A NOTE ON DISCRETE $q$-MODIFIED HERMITE POLYNOMIALS OF TYPE-I

3.1 Introduction

In this chapter we describe the discrete $q$-modified Hermite polynomials by means of generating functions. Explicit expression, recurrence relations and three terms recurrence relations for these polynomials are given. Furthermore, Rodrigue-type formula for the discrete $q$-modified Hermite polynomials are discussed.

A brief idea of $q$-Hermite polynomials are given in [6], [10], [66], [67], [94], [114]. Khan and Ahmad [61] established the classical families of modified Hermite polynomials

$$M_{\mathcal{H}}(\zeta; \lambda) = \begin{cases} m! & m \geq 0 \sum_{r=0}^{[m/2]} \frac{(-1)^r (2\zeta)^{m-2r} \lambda^{m-r}}{r!(m-2r)!}, m \geq 0. \end{cases}$$

For $m \geq 0$, $r \geq 0$, the following relation are satisfied in [82].

$$\sum_{m \geq 0} \sum_{r \geq 0} A(r, m) = \sum_{m \geq 0} \sum_{r=0}^{[m/2]} A(r, m - 2r),$$
The $q$-Hermite polynomials of type-I are defined by means of the generating function \[34, 66\]:

\[
(\mathfrak{z}, -\mathfrak{z}; q)_{\infty} (\zeta \mathfrak{z}; q)_{\infty}^{-1} = \sum_{m \geq 0} \frac{\mathcal{H}_m(\zeta; q) \mathfrak{z}^m}{m!},
\]

and explicit formula

\[
\mathcal{H}_m(\zeta; q) = \sum_{r=0}^{\lfloor m/2 \rfloor} \frac{(-1)^r(q; q)_m q^{r(r-1)}}{(q^2; q^2)_r (q; q)_{m-2r}} \zeta^{m-2r}, \quad m \in \mathbb{N}_0.
\]

Some $q$-analogues of special functions are given in \[66\], two different natural $q$-extension of exponential function are

\[
e_q(\zeta) = \varphi_0 \left( \frac{0}{-} ; q, \zeta \right) = \sum_{m \geq 0} \frac{\zeta^m}{(q; q)_m}
= \frac{1}{(\zeta; q)_\infty}, \quad 0 < |q| < 1, \quad |\zeta| < 1
\]

and

\[
E_q(\zeta) = \varphi_0 \left( \frac{-}{-} ; q, -\zeta \right) = \sum_{m \geq 0} \frac{q \binom{m}{2} \zeta^m}{(q; q)_m}
= (-\zeta; q)_{\infty}, \quad 0 < |q| < 1.
\]

The $q$-factorial function $[m]_q!$ is defined for a positive integer $m$ as

\[
[m]_q! = [m]_q[m - 1]_q \cdots [1]_q, \quad \text{with} \quad [0]_q! = 1,
\]

where

\[
[m]_q = \frac{(1 - q^m)}{(1 - q)}
\]

denotes the basic number.
3.2 The Definition of Discrete $q$-Modified Hermite Polynomials

The $q$-modified Hermite polynomials $M\mathcal{H}_m(\zeta; \lambda; q)$ of type-I can be defined by means of generating relation

$$F_q(\zeta, 3) = (3\sqrt{\ln\lambda}, -3\sqrt{\ln\lambda}; q)_\infty (2\zeta\ln\lambda; q)_\infty^{-1}$$

$$= \sum_{m \geq 0} \frac{M\mathcal{H}_m(\zeta; \lambda; q)3^m}{m!}, \lambda > 0, \lambda \neq 1,$$  \hspace{1cm} (3.2.1)$$

which on using Eqs. (3.1.7) and (3.1.8), becomes

$$(3\sqrt{\ln\lambda}, -3\sqrt{\ln\lambda}; q)_\infty (2\zeta\ln\lambda; q)_\infty^{-1} = (3^2\ln\lambda; q^2)_\infty (2\zeta\ln\lambda; q)_\infty^{-1}$$

$$= \sum_{m \geq 0} \sum_{r \geq 0} \frac{(2\zeta\ln\lambda)^m(-1)^r q^r(r-1) (3^2\ln\lambda)^r}{(q; q)_m (q^2; q^2)_r},$$

