), the \((P, Q)\)-binomial coefficients is similar to the \(Q\)-binomial coefficients. The \((P, Q)\)-powers are introduced as
\[
((A, B); (P, Q))_0 = 1
\]
and
\[
((A, B); (P, Q))_k = \prod_{j=1}^{k} (AP^{j-1} - BQ^{j-1}), \quad 0 < Q < P \leq 1.
\]
The symbols \(((A, B); (P, Q))_k\) are called \((P, Q)\)-shifted factorials.

The \((P, Q)\)-derivative operator \(D_{P, Q}\) is defined by
\[
D_{P, Q}f(z) := \begin{cases} 
\frac{f(Pz) - f(Qz)}{(P - Q)z}, & \text{when } z \neq 0 \\
 f'(0), & \text{when } z = 0,
\end{cases}
\]
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\[
D^0_{P, Q}f := f \quad \text{and} \quad D^n_{P, Q}f := D_{P, Q}(D^{n-1}_{P, Q}f), \quad n = 1, 2, 3, \ldots,
\]
it is not very difficult to see that
\[
\lim_{P \to 1} D_{P, Q}f(z) = D_Qf(z),
\]
if the function is differentiable at \(z\). Further \(D_{P, Q}\) is a linear operator and satisfies the following property
\[
D_{P, Q}(f(z)g(z)) = f(Pz)D_{P, Q}g(z) + g(Qz)D_{P, Q}f(z) \quad (4.1.7)
\]
or
\[
D_{P, Q}(f(z)g(z)) = g(Pz)D_{P, Q}f(z) + f(Qz)D_{P, Q}g(z), \quad (4.1.8)
\]
which is often referred to as the \((P, Q)\)-product rule. This can be generalized to a \((P, Q)\)-analogue of Leibniz’s rule
\[
D^n_{P, Q}(f(z)g(z)) = \sum_{m=0}^{n} \binom{n}{m}_{P, Q}(D^{n-m}_{P, Q}f)(P^{n-m}Q^m z)(D^m_{P, Q}g)(z), \quad n = 0, 1, 2, \ldots. \quad (4.1.9)
\]
The following definition of \((P, Q)\)-integral due to [87] is
\[
\int_0^A f(t)d_{P, Q}t = (P - Q)A \sum_{m \geq 0} \frac{P^m}{Q^{m+1}} f\left(\frac{P^m}{Q^{m+1}} t\right) \quad (4.1.10)
\]
with
\[
\int_A^B f(t)d_{P, Q}t = \int_0^B f(t)d_{P, Q}t - \int_0^A f(t)d_{P, Q}t
\]
For detailed studies on \((P, Q)\)-calculus, one can look at [87–89] and references therein.

### 4.2 The \((P, Q)\)-Shifted Factorial

The symbols \(((A, B); (P, Q))_k\) are called \((P, Q)\)-shifted factorials, for negative subscripts, we define
\[
((A, B); (P, Q))_{-k} = \frac{1}{\prod_{j=1}^{k} (A^{P^j} - B^{Q^j})}, \quad A \neq P, P^2, P^3, \ldots, P^k, \quad B \neq Q, Q^2, Q^3, \ldots, Q^k, \quad k = 1, 2, 3, \ldots. \quad (4.2.1)
\]
Remark 4.2.1. If we take $A = 1$ and $P = 1$ we obtain $((A, B); (P, Q))_n = \prod_{i=1}^{k}(1 \cdot BQ^{i-1}) = (B; Q)_k$ that is $Q$-analogue.

Proposition 4.2.1. The $(P, Q)$-shifted factorials for negative subscripts satisfies the relation

$$(A, B); (P, Q))_n = \left[\left((AP^{-n}, BQ^{-n}); (P, Q)\right)_n\right]^{-1} = \frac{(PQ)^{\binom{n+1}{2}}}{((AQ, BP); (Q, P))_n}, \quad n = 0, 1, 2 \cdots$$

(4.2.2)

Proof. From the definition of $(P, Q)$-shifted factorials for negative subscripts, we have

$$(A, B); (P, Q))_n = \left(\prod_{j=1}^{n} (AP^{-j} - bQ^{-j})\right)^{-1} = \left((AP^{-1} - BQ^{-1})(AP^{-2} - BQ^{-2}) \cdots (AP^{-n} - BQ^{-n})\right)^{-1} = \left[\left((AP^{-n}, BQ^{-n}); (P, Q)\right)_n\right]^{-1} = \frac{1}{\left((AQ, BP); (Q, P)\right)_n} = \frac{(PQ)^{\binom{n+1}{2}}}{((AQ, BP); (Q, P))_n}.$$ 

This proves (4.2.2).

