ABSTRACT

Special functions consist with a great historical background with enormous literature. In the seventeenth-century England was the birthplace of special functions. John Wallis at Oxford took two first steps towards the theory of the gamma function long before Euler reached it. Wallis had also the first encounter with elliptic integrals while using Cavalieri’s primitive forerunner of the calculus. [It is curious that two kinds of special functions encountered in the seventeenth century, Wallis’ elliptic integral and Newton’s elementary symmetric functions, belongs to the class of hypergeometric functions of several variables, which was not studied systematically nor even defined formally until the end of the nineteenth century]. A more sophisticated calculus, which made possible the real flowering of special functions, was developed by Newton at Cambridge and by Leibnitz in Germany during the period 1665-1685. Taylor’s theorem was found by Scottish mathematician Gregory in 1670, although it was not published until 1715 after rediscovery by Taylor.

In 1703 James Bernoulli solved a differential equation by an infinite series which would now be called the series representation of a Bessel function. Although Bessel functions were met by Euler and others in various mechanics problems, no systematic study of the functions was made until 1824, and the principal achievements in the eighteenth century were the gamma function and the theory of elliptic integrals. Euler found most of the major properties of the gamma functions around 1730. In 1772 Euler evaluated the Beta-function integral in terms of the gamma function. Only the duplication and multiplication theorems remained to be discovered by Legendre and Gauss, respectively, early in the next century. Other significant developments were the discovery of Vandermonde’s theorem in 1722 and the definition of Legendre polynomials and the discovery of their addition theorem by Laplace and Legendre during 1782-1785. In a slightly different form the polynomials had
already been met by Liouville in 1722.

The golden age of special functions, which was centred in nineteenth century German and France, was the result of developments in both mathematics and physics: the theory of analytic functions of a complex variable on one hand, and on the other hand, the field theories of physics (e.g. heat and electromagnetism) and their property of double periodicity was published by Abel in 1827. Elliptic functions grew up in symbiosis with the general theory of analytic functions and flourished throughout the nineteenth century, especially in the hands of Jacobi and Weierstrass.

Another major development was the theory of hypergeometric series which began in a systematic way (although some important results had been found by Euler and Pfaff) with Gauss’s memoir on the \( _2F_1 \) series in 1812, a memoir which was a landmark also on the path towards rigour in mathematics. The \( _3F_2 \) series was studied by Clausen 1928 and the \( _1F_1 \) series by Kummer 1836. The functions which Bessel considered in his memoir of 1824 are \( _0F_1 \) series; Bessel started from a problem in orbital mechanics, but the functions have found a place in every branch of mathematical physics, near the end of the century Appell 1880 introduced hypergeometric functions of two variables, and Lauricella generalized them to several variables in 1893.

The subject was considered to be part of pure mathematics in 1900, applied mathematics in 1950. In physical science special functions gained added importance as solutions of the Schrodinger equation of quantum mechanics, but there were important developments of a purely mathematical nature also. In 1907 Barnes used gamma function to develop a new theory of Gauss’s hypergeometric function \( _2F_1 \) Various generalizations of \( _2F_1 \) were introduced by Horn, Kampé de Fériet, MacRobert and Mijer. From another new view point that of a differential difference equation discussed much earlier for polynomials by Appell 1880, Truesdell 1948 made a partly successful effort at unification by fitting a number of special functions into a single framework.

In this thesis, certain classical orthogonal families are studied. The results for the classical families of special polynomials are established using matrix approach. Using
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$q$-analogue techniques, a new classical polynomial families are introduced. The characterization properties and basic hypergeometric functions of $(p, q)$-analogue are also established. Certain generalized special functions are studied through $q$-analogue techniques.

The objectives of the work presented in this thesis are:

1. The classical orthogonal polynomials families have different types of integral representation.

2. To derive the recurrence relations, differential equations and certain identities for discrete $q$-modified Hermite polynomials of type-I.

