Chapter 4

Results

4.1 4-cordial Labeling

M. Hovey[26] introduced the concept of $A$-cordial labeling as a simultaneous generalization of cordial labeling and harmonious labeling. In this chapter, 4-cordial labeling of standard graphs was discussed and analyzed and derived new results.

**Graph Labeling** is an assignment of numbers to the vertices or edges or both subject to certain conditions. Any graph labeling will have the following three characteristics:

- a set of numbers from which vertex labels are assigned;
- a rule that assigns a value to each edge;
- a condition that these values must satisfy.

**4-cordial Labeling**

Let $< A, *>$ be any Abelian group. A graph $G = (V(G), E(G))$ is said to be $A$-cordial if there is a mapping $f : V(G) \rightarrow A$ which satisfies the following two conditions when the edge $e = uv$ is labelled as $f(u) * f(v)$,

(i) $|v_f(a) - v_f(b)| \leq 1$; for all $a, b \in A$,

(ii) $|e_f(a) - e_f(b)| \leq 1$; for all $a, b \in A$.

Where,

$v_f(a) =$ the number of vertices with label $a$;
$v_f(b) =$ the number of vertices with label $b$;
$e_f(a) =$ the number of edges with label $a$;
$e_f(b) =$ the number of edges with label $b$.

If $A = < Z_k, +_k >$, that is additive group of modulo $k$ then the labeling is known as $k$-cordial labeling. A graph $G$ is $k$-cordial if it admits $k$-cordial labeling. Here consider $A = < Z_4, +_4 >$, that is additive group of modulo 4 then the labeling is known as 4-cordial labeling.
Some Existing Results of $4$-cordial and $k$-cordial Labeling

The concept of $A$-cordial labeling was introduced by Hovey[26] and proved the following results.

- All the connected graphs are $3$-cordial.
- All the trees are $3, 4, 5$-cordial.
- Cycles are $k$-cordial for all odd $k$.

In [57] and [58] Youssef obtained the following results.

- $K_n$ is $4$-cordial $\iff n \leq 6$.
- $C_n^2$ is $4$-cordial $\iff n \neq 2(\text{mod} \ 4)$.
- $K_{m,n}$ is $4$-cordial $\iff m \text{ or } n \neq 2(\text{mod} \ 4)$.
- $K_{m_1,m_2,...,m_r}$ is $4$-cordial if and only if $(m, n, p)(\text{mod} \ 4) \neq (0, 2, 2), (2, 2, 2)$, where $m(\text{mod} \ 4) \leq n(\text{mod} \ 4) \leq p(\text{mod} \ 4)$.
- $mC_{4n}$ is $4$-cordial if and only if $m$ or $n$ is even.
- $C_n^3$ is $4$-cordial if and only if $n \neq 8t + 4$, $t \geq 1$ and $n \neq 7$.
- If the ladder $L_n$ has $4$-cordial labeling $f$ such that $(f(u_n), f(v_n)) = (1, 2)$, then $L_{n+4}$ is $4$-cordial.
- $L_n$ is $4$-cordial for all $n \geq 2$.
- $M_n$ is $4$-cordial if and only if $n \neq 4(\text{mod} \ 8)$.
- $CL_n$ is $4$-cordial if and only if $n \neq 4(\text{mod} \ 8)$.
- If $L_n$ has a $4$-cordial labeling $f$ such that $(f(u_n), f(v_n)) = (1, 2)$, then so is $L_{n+2}$.
- $K_{m,n,p,r}$ is $4$-cordial if and only if $(m, n, p, r)(\text{mod} \ 4) \neq (0, 0, 2, 2)$.
- Let $G$ and $H$ be $(p_1, q_1)$ and $(p_2, q_2)$ are $4$-cordial graphs with $(q_1 \text{ and } q_2 \equiv 0(\text{mod} \ 4))$. If $(p_1 \text{ or } p_2 \neq 2(\text{mod} \ 4))$, then $G + H$ is $4$-cordial.
- $K_{m,n,p}$ is $4$-cordial if and only if $(m, n, p)(\text{mod} \ 4) \neq (0, 2, 2), (2, 2, 2)$.

Here are new results of $4$-cordial labeling.
4-cordial Labeling of Standard Graphs

**Theorem 4.1.1** The wheel $W_n$ is 4-cordial.

**Proof:** Let $G = W_n$ be the wheel. Let $v_1, v_2, ..., v_n$ be the rim vertices of and $v_0$ be the apex vertex of wheel $W_n$. It is noted that $|V(G)| = n + 1$ and $|E(G)| = 2n$.