Proposition 4.2.2. If we replace $P$ by $P^{-1}$ and $Q$ by $Q^{-1}$ the $(P, Q)$-shifted factorials satisfies the relation

$$\left((A, B); \left(\frac{1}{P}, \frac{1}{Q}\right)\right)_n = \frac{((A, B); (Q, P))_n}{(PQ)^{\binom{n}{2}}}.$$ 

(4.2.3)

Proof. By using the definition (4.1.5) we obtain

$$\left((A, B); \left(\frac{1}{P}, \frac{1}{Q}\right)\right)_n = \prod_{i=1}^{n}(AP^{-i+1} - BQ^{-i+1}) = \frac{(A - B)(AQ - BP)(AQ^2 - BP^2) \cdots (AQ^{n-1} - BP^{n-1})}{(PQ)^{\binom{n}{2}}}$$

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\[
= \binom{(A, B); (Q, P)}{(PQ)}_n. \tag{4.2.4}
\]

Hence the proof of proposition (4.2.2).

**Proposition 4.2.3.** The \((P, Q)\)-shifted factorials satisfies the relation

\[
((rA, rB); (P, Q))_n = \prod_{i=1}^{n} (rA^{P^i-1} - rB^{Q^i-1}) = r^n ((A, B); (P, Q))_n \tag{4.2.5}
\]

**Proof.** The proof follows easily by definition of \((P, Q)\)-shifted factorials.

**Proposition 4.2.4.** The relationship between \(Q\) and \((P, Q)\)-analogue, are given by the

\[
\left( \frac{B}{A}; \frac{Q}{P} \right)_n = \frac{((A, B); (P, Q))_n}{A^n P^{(\frac{n}{2})}}. \tag{4.2.6}
\]

**Proof.** By using the definition of \(Q\)-analogue [34, page 6]

\[
\left( \frac{B}{A}; \frac{Q}{P} \right)_n = \prod_{i=1}^{n} \left( 1 - B \left( \frac{Q}{P} \right)^{i-1} \right)
= \prod_{i=1}^{n} \frac{(AP^{i-1} - BQ^{i-1})}{AP^{i-1}}
= \frac{((A, B); (P, Q))_n}{A^n P^{(\frac{n}{2})}}.
\]

**Definition 4.2.1.** Here, we can also define

\[
((A, B); (P, Q))_\infty = \prod_{j=1}^{\infty} (AP^{j-1} - BQ^{j-1}), \quad 0 < Q < P \leq 1. \tag{4.2.7}
\]

This implies that

\[
((A, B); (P, Q))_n = \frac{((A, B); (P, Q))_\infty}{((AP^n, BQ^n); (P, Q))_\infty}, \quad 0 < Q < P \leq 1 \tag{4.2.8}
\]

and for any complex number \(\lambda\)

\[
((A, B); (P, Q))_\lambda = \frac{((A, B); (P, Q))_\infty}{((AP^\lambda, BQ^\lambda); (P, Q))_\infty}, \quad 0 < Q < P \leq 1. \tag{4.2.9}
\]

We list a number of transformation formulas for the \((P, Q)\)-shifted factorials, where \(k\) and \(n\) are non-negative integers.
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Proposition 4.2.5.

\[ ((A, B); (P, Q))_{n+k} = ((A, B); (P, Q))_n ((AP^n, BQ^n); (P, Q))_k. \] (4.2.10)

Proof. The proof follows easily by definition of \((P, Q)\)-shifted factorials.