3. To derive the characterization properties and basic hypergeometric functions of $(p, q)$-analogue.

4. To introduce the analytical properties of $(q, k)$-Mittag Leffler function.

5. To study certain classical and generalized special polynomials from matrix approach.

The present thesis entitled “A Study of Polynomials and their $q$-Analogues using Classical and Fractional Calculus Techniques” is a part of research work carried out by the author during last three years. The thesis comprises of ten chapters and each chapter is subdivided into number of sections.

A brief review of some important special functions, $q$-analogues, operators, formulae, the definitions, notations and various results, that commonly arise in practice and exploration on many of their salient properties, are given in chapter-I.

Chapter II deals with different types of integral representations such as, contour integral representation, real integral representation, infinite single integral representation, finite single integral representation, finite double, triple and multiple integral representations, of
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the classical orthogonal polynomials.

In Chapter III the discrete $q$-modified Hermite polynomials is described by means of generating functions. Explicit expression, recurrence relations and three terms recurrence relations for these polynomials are given. Furthermore, Rodrigues-type formulas for the discrete $q$-modified Hermite polynomials are discussed.

In Chapter IV some new results on $(P, Q)$-analogue are acquainted by giving definition of $(P, Q)$-shifted factorials. Some new relations on negative shifted factorial are reported. Proofs of new relations for $(P, Q)$-shifted factorials and $(P, Q)$-hypergeometric functions are produced.

In Chapter V the $q$ analogue, which play significant role in number theory and combinatorics of annihilation are discussed. The objective is to present a new definition of $(q, k)$-Mittag Leffler function and discuss its various analytical properties. Certain special cases are derived, from the obtained results and their relevance with known results is established.

Chapter VI introduces modified Hermite matrix polynomials of one variable denoted by $M\mathcal{H}_n(\zeta_1, \lambda; \mathcal{A})$ and finding of some important results such as generating functions, recurrence relations, Rodrigues formula, orthogonality conditions, expansion formula, integrals, fractional integrals, fractional derivatives and some other properties of the modified Hermite matrix polynomials $M\mathcal{H}_n(\zeta_1, \lambda; \mathcal{A})$.

The main aim of Chapter VII is to investigate the polynomials $M\mathcal{H}_m(\zeta_1, \zeta_2, \lambda; \mathcal{A})$ by finding some important results such as generating functions, recurrence relations, Rodrigues formula, expansion formula, integrals, Volterra integral equation and some other properties of the modified Hermite matrix polynomials of two variables $M\mathcal{H}_m(\zeta_1, \zeta_2, \lambda; \mathcal{A})$.

In Chapter VIII, a new generalization of the modified Hermite matrix polynomials is acquainted. An explicit representation and an expansion of the matrix exponential in a series of these matrix polynomials is obtained. Some important properties of modified
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Hermite matrix polynomials such as generating functions, recurrence relations which allow us a mathematical operations. Also we drive expansion formulae and some operational representations.

Chapter IX deals with a new kind of Gegenbauer matrix polynomials and some special cases. The chapter contains a three term matrix recurrence relation, hypergeometric representation, their Rodrigues formula and orthogonal properties.

Chapter X deals with the series expansion formula, summation formula and recurrence relations for Laguerre matrix polynomials of several variables. New generating functions and several new relations between Laguerre matrix polynomials of different number of variables are also presented.

In the end an exhaustive and up to date list of original papers related to the subject matters of this thesis have been provided in the form of a bibliography.

A part of this research work has been published/accepted/communicated for publication in the form of research papers listed below:

Published Work:

Journals:


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Conferences:

1. A note on Discrete q-Modified Hermite polynomials of Type-I, IEEE conference, Nanotechnology for Instrumentation and Measurement Workshop (III International Conference) Nov 16-17, 2017 at Electrical Engineering Department, School of Engineering Gautam Buddha University, Greater Noida, Uttar Pradesh, India.

Papers Accepted for publication


Papers Communicated for publication

1. Integral Representation of Classical Orthogonal Polynomials. Thai journal of Mathematics.


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