To define 4-cordial labeling $f : V(G) \to \mathbb{Z}_4$ following cases are considered:

**Case 1:** $n \equiv 0, 1, 3, 5, 6(\text{mod } 8)$

- $f(v_0) = 0$;
- $f(v_i) = 0; \quad i \equiv 0, 4(\text{mod } 8)$;
- $f(v_i) = 1; \quad i \equiv 1, 6(\text{mod } 8)$;
- $f(v_i) = 2; \quad i \equiv 3, 7(\text{mod } 8)$;
- $f(v_i) = 3; \quad i \equiv 2, 5(\text{mod } 8); \quad 1 \leq i \leq n$.

**Case 2:** $n \equiv 2, 4(\text{mod } 8)$

- $f(v_0) = 0$;
- $f(v_1) = 1$;
- $f(v_2) = 3$;
- $f(v_3) = 2$;
- $f(v_4) = 0$;
- $f(v_i) = 0; \quad i \equiv 0, 4(\text{mod } 8)$;
- $f(v_i) = 1; \quad i \equiv 2, 5(\text{mod } 8)$;
- $f(v_i) = 2; \quad i \equiv 3, 7(\text{mod } 8)$;
- $f(v_i) = 3; \quad i \equiv 1, 6(\text{mod } 8); \quad 5 \leq i \leq n$.

**Case 3:** $n \equiv 7(\text{mod } 8)$

- $f(v_0) = 0$;
- $f(v_1) = 1$;
- $f(v_2) = 3$;
- $f(v_3) = 2$;
- $f(v_4) = 0$;
- $f(v_5) = 2$;
- $f(v_i) = 0; \quad i \equiv 1, 5(\text{mod } 8)$;
- $f(v_i) = 1; \quad i \equiv 3, 6(\text{mod } 8)$;
- $f(v_i) = 2; \quad i \equiv 0, 4(\text{mod } 8)$;
- $f(v_i) = 3; \quad i \equiv 2, 7(\text{mod } 8); \quad 6 \leq i \leq n$.

The labeling pattern defined above covers all possible arrangements of vertices. In each possibility the graph under consideration satisfies the vertex conditions and edge conditions of 4-cordial labeling as shown in table 4.1.1. That is $G$ admits 4-cordial labeling.

Let $n = 8a + b, a, b \in \mathbb{N} \cup \{0\}$. 

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<table>
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<tr>
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Table 4.1.1: 4-cordial labeling of wheel $W_n$.

**Illustration 4.1.1** The wheel $W_{10}$ and its 4-cordial labeling is shown in Figure 4.1.1.

![Figure 4.1.1: 4-cordial labeling of wheel $W_{10}$](image)

**Theorem 4.1.2** The fan $f_n$ is 4-cordial.

**Proof:** Let $G = f_n$ be the fan. Let $v_1, v_2, ..., v_n$ be the path vertices and $v_0$ be the apex vertex of fan $f_n$. It is noted that $|V(G)| = n + 1$ and $|E(G)| = 2n - 1$.

Define 4-cordial labeling $f : V(G) \to \mathbb{Z}_4$ as follows:

- $f(v_0) = 0$;
- $f(v_i) = 0; \quad i \equiv 0, 4(\text{mod } 8)$;
- $f(v_i) = 1; \quad i \equiv 2, 5(\text{mod } 8)$;
- $f(v_i) = 2; \quad i \equiv 3, 7(\text{mod } 8)$;
- $f(v_i) = 3; \quad i \equiv 1, 6(\text{mod } 8); \quad 1 \leq i \leq n$.

The labeling pattern defined above covers all possible arrangements of vertices. In each possibility the graph under consideration satisfies the vertex conditions and edge conditions of 4-cordial labeling as shown in *table 4.1.2*. That is $G$ admits 4-cordial labeling. Let $n = 8a + b, a, b \in \mathbb{N} \cup \{0\}$. 
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Table 4.1.2: 4-cordial labeling of fan $f_n$.

Illustration 4.1.2 The fan $f_8$ and its 4-cordial labeling is shown in Figure 4.1.2.

![Figure 4.1.2](image)

Figure 4.1.2: 4-cordial labeling of fan $f_8$.

Theorem 4.1.3 The friendship graph $F_n$ is 4-cordial.

Proof: Let $G = F_n$ be the friendship graph. Let $v_1, v_2, ..., v_{2n}$ be partition vertices of $n$ triangles consecutively and $v_0$ be the apex vertex of friendship graph $F_n$. It is noted that $|V(G)| = 2n + 1$ and $|E(G)| = 3n$.

To define 4-cordial labeling $f : V(G) \rightarrow \mathbb{Z}_4$ following cases are considered:

Case 1: $n \equiv 0, 1, 2, 3, 4 (\text{mod } 8)$

$f(v_0) = 0$;

$f(v_i) = 0; \quad i \equiv 5, 9, 13, 15 (\text{mod } 16)$;

$f(v_i) = 1; \quad i \equiv 1, 4, 8, 14 (\text{mod } 16)$;

$f(v_i) = 2; \quad i \equiv 2, 6, 10, 11 (\text{mod } 16)$;

$f(v_i) = 3; \quad i \equiv 0, 3, 7, 12 (\text{mod } 16); \quad 1 \leq i \leq 2n$.