Proposition 4.2.6.

\[ ((AP^n, BQ^n); (P, Q))_k \left( ((AP^k, BQ^k); (P, Q))_n \right)^{-1} = ((A, B); (P, Q))_k \left( ((A, B); (P, Q))_n \right)^{-1} \] (4.2.11)

Proof.

\[
\frac{((AP^n, BQ^n); (P, Q))_k}{((AP^k, BQ^k); (P, Q))_n} = \frac{\prod_{j=1}^{k} (AP^n P^{-j-1} - BQ^n Q^{-j-1})}{\prod_{j=1}^{n} (AP^k P^{-j-1} - BQ^k Q^{-j-1})} = \frac{(AP^n - BQ^n)(AP^{n+1} - BQ^{n+1}) \cdots (AP^{n+k-1} - BQ^{n+k-1})}{(AP^k - BQ^k)(AP^{k+1} - BQ^{k+1}) \cdots (AP^{k+n-1} - BQ^{k+n-1})} = \frac{((A, B); (P, Q))_k}{((A, B); (P, Q))_n}.
\]

Therefore, (4.2.11) is true for any non-negative integers \(n\) and \(k\).

Proposition 4.2.7.

\[ ((AP^k, BQ^k); (P, Q))_{n-k} = ((A, B); (P, Q))_n \left( ((A, B); (P, Q))_k \right)^{-1}. \] (4.2.12)

Proof. The proof follows easily by definition of \((P, Q)\)-shifted factorials.

Proposition 4.2.8.

\[ ((A, B); (P, Q))_n = (-AB)^n (PQ)^{\frac{n}{2}} ((A^{-1} P^{1-n}, B^{-1} Q^{1-n}); (P, Q))_n. \] (4.2.13)

Proof.

\[
((A, B); (P, Q))_n = \prod_{j=1}^{n} (AP^j - BQ^j) = (A - B)(AP - BQ) \cdots (AP^{n-1} - BQ^{n-1})
\]
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\[
= (-AB)^n(PQ)^{\binom{n}{2}}(A^{-1} - B^{-1})(A^{-1}P^{-1} - B^{-1}Q^{-1}) \cdots (A^{-1}P^{1-n} - B^{-1}Q^{1-n})
\]

\[
= (-AB)^n(PQ)^{\binom{n}{2}}((A^{-1}P^{1-n}, B^{-1}Q^{1-n}); (P, Q))_n.
\]

Proposition 4.2.9.

\[
((AP^{-n}, BQ^{-n}); (P, Q))_n = (-AB)^n(PQ)^{-n-\binom{n}{2}}((A^{-1}P, B^{-1}Q); (P, Q))_n, \quad A \neq 0, B \neq 0.
\]

(4.2.14)

Proof. The proof follows easily by taking ratio of two \((P, Q)\)-shifted factorials.

Proposition 4.2.10.

\[
\frac{((AP^{-n}, BQ^{-n}); (P, Q))_n}{((CP^{-n}, DQ^{-n}); (P, Q))_n} = \frac{((A^{-1}P, B^{-1}Q); (P, Q))_n}{((C^{-1}P, D^{-1}Q); (P, Q))_n} \left(\frac{AB}{CD}\right)^n, \quad A, B, C, D \neq 0
\]

(4.2.15)

Proof. The proof follows easily by taking ratio of two \((P, Q)\)-shifted factorials.

Proposition 4.2.11.

\[
((A, B); (P, Q))_{n-k} = \frac{((A, B); (P, Q))_n}{((A^{-1}P^{1-n}, B^{-1}Q^{1-n}); (P, Q))_k} \left(\frac{-PQ}{AB}\right)^k (PQ)^{\binom{k}{2} - nk},
\]

\[
A, B \neq 0, \quad k = 0, 1, 2, \cdots n.
\]

(4.2.16)
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Proof.

\[(A, B); (P, Q)\]_{n-k} = (A - B)(AP - BQ)(AP^2 - BQ^2) \cdots (AP^{n-k-1} - BQ^{n-k-1})
\[
= ((A, B); (P, Q))_n
\]
\[
= (AP^{n-k} - BQ^{n-k})(AP^{n-k+1} - BQ^{n-k+1}) \cdots (AP^{n-1} - BQ^{n-1})
\]
\[
= ((A, B); (P, Q))_n \left(\frac{-PQ}{AB}\right)^k (PQ)^{\binom{k}{2}}^{-nk}.
\]

\[\Box\]

Proposition 4.2.12.