Using Eq. (3.1.3) in the above equation, then comparing the coefficients of $3^n$ on both side of the resultant equation, we obtain an explicit representation for the discrete $q$-modified Hermite polynomials in the form

$$M\mathcal{H}_m(\zeta; \lambda; q) = \sum_{r=0}^{[m/2]} (-1)^r (q; q)_m q^r(r-1) (2\zeta)^{m-2r} (\ln\lambda)^{m-r}. \hspace{1cm} (3.2.2)$$

For $\lambda = e$, it reduces to $q$-Hermite polynomials of type-I i.e $\mathcal{H}_m(\zeta; q)$ which have been considered by Al-Salam and Carlitz [3]. If we replace by $3$ by $-3$ and $\zeta$ by $-\zeta$ in Eq. (3.2.1) then left hand side of the Eq. (3.2.1) does not change, so

$$M\mathcal{H}_m(\zeta; \lambda; q) = (-1)^m M\mathcal{H}_m(\zeta; \lambda; q), \hspace{1cm} (3.2.3)$$

it shows that $M\mathcal{H}_m(\zeta; \lambda; q)$ is an odd function of $\zeta$ for odd $m$, an even function of $\zeta$ for even $m$. Also,

$$M\mathcal{H}_{2m}(0; \lambda; q) = (-1)^m (q; q^2)_m q^{m(m-1)} (\ln\lambda)^m,$$

$$M\mathcal{H}_{2m+1}(0; \lambda; q) = 0.$$

The first two $q$-Modified Hermite polynomials are

$$M\mathcal{H}_0(\zeta; \lambda; q) = 1,$$

$$M\mathcal{H}_1(\zeta; \lambda; q) = 2\zeta (\ln\lambda).$$
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3.3 Recurrence Relations of $M\mathcal{H}_m(\zeta; \lambda; q)$

To produce the recurrence relations for the discrete $q$-Modified Hermite polynomials, we recall the definitions of the $q$-derivative $D_q g$ of a function $g$ given by

$$D_q g(\zeta) = \frac{g(\zeta) - g(\zeta q)}{(1 - q)\zeta}, \, q \neq 1, \, \zeta \neq 0. \quad (3.3.1)$$

Further, we have

$$D_q [g_1(\zeta)g_2(\zeta)] = g_1(\zeta)D_q g_2(\zeta) + g_2(\zeta)D_q g_1(\zeta) \quad (3.3.2)$$

which is often referred as the $q$-product rule, now from Eq. (3.3.1), we have

$$D_q (\mu \zeta q^r; q)_{\infty} = -\frac{\mu}{1 - q} (\mu \zeta q^r)_{\infty}, \quad (3.3.3)$$

$$D_q (\mu \zeta^{-1} q^r; q)_{\infty} = \frac{\mu}{1 - q} (\mu \zeta^{-1} q^r)_{\infty}^{-1}, \quad (3.3.4)$$

$$D_q (\mu \zeta^2; q^2)_{\infty} = -\frac{\mu}{1 - q} (\mu \zeta^2 q^2)_{\infty}, \quad (3.3.5)$$

$$D_q \zeta^{-\mu} = [\mu]_q \zeta^{-\mu - 1}. \quad (3.3.6)$$

**Theorem 3.3.1.** The discrete $q$-modified Hermite polynomials $M\mathcal{H}_m(\zeta; \lambda; q)$ satisfy

$$D_q^n M\mathcal{H}_m(\zeta; \lambda; q) = \frac{[m]_q! (2\ln\lambda)^r}{[m - r]_q!} M\mathcal{H}_{m-r}(\zeta; \lambda; q), \quad 0 \leq r \leq m. \quad (3.3.7)$$

**Proof.** On $q$-differentiating Eq. (3.2.1) with respect to $\zeta$ by using product rule (3.3.2), we obtain

$$D_{q,\zeta} F_q(\zeta, z) = (3\sqrt{\ln\lambda}; q)_{\infty} (-3\sqrt{\ln\lambda}; q)_{\infty} (2\zeta \ln\lambda; q)_{\infty}^{-1} \left(\frac{2\ln\lambda}{1 - q}\right)$$

$$= \sum_{m \geq 1} \frac{D_q M\mathcal{H}_m(\zeta; \lambda; q)}{(q; q)_m} z^m. \quad (3.3.8)$$