Case 2: $n \equiv 5 (\text{mod } 8)$

$f(v_0) = 0$;

$f(v_i) = 0; \quad i \equiv 5, 9, 13, 15 (\text{mod } 16)$;

$f(v_i) = 1; \quad i \equiv 1, 4, 8, 14 (\text{mod } 16)$;

$f(v_i) = 2; \quad i \equiv 2, 6, 10, 11 (\text{mod } 16)$;

$f(v_i) = 3; \quad i \equiv 0, 3, 7, 12 (\text{mod } 16); \quad 1 \leq i \leq 2n - 2$.

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\[
f(v_{2n-1}) = 2; \\
f(v_{2n}) = 3.
\]

**Case 3:** \( n \equiv 6 (\text{mod } 8) \\
f(v_0) = 0; \\
f(v_i) = 0; \quad i \equiv 5, 9, 13, 15 (\text{mod } 16); \\
f(v_i) = 1; \quad i \equiv 1, 4, 8, 14 (\text{mod } 16); \\
f(v_i) = 2; \quad i \equiv 2, 6, 10, 11 (\text{mod } 16); \\
f(v_i) = 3; \quad i \equiv 0, 3, 7, 12 (\text{mod } 16); \quad 1 \leq i \leq 2n - 2; \\
f(v_{2n-1}) = 3; \\
f(v_{2n}) = 1.
\]

**Case 4:** \( n \equiv 7 (\text{mod } 8) \\
f(v_0) = 0; \\
f(v_i) = 0; \quad i \equiv 5, 9, 13, 15 (\text{mod } 16); \\
f(v_i) = 1; \quad i \equiv 1, 4, 8, 14 (\text{mod } 16); \\
f(v_i) = 2; \quad i \equiv 2, 6, 10, 11 (\text{mod } 16); \\
f(v_i) = 3; \quad i \equiv 0, 3, 7, 12 (\text{mod } 16); \quad 1 \leq i \leq 2n - 4; \\
f(v_{2n-3}) = 3; \\
f(v_{2n-2}) = 1; \\
f(v_{2n-1}) = 2; \\
f(v_{2n}) = 3.
\]

The labeling pattern defined above covers all possible arrangements of vertices. In each possibility the graph under consideration satisfies the vertex conditions and edge conditions of 4-cordial labeling as shown in table 4.1.3. That is \( G \) admits 4-cordial labeling.

Let \( n = 8a + b, \ a, b \in N \cup \{0\}. \)

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Table 4.1.3: 4-cordial labeling of friendship graph \( F_n \).
Illustration 4.1.3 The friendship graph $F_5$ and its 4-cordial labeling is shown in Figure 4.1.3.

Figure 4.1.3: 4-cordial labeling of friendship graph $F_5$.

**Theorem 4.1.4** The helm $H_n$ is 4-cordial.

**Proof:** Let $G = H_n$ be the helm. Let $v_1, v_2, ..., v_n$ be the vertices of degree 3 of $H_n$. Let $v'_1, v'_2, ..., v'_n$ be the pendant vertices and $v_0$ be the apex vertex of helm $H_n$. It is noted that $|V(G)| = 2n + 1$ and $|E(G)| = 3n$.

To define 4-cordial labeling $f : V(G) \rightarrow \mathbb{Z}_4$ following cases are considered:

**Case 1:** $n \equiv 0, 5 (mod 8)$
- $f(v_0) = 0$;
- $f(v_i) = 0$; $i \equiv 0, 4 (mod 8)$;
- $f(v_i) = 1$; $i \equiv 1, 6 (mod 8)$;
- $f(v_i) = 2$; $i \equiv 3, 7 (mod 8)$;
- $f(v_i) = 3$; $i \equiv 2, 5 (mod 8)$; $1 \leq i \leq n$,
- $f(v'_i) = 0$; $i \equiv 2, 6 (mod 8)$;
- $f(v'_i) = 1$; $i \equiv 1, 5 (mod 8)$;
- $f(v'_i) = 2$; $i \equiv 4, 7 (mod 8)$;
- $f(v'_i) = 3$; $i \equiv 0, 3 (mod 8)$; $1 \leq i \leq n$.