\[((P^{-n}, Q^{-n}); (P, Q))_k = (-1)^k (PQ)^{\binom{k}{2}}^{-nk} ((P, Q); (P, Q))_n \]

\[\text{Proof.}\]

\[((P^{-n}, Q^{-n}); (P, Q))_k = (P^{-n}, Q^{-n})(P^{-n+1}, Q^{-n+1}) \cdots (P^{-n+k-1}, Q^{-n+k-1})
\[
= (-1)^k (PQ)^{-nk}(PQ)^{\sum(k-1)}(P^{-n-k+1}, Q^{-n-k+1}) \cdots (P^n - Q^n)
\]
\[
= (-1)^k (PQ)^{\binom{k}{2}}^{-nk} ((P, Q); (P, Q))_n.
\]

\[\Box\]

Proposition 4.2.13.

\[((AP^{-n}, BQ^{-n}); (P, Q))_k = \frac{((A^{-1}P, B^{-1}Q); (P, Q))_n}{((A^{-1}P^{1-k}, B^{-1}Q^{1-k}); (P, Q))_n} ((A, B); (P, Q))_k (PQ)^{-nk},\]

\[\text{Proof.}\]

\[((AP^{-n}, BQ^{-n}); (P, Q))_k = (AP^{-n}, BQ^{-n})(AP^{-n+1}, BQ^{-n+1}) \cdots (AP^{-n+k-1}, BQ^{-n+k-1})
\[
= (-AB)^k (PQ)^{-nk}(PQ)^{\sum(k-1)}(A^{-1}P^{-n-k+1}, B^{-1}Q^{-n-k+1}) \cdots (A^{-1}P^n - B^{-1}Q^n)
\]
\[
= (-AB)^k (PQ)^{-nk}(PQ)^{\sum(k-1)}((A^{-1}P, B^{-1}Q); (P, Q))_n.
\]

\[A, B \neq 0.\]
Remark 4.2.2. When $k$ is replace by $n - k$ in above proposition then, we get

\[(\mathcal{A}P^{-n}, BQ^{-n}); (\mathcal{P}, Q)\] -parity \(-\mathcal{A} B \) \(n-k\) = \((-AB)^{n-k}(PQ)\Sigma^{k-1}(A^{-1}P, B^{-1}Q); (P, Q))_{n}(A^{-1}B^{-1}) \cdots (A^{-1}P^{1-k}, B^{-1}Q^{1-k})
\]

\[
= \frac{((A^{-1}P, B^{-1}Q); (P, Q))_{n}}{((A^{-1}P^{1-k}, B^{-1}Q^{1-k}); (P, Q))_{n}}(\mathcal{A}; (P, Q)) (PQ)^{-n-k}, \quad \mathcal{A}, \mathcal{B} \neq 0. \]

\[\text{(4.2.19)}\]

Proposition 4.2.14.

\[(\mathcal{A}; (P, Q))_{2n} = (\mathcal{A}; (P^2, Q^2))_{n}((\mathcal{A}; (P, BQ); (P^2, Q^2))_{n}. \]

\[\text{(4.2.20)}\]

Proof. The proof follows easily by definition of \((\mathcal{P}, Q)\)-shifted factorials.

Remark 4.2.3. Similarly, we can prove that

\[(\mathcal{A}; (P, Q))_{3n} = (\mathcal{A}; (\mathcal{A}; (P, BQ), (AP, BQ); (P^3, Q^3))_{n}, \]

where

\[(\mathcal{A}, (AP, BQ), (AP^2, BQ^2); (P^3, Q^3))_{n} = ((\mathcal{A}, (P^3, Q^3))_{n}((\mathcal{A}; (P, BQ); (P^3, Q^3))_{n}. \]

Remark 4.2.4. So, the generalization of above two equations are can be prove by the principal of mathematical induction

\[(\mathcal{A}; (P, Q))_{kn} = (\mathcal{A}; (AP, BQ), \cdots (AP^{k-1}, BQ^{k-1}); (P^k, Q^k))_{n}, \]

where

\[(\mathcal{A}, (AP, BQ), \cdots (AP^{k-1}, BQ^{k-1}); (P^k, Q^k))_{n} = ((\mathcal{A}, (P^k, Q^k))_{n}((\mathcal{A}; (P, BQ); (P^k, Q^k))_{n} \cdots ((\mathcal{A}; (P^{k-1}, BQ^{k-1}); (P^k, Q^k))_{n.} \]

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Proposition 4.2.15.

\[ ((A^2, B^2); (P^2, Q^2))_n = ((A, B); (P, Q))_n ((A, -B); (P, Q))_n. \]  \hspace{1cm} (4.2.23)