By the aid of (3.2.1) and (3.3.8), we have

$$\left(\frac{2\ln\lambda}{1 - q}\right) \sum_{m \geq 0} \frac{M\mathcal{H}_m(\zeta; \lambda; q)z^m}{m!} = \sum_{m \geq 1} \frac{D_q M\mathcal{H}_m(\zeta; \lambda; q)}{(q; q)_m} z^m.$$
Comparison of the coefficients of $z^m$ on both side of the above equation, we have

$$\mathcal{D}_q M \mathcal{H}_m(\zeta; \lambda; q) = [m]_q (2 \ln \lambda)_M \mathcal{H}_{m-1}(\zeta; \lambda; q)$$  \hspace{1cm} (3.3.9)

or likewise the forward shift operator, we get

$$M \mathcal{H}_m(\zeta; \lambda; q) - M \mathcal{H}_m(q \zeta; \lambda; q) = (1 - q^m)(2 \zeta \ln \lambda)_M \mathcal{H}_{m-1}(\zeta; \lambda; q).$$ \hspace{1cm} (3.3.10)

On iterating Eq. (3.3.9), for $0 \leq r \leq m$, we obtain the result (3.3.7).

**Theorem 3.3.2.** The discrete $q$-modified Hermite polynomials $M \mathcal{H}_m(\zeta; \lambda; q)$ satisfies the recurrence relation

$$2 \zeta \ln \lambda q^m M \mathcal{H}_m(q^{-1}\zeta; \lambda; q) - \ln \lambda q^{m-1}(1 - q^m) = M \mathcal{H}_{m+1}(\zeta; \lambda; q),$$ \hspace{1cm} (3.3.11)

$$(4 \zeta^2 \ln \lambda - 1) M \mathcal{H}_m(q^{-1}\zeta; \lambda; q) + M \mathcal{H}_m(\zeta; \lambda; q) = q^{-m}(2 \zeta) M \mathcal{H}_{m+1}(\zeta; \lambda; q).$$ \hspace{1cm} (3.3.12)

**Proof.** On $q$-differentiating Eq. (3.2.1) with respect to $\zeta$ by using product rule (3.3.2) and Eqs. (3.3.3), (3.3.4), gives

$$\mathcal{D}_q F_q(\zeta, \zeta) = \frac{2 \zeta \ln \lambda q^m M \mathcal{H}_m(q^{-1}\zeta; \lambda; q) - \ln \lambda q^{m-1}(1 - q^m)}{(1 - q) (\zeta; q)_m} \zeta^{m-1}. \hspace{1cm} (3.3.13)$$

Also, we can deduce that

$$(\zeta \ln \lambda q; q)_\infty (\zeta q \ln \lambda q; q)_\infty (2 \zeta \ln \lambda q; q)_\infty^{-1} = \sum_{m \geq 0} \frac{M \mathcal{H}_m(q^{-1}\zeta; \lambda; q)}{(q; q)_m} \zeta^m,$$ \hspace{1cm} (3.3.14)

using Eq. (3.3.14) in Eq. (3.3.13), we get

$$\left(\frac{2 \zeta \ln \lambda - \ln \lambda}{1 - q}\right) \sum_{m \geq 0} \frac{M \mathcal{H}_m(q^{-1}\zeta; \lambda; q)}{(q; q)_m} \zeta^m = \sum_{m \geq 1} \frac{M \mathcal{H}_m(\zeta; \lambda; q)}{(1 - q)(q; q)_{m-1}} \zeta^{m-1}.$$

Identification the coefficients of $\zeta^m$ on both side gives Eq. (3.3.11). In order to prove Eq. (3.3.12), we replace $\zeta$ by $\zeta q^{-1}$ in Eq. (3.3.10) that produce

$$M \mathcal{H}_m(q^{-1}\zeta; \lambda; q) - M \mathcal{H}_m(\zeta; \lambda; q)$$
using the Eq. (3.3.15) and (3.3.11) yields (3.3.12).

**Theorem 3.3.3.** The discrete $q$-modified Hermite polynomial $M_{\mathcal{H}}(\zeta; \lambda; q)$ satisfies the following expansion relation

$$(2\zeta\ln\lambda)^m = \left[\sum_{r=0}^{[m/2]} \frac{(q; q)_m M_{\mathcal{H}}(\zeta; \lambda; q)(\ln\lambda)^r}{(q; q)_{m-2r}(q^2; q^2)_r}\right].$$

**Proof.** In order to prove this relation, we required to rearrange Eq. (3.2.1) in the form

$$(2\zeta\ln\lambda; q)_{\infty}^{-1} = \frac{1}{(\zeta^2\ln\lambda; q^2)_{\infty}} \sum_{m \geq 0} \frac{M_{\mathcal{H}}(\zeta; \lambda; q) \zeta^m}{m!}$$

expansion of L.H.S., and then using Eq. (3.1.3) after that identification of the coefficients of $\zeta^m$ on both side yields Eq. (3.3.16).