**Case 2:** $n \equiv 1, 3, 6 (mod 8)$
- $f(v_0) = 0$;
- $f(v_i) = 0$; $i \equiv 0, 4 (mod 8)$;
- $f(v_i) = 1$; $i \equiv 1, 6 (mod 8)$;
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\[ f(v_i) = 2; \quad i \equiv 3, 7 (\text{mod } 8); \]
\[ f(v_i) = 3; \quad i \equiv 2, 5 (\text{mod } 8); \quad 1 \leq i \leq n, \]
\[ f(v_i') = 0; \quad i \equiv 2, 6 (\text{mod } 8); \]
\[ f(v_i') = 1; \quad i \equiv 1, 5 (\text{mod } 8); \]
\[ f(v_i') = 2; \quad i \equiv 4, 7 (\text{mod } 8); \]
\[ f(v_i') = 3; \quad i \equiv 0, 3 (\text{mod } 8); \quad 1 \leq i \leq n - 1, \]
\[ f(v_{n'}) = 2. \]

**Case 3:** \( n \equiv 2 (\text{mod } 8) \)

\[ f(v_0) = 0; \]
\[ f(v_1) = 1; \]
\[ f(v_2) = 3; \]
\[ f(v_3) = 2; \]
\[ f(v_4) = 2; \]
\[ f(v_i) = 0; \quad i \equiv 0, 4 (\text{mod } 8); \]
\[ f(v_i) = 1; \quad i \equiv 2, 5 (\text{mod } 8); \]
\[ f(v_i) = 2; \quad i \equiv 3, 7 (\text{mod } 8); \]
\[ f(v_i) = 3; \quad i \equiv 1, 6 (\text{mod } 8); \quad 5 \leq i \leq n, \]
\[ f(v_i') = 0; \quad i \equiv 2, 6 (\text{mod } 8); \]
\[ f(v_i') = 1; \quad i \equiv 1, 5 (\text{mod } 8); \]
\[ f(v_i') = 2; \quad i \equiv 4, 7 (\text{mod } 8); \]
\[ f(v_i') = 3; \quad i \equiv 0, 3 (\text{mod } 8); \quad 1 \leq i \leq n - 2, \]
\[ f(v_{n-1'}) = 2; \]
\[ f(v_{n'}) = 0. \]

**Case 4:** \( n \equiv 4 (\text{mod } 8) \)

\[ f(v_0) = 0; \]
\[ f(v_1) = 1; \]
\[ f(v_2) = 3; \]
\[ f(v_3) = 2; \]
\[ f(v_4) = 2; \]
\[ f(v_i) = 0; \quad i \equiv 0, 4 (\text{mod } 8); \]
\[ f(v_i) = 1; \quad i \equiv 2, 5 (\text{mod } 8); \]
\[ f(v_i) = 2; \quad i \equiv 3, 7 (\text{mod } 8); \]
\[ f(v_i) = 3; \quad i \equiv 1, 6 (\text{mod } 8); \quad 5 \leq i \leq n, \]
\[ f(v_i') = 0; \quad i \equiv 2, 6 (\text{mod } 8); \]
\[ f(v_i') = 1; \quad i \equiv 1, 5 (\text{mod } 8); \]
\[ f(v_i') = 2; \quad i \equiv 4, 7 (\text{mod } 8); \]
\[ f(v_i') = 3; \quad i \equiv 0, 3 (\text{mod } 8); \quad 1 \leq i \leq n. \]
Case 5: $n \equiv 7 (\text{mod } 8)$.

$f(v_0) = 0$;
$f(v_1) = 1$;
$f(v_2) = 3$;
$f(v_3) = 2$;
$f(v_4) = 0$;
$f(v_5) = 2$;
$f(v_i) = 0; \quad i \equiv 1, 5 (\text{mod } 8)$;
$f(v_i) = 1; \quad i \equiv 3, 6 (\text{mod } 8)$;
$f(v_i) = 2; \quad i \equiv 0, 4 (\text{mod } 8)$;
$f(v_i) = 3; \quad 6 \leq i \leq n$;
$f(v')_i = 1$;
$f(v'_2) = 2$;
$f(v'_3) = 3$;
$f(v'_4) = 3$;
$f(v'_5) = 2$;
$f(v'_i) = 0; \quad i \equiv 3, 7 (\text{mod } 8)$;
$f(v'_i) = 1; \quad i \equiv 2, 6 (\text{mod } 8)$;
$f(v'_i) = 2; \quad i \equiv 1, 4 (\text{mod } 8)$;
$f(v'_i) = 3; \quad 6 \leq i \leq n$.

The labeling pattern defined above covers all possible arrangements of vertices. In each possibility the graph under consideration satisfies the vertex conditions and edge conditions of 4-cordial labeling as shown in table 4.1.4. That is $G$ admits 4-cordial labeling.

Let $n = 8a + b, a, b \in N \cup \{0\}$.