**Proof.** The proof follows easily by definition of \((P, Q)\)-shifted factorials and using the formula \((A^2 - B^2) = (A - B)(A + B)\), we get the result.  \[ \blacksquare \]

**Remark 4.2.5.** Also, we can prove that

\[ ((A^3, B^3); (P^3, Q^3))_n = ((A, B), (A, B\omega), (A, B\omega^2); (P, Q))_n, \]  \hspace{1cm} (4.2.24)

where

\[ ((A, B), (A, B\omega), (A, B\omega^2); (P, Q))_n = ((A, B); (P, Q))_n ((A, B\omega); (P, Q))_n ((A, B\omega^2); (P, Q))_n \]

and \( \omega = e^{2\pi i/3}. \)

**Remark 4.2.6.** So, the generalization of above two equations can be proved by the principal of mathematical induction

\[ ((A^k, B^k); (P^k, Q^k))_n = ((A, B), (A, \omega B), \cdots (A, \omega^{k-1} B); (P, Q))_n, \]  \hspace{1cm} (4.2.25)

where

\[ ((A, B), (A, \omega B), \cdots (A, \omega^{k-1} B); (P, Q))_n = ((A, B); (P, Q))_n ((A, \omega B); (P, Q))_n \cdots ((A, \omega^{k-1} B); (P, Q))_n \]

and \( \omega_k = e^{2\pi i/k}. \)

**Proposition 4.2.16.** By using above proposition we have

\[ \frac{(A^2P^2n - B^2Q^2n)}{(A^2 - B^2)} = \frac{((A^2P^2, B^2Q^2); (P^2, Q^2))_n}{((A^2, B^2); (P^2, Q^2))_n} = \frac{((AP, BQ); (P, Q))_n ((AP, -BQ); (P, Q))_n}{((A, B); (P, Q))_n ((A, -B); (P, Q))_n}. \]  \hspace{1cm} (4.2.26)

**Proof.** The proof follows easily by definition.  \[ \blacksquare \]
Remark 4.2.7. By the similarity of above result, we can prove by using proposition
\[
\left( A^3 P^{3n} - B^3 Q^{3n} \right) \left( A^3 B^3 \right) = \left( (A^3 P^3, B^3 Q^3) \right)_n = \left( (A, B) \right)_n,
\]
where \( \omega = e^{2\pi i/3} \).

Remark 4.2.8. So, the generalization of above two equations can be written as
\[
\left( A^k P^{kn} - B^k Q^{kn} \right) \left( A^k B^k \right) = \left( (A^k P^k, B^k Q^k) \right)_n = \left( (A, B) \right)_n,
\]
where \( \omega_k = e^{2\pi i/k} \).

4.3 \((\mathcal{P}, \mathcal{Q})\)-Hypergeometric Functions

First of all, we have needed to announce a \((\mathcal{P}, \mathcal{Q})\)-series containing several parameters. For that purpose, we recall that the Gauss 1813 hypergeometric series defined by
\[
\mathcal{F}(\mathcal{A}, \mathcal{B}; \mathcal{C}; z) = \sum_{n \geq 0} \frac{(\mathcal{A})_n (\mathcal{B})_n}{(\mathcal{C})_n n!} z^n,
\]
where \( \mathcal{C} \neq 0, -1, -2, \cdots \) so that no zero factors appear in the Eq. (4.3.1). Gauss’ series or Eq. (4.3.1) converges absolutely for \( |z| < 1 \) and for \( |z| = 1 \) when \( \Re(\mathcal{C} - \mathcal{A} - \mathcal{B}) > 0 \). Heine in 1847, 1878 introduced the series
\[
\Phi(\lambda, \mu, \delta, Q, z) = \sum_{n \geq 0} \frac{(\mathcal{A} Q^\lambda, \mathcal{B} Q^\mu; \mathcal{Q}^\delta; Q_1, Q_2)_{(\mathcal{Q}^m)}}{(\mathcal{C} Q^\lambda, \mathcal{Q}^\mu; \mathcal{Q}^\delta; Q_1, Q_2)_{(\mathcal{Q}^m)}} z^n,
\]
where \( \delta \neq -m \) and \( \mathcal{C} \neq \mathcal{Q}^{-m} \) for \( m = 0, 1, 2, \cdots \). Heine’s series or Eq. (4.3.2) converges absolutely for \( |z| < 1 \) when \( |Q| < 1 \), and it is a \( \mathcal{Q} \)-analogue of Gauss’ series or Eq. (4.3.1) because, by taking a formal term-wise limit,
\[
\lim_{Q \to 1} \Phi(\lambda, \mu, \delta, Q, z) = \mathcal{F}(\lambda, \mu; \delta; z).
\]
Here, we introduce a \((P, Q)\)-series as