**Theorem 3.3.4.** The discrete $q$-modified Hermite polynomial $M_{\mathcal{H}}(\zeta; \lambda; q)$ satisfies the three terms recurrence relation

$$M_{\mathcal{H}}(\zeta; \lambda; q) = 2\zeta\ln\lambda M_{\mathcal{H}}(\zeta; \lambda; q) - \ln\lambda(1 - q^{m-1})(q^{m-2})_M M_{\mathcal{H}}(\zeta; \lambda; q).$$

**Proof.** Using Eq. (3.3.8) and (3.13) we get

$$(2\zeta - \zeta)\ln\lambda D_{q,\zeta}(F_q(\zeta, \zeta)) = (2\zeta\ln\lambda)(1 - \zeta^2\ln\lambda) D_{q,\zeta}(F_q(\zeta, \zeta)).$$

Also,

$$(2\zeta - \zeta)\ln\lambda \sum_{m \geq 1} \frac{D_{q,\zeta} M_{\mathcal{H}}(\zeta; \lambda; q) \zeta^m}{(q; q)_{m-1}} = (2\zeta\ln\lambda)(1 - \zeta^2\ln\lambda) \sum_{m \geq 1} \frac{M_{\mathcal{H}}(\zeta; \lambda; q)}{1 - q(q; q)_{m-1}} \zeta^{m-1},$$

after some simplification and identification of the coefficients of $\zeta^m$ on both side we get

$$2\ln\lambda \left[\zeta D_{q,\zeta} M_{\mathcal{H}}(\zeta; \lambda; q) - [m]_q M_{\mathcal{H}}(\zeta; \lambda; q)\right] = \ln\lambda(1 - q^m) \left[D_{q,\zeta} M_{\mathcal{H}}(\zeta; \lambda; q)\right]$$

$$- 2(\ln\lambda)^2[m - 2]_q (1 - q^m)(1 - q^{m-1})_M M_{\mathcal{H}}(\zeta; \lambda; q),$$

using Eq. (3.3.9), we get the result.
Corollary 3.3.1. Using three terms recurrence relation (3.3.17) and (3.3.9) we can achieve the following $q$-differential equation for the discrete $q$-Modified Hermite polynomials

$$
\left( \mathcal{D}_q^2 - \frac{4\zeta \ln \lambda q^{2-m}}{1 - q} \mathcal{D}_q + \frac{4\ln \lambda q^{2-m}[m]_q}{1 - q} \right) M_{\mathcal{H}}(\zeta; \lambda; q) = 0.
$$

(3.3.18)

3.4 Rodrigues Type Formula

For finding Rodrigues type formula consider the following $q$-counterpart of Taylor series given by Jackson [41]

$$
f(\zeta) = \sum_{m \geq 0} \zeta^m \left( \frac{\mu/\zeta; q}{[m]_q} \right)_m \mathcal{D}_q^m f(\mu).
$$

(3.4.1)

and the Cauchy’s integral formula for the $m^{th}$ $q$-derivative given in [92]

$$
\mathcal{D}_q^m f(\mu) = \frac{[m]_q!}{2\pi i} \int_c \frac{f(w)}{w^{m+1}(\mu/w; q)_{m+1}} dw.
$$

(3.4.2)

And,

$$
\mathcal{D}_q^m f(\mu) = \frac{(-1)^{m+1} q^m [m]_q!}{2\pi i (\mu/w; q)_{m+1}} \int_c \frac{f(w)dw}{w^{m+1}}.
$$

(3.4.3)

The generating function (3.2.1) and with the help of $q$-Taylor formula (3.4.1), we get

$$
M_{\mathcal{H}}(\zeta; \lambda; q) = (1 - q)^m \times \left[ \mathcal{D}_q^m (\zeta \sqrt{\ln \lambda}, -\zeta \sqrt{\ln \lambda}; q)_\infty (2\zeta \ln \lambda; q)^{-1} \right]_{\zeta = 0},
$$

(3.4.4)

then by using the Cauchy’s integral formula for $m^{th}$ $q$-derivative (3.4.2) can be written as

$$
M_{\mathcal{H}}(\zeta; \lambda; q) = \frac{(q; q)_m}{2\pi i} \times \int_c \frac{(w \sqrt{\ln \lambda}, -w \sqrt{\ln \lambda}; q)_\infty (2\zeta w \ln \lambda; q)^{-1}}{w^{m+1}} dw.
$$