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<td>$e_f(0) = e_f(1) = e_f(2) = e_f(3)$</td>
</tr>
<tr>
<td>1</td>
<td>$v_f(0) = v_f(1) = v_f(2) = v_f(3) + 1$</td>
<td>$e_f(0) + 1 = e_f(1) = e_f(2) = e_f(3)$</td>
</tr>
<tr>
<td>2</td>
<td>$v_f(0) + 1 = v_f(1) + 1 = v_f(2) = v_f(3) + 1$</td>
<td>$e_f(0) + 1 = e_f(1) = e_f(2) + 1 = e_f(3)$</td>
</tr>
<tr>
<td>3</td>
<td>$v_f(0) = v_f(1) = v_f(2) = v_f(3) + 1$</td>
<td>$e_f(0) + 1 = e_f(1) + 1 = e_f(2) + 1 = e_f(3)$</td>
</tr>
<tr>
<td>4</td>
<td>$v_f(0) + 1 = v_f(1) + 1 = v_f(2) = v_f(3) + 1$</td>
<td>$e_f(0) = e_f(1) = e_f(2) = e_f(3)$</td>
</tr>
<tr>
<td>5</td>
<td>$v_f(0) = v_f(1) = v_f(2) + 1 = v_f(3)$</td>
<td>$e_f(0) = e_f(1) + 1 = e_f(2) = e_f(3)$</td>
</tr>
<tr>
<td>6</td>
<td>$v_f(0) + 1 = v_f(1) = v_f(2) + 1 = v_f(3) + 1$</td>
<td>$e_f(0) + 1 = e_f(1) + 1 = e_f(2) = e_f(3)$</td>
</tr>
<tr>
<td>7</td>
<td>$v_f(0) + 1 = v_f(1) = v_f(2) = v_f(3)$</td>
<td>$e_f(0) + 1 = e_f(1) + 1 = e_f(2) = e_f(3) + 1$</td>
</tr>
</tbody>
</table>

Table 4.1.4: 4-cordial labeling of helm $H_n$. 
Illustration 4.1.4 The helm $H_{13}$ and its 4-cordial labeling is shown in Figure 4.1.4.

![Figure 4.1.4: 4-cordial labeling of the helm $H_{13}$.

Theorem 4.1.5 The flower graph $Fl_n$ is 4-cordial.

Proof: Let $G = Fl_n$ be the flower graph. Let $v_0$ be the apex vertex and $v_1, v_2, ..., v_n$ be the $n$ vertices of cycle and $v'_1, v'_2, ..., v'_n$ be the vertices of degree 2. It is noted that $|V(G)| = 2n + 1$ and $|E(G)| = 4n$.

To define 4-cordial labeling $f : V(G) \rightarrow \mathbb{Z}_4$ following cases are considered:

Case 1: $n \equiv 0, 5 \pmod{8}$

$f(v_0) = 0$;

$f(v_i) = 0; \quad i \equiv 0, 4 \pmod{8}$;

$f(v_i) = 1; \quad i \equiv 1, 6 \pmod{8}$;

$f(v_i) = 2; \quad i \equiv 3, 7 \pmod{8}$;

$f(v_i) = 3; \quad i \equiv 2, 5 \pmod{8}, \quad 1 \leq i \leq n$,

$f(v'_i) = 0; \quad i \equiv 2, 6 \pmod{8}$;

$f(v'_i) = 1; \quad i \equiv 1, 5 \pmod{8}$;

$f(v'_i) = 2; \quad i \equiv 4, 7 \pmod{8}$;

$f(v'_i) = 3; \quad i \equiv 3, 0 \pmod{8}, \quad 1 \leq i \leq n$.

Case 2: $n \equiv 1 \pmod{8}$

$f(v_0) = 0$;
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\( f(v_i) = 0; \quad i \equiv 0, 4(\text{mod } 8); \)
\( f(v_i) = 1; \quad i \equiv 1, 6(\text{mod } 8); \)
\( f(v_i) = 2; \quad i \equiv 3, 7(\text{mod } 8); \)
\( f(v_i) = 3; \quad i \equiv 2, 5(\text{mod } 8), \quad 1 \leq i \leq n, \)
\( f(v'_i) = 0; \quad i \equiv 2, 6(\text{mod } 8); \)
\( f(v'_i) = 1; \quad i \equiv 1, 5(\text{mod } 8); \)
\( f(v'_i) = 2; \quad i \equiv 4, 7(\text{mod } 8); \)
\( f(v'_i) = 3; \quad i \equiv 3, 0(\text{mod } 8), \quad 1 \leq i \leq n - 1, \)
\( f(v'_n) = 3. \)