\[
_{2}\Phi_{1} \left( \begin{array}{c}
(P^{\lambda}, Q^{\lambda}), (P^{\mu}, Q^{\mu}) \\
(P^{\delta}, Q^{\delta})
\end{array} \mid (P, Q), z \right),
\]

with

\[
_{2}\Phi_{1} \left( \begin{array}{c}
(A, B), (C, D) \\
(E, F)
\end{array} \mid (P, Q), z \right) = \sum_{n \geq 0} \left( \frac{(A, B); (P, Q)}{(C, D); (P, Q)} \right)_{n} z^{n},
\]

where \(\delta \neq -m\) and \((E, F) \neq (P^{-m}, Q^{-m})\) for \(m = 0, 1, \cdots\), the Eq. (4.3.3) converges absolutely for \(|z| < 1\) when \(\frac{Q_{/m}}{P_{/n}} < 1\) and it is a \((P, Q)\)-analogue of Gauss’ series or Eq. (4.3.1) because, by taking a formal termwise limit,

\[
\lim_{P \to 1-} _{2}\Phi_{1} \left( \begin{array}{c}
(P^{\lambda}, Q^{\lambda}), (P^{\mu}, Q^{\mu}) \\
(P^{\delta}, Q^{\delta})
\end{array} \mid (P, Q), z \right) = _{2}\Phi_{1}(Q^{\lambda}, Q^{\mu}; Q^{\delta}, Q, z)
\]

and

\[
\lim_{Q \to 1-} _{2}\Phi_{1}(Q^{\lambda}, Q^{\mu}; Q^{\delta}, Q, z) = _{2}\Phi_{1}(\lambda, \mu; \delta; z).
\]

The generalized hypergeometric series with \(i\) numerator parameters \((A_{1P}, A_{1Q}), \cdots, (A_{iP}, A_{iQ})\) and \(j\) denominator parameters \((B_{1P}, B_{1Q}), \cdots, (B_{jP}, B_{jQ})\) is defined by

\[
_{i}\Phi_{j} \left( \begin{array}{c}
(A_{1P}, A_{1Q}); \cdots; (A_{iP}, A_{iQ}) \\
(B_{1P}, B_{1Q}); \cdots; (B_{jP}, B_{jQ})
\end{array} \mid (P, Q); z \right)
\]

\[
= \sum_{n \geq 0} \frac{(A_{1P}, A_{1Q}), \cdots, (A_{iP}, A_{iQ}); (P, Q))_{n}}{(B_{1P}, B_{1Q}), \cdots, (B_{jP}, B_{jQ}); (P, Q))_{n}} \left( (-1)^{n} \left( \frac{Q}{P} \right) \right)^{1+j-i} z^{n},
\]

where \((A_{1P}, A_{1Q}), \cdots, (A_{iP}, A_{iQ}); (P, Q))_{n} = ((A_{1P}, A_{1Q}); (P, Q))_{n} \cdots ((A_{iP}, A_{iQ}); (P, Q))_{n}.\)

Also, for any \(j, ((B_{jP}, B_{jQ}); (P, Q))_{n} \neq 0\), if one of \(r\) is such that \((A_{rP}, A_{rQ}) = (P^{-n}, Q^{-n})\) where \(n\) is a non-negative integer, this \((P, Q)\)-hypergeometric function is a polynomials in \(z\), otherwise the radius of convergence \(l\) of the \((P, Q)\)-hypergeometric series is given by

\[
l = \begin{cases} 
\infty, & i < j + 1 \\
1, & i = j + 1 \\
0, & i > j + 1.
\end{cases}
\]
Remark 4.3.1. The special case when \( i = j + 1 \) we get

\[
j+1\Phi_j \left( \begin{array}{c} (A_{iP}, A_{iQ}); \cdots; (A_{(j+1)P}, A_{(j+1)Q}) \\ (B_{iP}, B_{iQ}); \cdots; (B_{jP}, B_{jQ}) \end{array} \right) \biggr| (P, Q); z \right) = \sum_{n \geq 0} \frac{((A_{1P}, A_{1Q}), \cdots, (A_{(j+1)P}, A_{(j+1)Q}); (P, Q))_n}{((B_{1P}, B_{1Q}), \cdots, (B_{jP}, B_{jQ}); (P, Q))_n} z^n.
\]