(3.4.5)

Theorem 3.4.1. The Rodrigues type formula for the discrete $q$-modified Hermite polynomials $M_{\mathcal{H}}(\zeta; \lambda; q)$ can be derived as

$$
\mathcal{W} (\zeta; \lambda; q) M_{\mathcal{H}}(\zeta; \lambda; q) = (-1)^m (1 - q)^m q^{m(m-3)/4} \left( \frac{\ln \lambda}{2} \right)^{-m/2} \mathcal{D}_q^{m-1} \left( \mathcal{W} (\zeta; \lambda; q) \right).
$$

(3.4.6)

where

$$
\mathcal{W} (\zeta; \lambda; q) = (q \zeta \sqrt{\ln \lambda}; q)_\infty (-q \zeta \sqrt{\ln \lambda}; q)_\infty.
$$

(3.4.7)
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Proof. Using backward shifting relation we have
\[
D_{q^{-1}} \left( M_H^m(z; \lambda; q) \right) = \frac{q^{1-m}}{q-1} \sqrt{\frac{\ln \lambda}{2}} \mathcal{W}(z; \lambda; q)_{M_{H^{m+1}}(z; \lambda; q)}
\]  
(3.4.8)

iterating Eq. (3.4.8) \( r \)-times gives
\[
D_{q^{-1}}^r \left( M_H^m(z; \lambda; q) \right) = \frac{q^{-rm-\frac{r(r-1)}{2}}}{(q-1)^r} \left( \frac{\sqrt{\ln \lambda}}{2} \right)^r \mathcal{W}(z; \lambda; q)_{M_{H^{m+r}}(z; \lambda; q)}.
\]  
(3.4.9)

The Rodrigues-type formula (3.4.6) for the discrete \( q \)-modified Hermite polynomials \( M_H^m(z; \lambda; q) \) can be obtained from Eq. (3.4.9) by substituting \( r = 0 \) and replacing \( r \) by \( m \) and putting the value of \( M_H^0(z; \lambda; q) = 1 \). Hence we get the result.

By using Eq. (3.4.3) and (3.4.6), the discrete \( q \)-modified Hermite polynomials can be written in the form of contour integral
\[
\mathcal{W}(z; \lambda; q)_{M_H^m(z; \lambda; q)} = -\frac{q^{m(m-1)}(q; q)_m}{{\zeta}^{m+1}} \times \left( \frac{\ln \lambda}{2} \right)^{-m} \frac{1}{2\pi i} \int_c \frac{\mathcal{W}(w; \lambda; q)}{(w/\zeta; q)_{m+1}} dw.
\]  
(3.4.10)

\[\blacksquare\]

**Theorem 3.4.2.** The discrete \( q \)-modified Hermite polynomials \( M_H^m(z; \lambda; q) \) may be represented as
\[
M_H^m(z; \lambda; q) = \left( \frac{(1-q)^2}{4ln\lambda} D_{q^{-1}}^2 q^2 \right)_{\infty} (2\zeta ln\lambda)^m
\]  
(3.4.11)

Proof. From the generating relation (3.2.1) we have
\[
\sum_{m \geq 0} \frac{M_H^m(z; \lambda; q)z^m}{m!} = \left( \frac{z^2ln\lambda; q}{q} \right)_{\infty} (2\zeta ln\lambda; q)_{\infty}^{-1}
\]
\[
= \sum_{m \geq 0} \frac{(-1)^m q^{m(m-1)}}{(q^2; q^2)_m} \left( \frac{z^2ln\lambda}{q} \right)^m (2\zeta ln\lambda; q)_{\infty}^{-1}
\]
\[
= \sum_{m \geq 0} \frac{(-1)^m q^{m(m-1)}(1-q)^{2m}}{(4ln\lambda)^m (q^2; q^2)_m} D_{q^{-1}}^{2m} (2\zeta ln\lambda; q)_{\infty}^{-1}
\]
\[
= \left( \frac{(1-q)^2}{4ln\lambda} D_{q^{-1}}^2 q^2 \right)_{\infty} (2\zeta ln\lambda; q)_{\infty}^{-1}
\]

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\[
\begin{align*}
&= \left( \frac{(1 - q)^2}{4 \ln \lambda} \mathcal{D}_q^2; q^2 \right) \sum_{m \geq 0} (2 \zeta \ln \lambda)^m (q; q)_m.
\end{align*}
\]

Identification of the coefficients of $z^m$ on both side, we get the result. ■

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