**Case 3:** \( n \equiv 2(\text{mod } 8) \)
\( f(v_0) = 0; \)
\( f(v_i) = 0; \quad i \equiv 0, 4(\text{mod } 8); \)
\( f(v_i) = 1; \quad i \equiv 1, 6(\text{mod } 8); \)
\( f(v_i) = 2; \quad i \equiv 3, 7(\text{mod } 8); \)
\( f(v_i) = 3; \quad i \equiv 2, 5(\text{mod } 8), \quad 1 \leq i \leq n - 2, \)
\( f(v_{n-1}) = 3, \)
\( f(v_n) = 1. \)
\( f(v'_i) = 0; \quad i \equiv 2, 6(\text{mod } 8); \)
\( f(v'_i) = 1; \quad i \equiv 1, 5(\text{mod } 8); \)
\( f(v'_i) = 2; \quad i \equiv 4, 7(\text{mod } 8); \)
\( f(v'_i) = 3; \quad i \equiv 3, 0(\text{mod } 8), \quad 1 \leq i \leq n - 2, \)
\( f(v'_{n-1}) = 2, \)
\( f(v'_n) = 0. \)

**Case 4:** \( n \equiv 3(\text{mod } 8) \)
\( f(v_0) = 0; \)
\( f(v_i) = 0; \quad i \equiv 0, 4(\text{mod } 8); \)
\( f(v_i) = 1; \quad i \equiv 1, 6(\text{mod } 8); \)
\( f(v_i) = 2; \quad i \equiv 3, 7(\text{mod } 8); \)
\( f(v_i) = 3; \quad i \equiv 2, 5(\text{mod } 8), \quad 1 \leq i \leq n, \)
\( f(v_i) = 0; \quad i \equiv 2, 6(\text{mod } 8); \)
\( f(v'_i) = 1; \quad i \equiv 1, 5(\text{mod } 8); \)
\( f(v'_i) = 2; \quad i \equiv 4, 7(\text{mod } 8); \)
\( f(v'_i) = 3; \quad i \equiv 3, 0(\text{mod } 8), \quad 1 \leq i \leq n - 1, \)
\( f(v'_n) = 2. \)

**Case 5:** \( n \equiv 4(\text{mod } 8) \)
\( f(v_0) = 0; \)
\( f(v_i) = 0; \quad i \equiv 0, 4(\text{mod } 8); \)
\( f(v_i) = 1; \quad i \equiv 1, 6(\text{mod } 8); \)
Case 6: $n \equiv 6(\text{mod } 8)$
\[
\begin{align*}
f(v_0) &= 0; \\
f(v_i) &= 0; \quad i \equiv 0, 4(\text{mod } 8); \\
f(v_i) &= 1; \quad i \equiv 1, 6(\text{mod } 8); \\
f(v_i) &= 2; \quad i \equiv 3, 7(\text{mod } 8); \\
f(v_i) &= 3; \quad i \equiv 2, 5(\text{mod } 8), \quad 1 \leq i \leq n, \\
f(v'_i) &= 0; \quad i \equiv 2, 6(\text{mod } 8); \\
f(v'_i) &= 1; \quad i \equiv 1, 5(\text{mod } 8); \\
f(v'_i) &= 2; \quad i \equiv 0, 3(\text{mod } 8); \\
f(v'_i) &= 3; \quad i \equiv 4, 7(\text{mod } 8), \quad 1 \leq i \leq n - 2, \\
f(v'_{n-1}) &= 2, \\
f(v'_n) &= 0.
\end{align*}
\]

Case 7: $n \equiv 7(\text{mod } 8)$
\[
\begin{align*}
f(v_0) &= 0; \\
f(v_i) &= 0; \quad i \equiv 0, 4(\text{mod } 8); \\
f(v_i) &= 1; \quad i \equiv 1, 6(\text{mod } 8); \\
f(v_i) &= 2; \quad i \equiv 3, 7(\text{mod } 8); \\
f(v_i) &= 3; \quad i \equiv 2, 5(\text{mod } 8), \quad 1 \leq i \leq n - 3, \\
f(v_{n-2}) &= 2, \\
f(v_{n-1}) &= 1, \\
f(v_n) &= 3. \\
f(v'_i) &= 0; \quad i \equiv 2, 6(\text{mod } 8); \\
f(v'_i) &= 1; \quad i \equiv 1, 5(\text{mod } 8); \\
f(v'_i) &= 2; \quad i \equiv 0, 3(\text{mod } 8); \\
f(v'_i) &= 3; \quad i \equiv 4, 7(\text{mod } 8), \quad 1 \leq i \leq n - 1, \\
f(v'_n) &= 2.
\end{align*}
\]

The labeling pattern defined above covers all possible arrangements of vertices. In each possibility the graph under consideration satisfies the vertex conditions and edge conditions of 4-cordial labeling as shown in table 4.1.5. That is $G$ admits 4-cordial labeling.

Let $n = 8a + b, a, b \in N \cup \{0\}$. 

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### Table 4.1.5: 4-cordial labeling of flower graph $F_l_n$.  