(4.3.6)

Remark 4.3.2. If \( A_{1P} = A_{2P} = \cdots = A_{iP} = B_{1P} = \cdots = B_{jP} = 1 \) and \( A_{1Q} = \cdots, A_{iQ} = A_i, B_{1Q} = B_1, \cdots, B_{jQ} = B_j \) then

\[
\lim_{P \to 1} i \Phi_j \left( \begin{array}{c} (1, A_1); \cdots; (1, A_i) \\ (1, B_1); \cdots; (1, B_j) \end{array} \right) \biggr| (P, Q); z \right) = i \Phi_j \left( \begin{array}{c} A_1; \cdots; A_i \\ B_1; \cdots; B_j \end{array} \biggr| (P, Q); z \right). \quad (4.3.7)
\]

Remark 4.3.3. We assume that each \((P, Q)\)-hypergeometric function is in fact a polynomials when

\[
\lim_{(A_{iP}, A_{iQ}) \to \infty} i \Phi_j \left( \begin{array}{c} (A_{1P}, A_{1Q}); \cdots; (A_{iP}, A_{iQ}) \\ (B_{1P}, B_{1Q}); \cdots; (B_{jP}, B_{jQ}) \end{array} \biggr| (P, Q); \frac{z}{(A_{iP}, A_{iQ})} \right) = i_{-1} \Phi_j \left( \begin{array}{c} (A_{1P}, A_{1Q}); \cdots; (A_{(i-1)P}, A_{(i-1)Q}) \\ (B_{1P}, B_{1Q}); \cdots; (B_{jP}, B_{jQ}) \end{array} \biggr| (P, Q); z \right). \quad (4.3.8)
\]

Many limit relations between \((P, Q)\)-hypergeometric orthogonal polynomials are based on the observations that

\[
\Phi_j \left( \begin{array}{c} (A_{1P}, A_{1Q}); \cdots; (A_{(i-1)P}, A_{(i-1)Q}); (\lambda, \mu) \\ (B_{1P}, B_{1Q}); \cdots; (B_{(j-1)P}, B_{(j-1)Q}); (\lambda, \mu) \end{array} \biggr| (P, Q); z \right) = i_{-1} \Phi_{j-1} \left( \begin{array}{c} (A_{1P}, A_{1Q}); \cdots; (A_{(i-1)P}, A_{(i-1)Q}) \\ (B_{1P}, B_{1Q}); \cdots; (B_{(j-1)P}, B_{(j-1)Q}) \end{array} \biggr| (P, Q); z \right). \quad (4.3.9)
\]

\[
\lim_{\lambda \to \infty} \Phi_j \left( \begin{array}{c} (A_{1P}, A_{1Q}); \cdots; (A_{(i-1)P}, A_{(i-1)Q}); \lambda(A_{iP}, A_{iQ}) \\ (B_{1P}, B_{1Q}); \cdots; (B_{jP}, B_{jQ}) \end{array} \biggr| (P, Q); \frac{z}{\lambda} \right)
\]
Chapter 4

\[ = i^{-1} \Phi_j \left( (A_1, A_1); \cdots; (A_{i-1}, A_{i-1}) \mid (P, Q); (A_i, A_i)z \right) \cdot \]

(4.3.10)

\[
\lim_{\lambda \to \infty} i^{-1} \Phi_j \left( (A_1, A_1); \cdots; (A_i, A_i) \mid (P, Q); \lambda z \right)
\]

(4.3.11)

\[
= i^{-1} \Phi_{j-1} \left( (A_1, A_1); \cdots; (A_i, A_i) \mid (P, Q); \frac{z}{(B_i, B_i)} \right)
\]

and

\[
\lim_{\lambda \to \infty} i^{-1} \Phi_j \left( (A_1, A_1); \cdots; (A_i, A_i) \mid (P, Q); z \right)
\]