<table>
<thead>
<tr>
<th>$b$</th>
<th>Vertex conditions</th>
<th>Edge conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>0,2,6</td>
<td>$v_f(0) = v_f(1) + 1 = v_f(2) + 1 = v_f(3) + 1$</td>
<td>$e_f(0) = e_f(1) = e_f(2) = e_f(3)$</td>
</tr>
<tr>
<td>1,5</td>
<td>$v_f(0) = v_f(1) = v_f(2) + 1 = v_f(3)$</td>
<td>$e_f(0) = e_f(1) = e_f(2) = e_f(3)$</td>
</tr>
<tr>
<td>4</td>
<td>$v_f(0) + 1 = v_f(1) + 1 = v_f(2) = v_f(3) + 1$</td>
<td>$e_f(0) = e_f(1) = e_f(2) = e_f(3)$</td>
</tr>
<tr>
<td>3,7</td>
<td>$v_f(0) = v_f(1) = v_f(2) = v_f(3) + 1$</td>
<td>$e_f(0) = e_f(1) = e_f(2) = e_f(3)$</td>
</tr>
</tbody>
</table>

**Illustration 4.1.5** The flower graph $F_l_5$ and its 4-cordial labeling is shown in Figure 4.1.5.

![Figure 4.1.5: 4-cordial labeling of flower graph $F_l_5$.](image)

**Theorem 4.1.6** The gear graph $G_n$ is 4-cordial.

**Proof:** Let $G = G_n$ be the gear graph. Let $v_1, v_2, \ldots, v_{2n}$ be the rim vertices and $v_0$ be the apex vertex of gear graph $G_n$. It is noted that $|V(G)| = 2n + 1$ and $|E(G)| = 3n$.

To define 4-cordial labeling $f : V(G) \to Z_4$ following cases are considered:

**Case 1:** $n \equiv 0, 3, 4, 7 (\text{mod } 8)$
- $f(v_0) = 0$
- $f(v_i) = 0; \quad i \equiv 4, 7 (\text{mod } 8)$
- $f(v_i) = 1; \quad i \equiv 2, 5 (\text{mod } 8)$
- $f(v_i) = 2; \quad i \equiv 0, 3 (\text{mod } 8)$
- $f(v_i) = 3; \quad i \equiv 1, 6 (\text{mod } 8); \quad 1 \leq i \leq 2n$.

**Case 2:** $n \equiv 1, 5 (\text{mod } 8)$
- $f(v_0) = 0$
- $f(v_i) = 0; \quad i \equiv 4, 7 (\text{mod } 8)$
- $f(v_i) = 1; \quad i \equiv 2, 5 (\text{mod } 8)$
- $f(v_i) = 2; \quad i \equiv 0, 3 (\text{mod } 8)$
- $f(v_i) = 3; \quad i \equiv 1, 6 (\text{mod } 8); \quad 1 \leq i \leq 2n - 2$. 

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\[ f(v_{2n-1}) = 1; \]
\[ f(v_{2n}) = 3. \]

**Case 3**: \( n \equiv 2, 6 \text{(mod 8)} \)

\[ f(v_0) = 0; \]
\[ f(v_i) = 0; \quad i \equiv 4, 7 \text{(mod 8)}; \]
\[ f(v_i) = 1; \quad i \equiv 2, 5 \text{(mod 8)}; \]
\[ f(v_i) = 2; \quad i \equiv 0, 3 \text{(mod 8)}; \]
\[ f(v_i) = 3; \quad i \equiv 1, 6 \text{(mod 8)}; \quad 1 \leq i \leq 2n - 4, \]
\[ f(v_{2n-3}) = 1; \]
\[ f(v_{2n-2}) = 3; \]
\[ f(v_{2n-1}) = 2; \]
\[ f(v_{2n}) = 0. \]

The labeling pattern defined above covers all possible arrangements of vertices. In each possibility the graph under consideration satisfies the vertex conditions and edge conditions of 4-cordial labeling as shown in table 4.1.6. That is \( G \) admits 4-cordial labeling.

Let \( n = 8a + b, a, b \in N \cup \{0\} \).

<table>
<thead>
<tr>
<th>( b )</th>
<th>Vertex conditions</th>
<th>Edge conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>0,4</td>
<td>( v_f(0) = v_f(1) + 1 = v_f(2) + 1 = v_f(3) + 1 )</td>
<td>( e_f(0) = e_f(1) = e_f(2) = e_f(3) )</td>
</tr>
<tr>
<td>1,5</td>
<td>( v_f(0) = v_f(1) = v_f(2) + 1 = v_f(3) )</td>
<td>( e_f(0) = e_f(1) + 1 = e_f(2) = e_f(3) )</td>
</tr>
<tr>
<td>2,6</td>
<td>( v_f(0) = v_f(1) + 1 = v_f(2) + 1 = v_f(3) + 1 )</td>
<td>( e_f(0) + 1 = e_f(1) + 1 = e_f(2) = e_f(3) )</td>
</tr>
<tr>
<td>3,7</td>
<td>( v_f(0) = v_f(1) = v_f(2) + 1 = v_f(3) )</td>
<td>( e_f(0) + 1 = e_f(1) + 1 = e_f(2) = e_f(3) + 1 )</td>
</tr>
</tbody>
</table>

Table 4.1.6: 4-cordial labeling of gear graph \( G_n \).