(4.3.12)

\[
= i^{-1} \Phi_{j-1} \left( (A_1, A_1); \cdots; (A_i, A_i) \mid (P, Q); \frac{z}{(B_i, B_i)} \right)
\]

Here, we also introduce some transformation formulas for $2 \Phi_1$ series

**Theorem 4.3.1.**

\[
2 \Phi_1 \left( (A, B), (C, D) \mid (P, Q), z \right) = \frac{((C, D), (E, F), (BD, z); (P, Q))}{((E, F), (ED, z); (P, Q))} \cdot 2 \Phi_1 \left( (E, F), (ED, z); (P, Q) \right) \cdot
\]

(4.3.13)

\[
= \frac{((E, F), (ED, z); (P, Q))}{((E, F), (ED, z); (P, Q))} \cdot 2 \Phi_1 \left( (E, F), (ED, z); (P, Q) \right) \cdot
\]

(4.3.14)

\[
= \frac{((E, F), (ED, z); (P, Q))}{((E, F), (ED, z); (P, Q))} \cdot 2 \Phi_1 \left( (E, F), (ED, z); (P, Q) \right) \cdot
\]

(4.3.15)

**Proof.**

\[
2 \Phi_1 \left( (A, B), (C, D) \mid (P, Q), z \right) = \sum_{n \geq 0} \frac{((A, B); (P, Q))_n ((C, D); (P, Q))_n}{((E, F); (P, Q))_n ((P, Q); (P, Q))_n} z^n
\]
The Characterization properties and Basic Hypergeometric functions of \( (\mathcal{P}, \mathcal{Q}) \)-analogue

\[\sum_{n \geq 0} \frac{A^n}{n!} \left[ \frac{P_n}{Q_n} \right] \frac{C^n}{n^n} (\frac{D}{C}) \frac{\mathcal{P}^{n}}{(\mathcal{P})^{n}} = (\frac{P}{C}) \frac{\mathcal{E}^{n}}{(\mathcal{E})^{n}} \ (\text{: Proposition 2.4)}
\]

\[\sum_{n \geq 0} \left[ \frac{P_n}{Q_n} \right] (\frac{D}{C}) \frac{\mathcal{P}^{n}}{(\mathcal{P})^{n}} \left[ \frac{A^n}{n!} \left( \frac{P_n}{Q_n} \right) \frac{C^n}{n^n} \right] = (\frac{P}{C}) \frac{\mathcal{E}^{n}}{(\mathcal{E})^{n}} \]

\[\sum_{n \geq 0} \left[ \frac{P_n}{Q_n} \right] (\frac{D}{C}) \frac{\mathcal{P}^{n}}{(\mathcal{P})^{n}} \left[ \frac{A^n}{n!} \left( \frac{P_n}{Q_n} \right) \frac{C^n}{n^n} \right] = (\frac{P}{C}) \frac{\mathcal{E}^{n}}{(\mathcal{E})^{n}} \]

An iterative way to prove the rest of the above equality is just to iterate the first equality. The latter formula is a \( (\mathcal{P}, \mathcal{Q}) \)-analogue of Euler's transformation formula:

\[\frac{(\mathcal{C}, \mathcal{E}, \mathcal{F}, \mathcal{G}); (\mathcal{P}, \mathcal{Q})}{(\mathcal{E}, \mathcal{F}, \mathcal{P}, \mathcal{A}; \mathcal{P}, \mathcal{Q})} \]

\[\sum_{m \geq 0} \left[ \frac{D^n}{C^n} \right] \frac{\mathcal{P}^{n}}{(\mathcal{P})^{n}} \left[ \frac{A^n}{n!} \left( \frac{P_n}{Q_n} \right) \frac{C^n}{n^n} \right] = (\frac{P}{C}) \frac{\mathcal{E}^{n}}{(\mathcal{E})^{n}} \]

A short way to prove the rest of the two equality is just to iterate the first equality. The latter formula is a \( (\mathcal{P}, \mathcal{Q}) \)-analogue of Euler's transformation formula:

\[\begin{array}{c}
2F_{1} \left[ \frac{A}{C} ; z \right] = (1-z)^{A-B} \ 2F_{1} \left[ \frac{C-A}{C} ; z \right]
\end{array}
\]

(4.3.16)

The contents of this chapter is communicated to Palestine journal of Mathematics for its possible publication.