**Illustration 4.1.6** The gear graph \( G_7 \) and its 4-cordial labeling is shown in Figure 4.1.6.

![Figure 4.1.6: 4-cordial labeling of gear graph G7.](image-url)
Theorem 4.1.7 The double fan $DF_n$ is 4-cordial.

Proof: Let $G = DF_n$ be the double fan graph. Let $v_1, v_2, ..., v_n$ are vertices of common path and $u_1, u_2$ be the apex vertices. It is noted that $|V(G)| = n + 2$ and $|E(G)| = 3n - 1$.

To define 4-cordial labeling $f : V(G) \rightarrow \mathbb{Z}_4$ following cases are considered:

Case 1: $n \equiv 0, 3, 4, 5 (\text{mod } 8)$

$f(u_1) = 0$;
$f(u_2) = 1$;
$f(v_i) = 0; \quad i \equiv 0, 4 (\text{mod } 8)$;
$f(v_i) = 1; \quad i \equiv 1, 6 (\text{mod } 8)$;
$f(v_i) = 2; \quad i \equiv 3, 7 (\text{mod } 8)$;
$f(v_i) = 3; \quad i \equiv 2, 5 (\text{mod } 8); \quad 1 \leq i \leq n$.

Case 2: $n \equiv 1, 7 (\text{mod } 8)$

$f(u_1) = 0$;
$f(u_2) = 3$;
$f(v_i) = 0; \quad i \equiv 0, 4 (\text{mod } 8)$;
$f(v_i) = 1; \quad i \equiv 1, 6 (\text{mod } 8)$;
$f(v_i) = 2; \quad i \equiv 3, 7 (\text{mod } 8)$;
$f(v_i) = 3; \quad i \equiv 2, 5 (\text{mod } 8); \quad 1 \leq i \leq n$.

Case 3: $n \equiv 2 (\text{mod } 8)$

$f(u_1) = 0$;
$f(u_2) = 3$;
$f(v_i) = 0; \quad i \equiv 0, 4 (\text{mod } 8)$;
$f(v_i) = 1; \quad i \equiv 1, 6 (\text{mod } 8)$;
$f(v_i) = 2; \quad i \equiv 3, 7 (\text{mod } 8)$;
$f(v_i) = 3; \quad i \equiv 2, 5 (\text{mod } 8); \quad 1 \leq i \leq n - 1$,

$f(v_n) = 2$.

Case 4: $n \equiv 6 (\text{mod } 8)$

$f(u_1) = 0$;
$f(u_2) = 1$;
$f(v_i) = 0; \quad i \equiv 0, 4 (\text{mod } 8)$;
$f(v_i) = 1; \quad i \equiv 1, 6 (\text{mod } 8)$;
$f(v_i) = 2; \quad i \equiv 3, 7 (\text{mod } 8)$;
$f(v_i) = 3; \quad i \equiv 2, 5 (\text{mod } 8); \quad 1 \leq i \leq n - 1$,

$f(v_n) = 2$. 
The labeling pattern defined above covers all possible arrangements of vertices. In each possibility the graph under consideration satisfies the vertex conditions and edge conditions of 4-cordial labeling as shown in table 4.1.7. That is $G$ admits 4-cordial labeling.

Let $n = 8a + b$, $a, b \in N \cup \{0\}$.

<table>
<thead>
<tr>
<th>b</th>
<th>Vertex conditions</th>
<th>Edge conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$v_f(0) = v_f(1) = v_f(2) + 1 = v_f(3) + 1$</td>
<td>$e_f(0) = e_f(1) + 1 = e_f(2) = e_f(3)$</td>
</tr>
<tr>
<td>1</td>
<td>$v_f(0) = v_f(1) = v_f(2) + 1 = v_f(3)$</td>
<td>$e_f(0) = e_f(1) = e_f(2) + 1 = e_f(3) + 1$</td>
</tr>
<tr>
<td>2</td>
<td>$v_f(0) = v_f(1) = v_f(2) = v_f(3)$</td>
<td>$e_f(0) + 1 = e_f(1) = e_f(2) + 1 = e_f(3) + 1$</td>
</tr>
<tr>
<td>3</td>
<td>$v_f(0) + 1 = v_f(1) = v_f(2) + 1 = v_f(3) + 1$</td>
<td>$e_f(0) = e_f(1) = e_f(2) = e_f(3)$</td>
</tr>
<tr>
<td>4</td>
<td>$v_f(0) = v_f(1) = v_f(2) + 1 = v_f(3) + 1$</td>
<td>$e_f(0) = e_f(1) = e_f(2) = e_f(3) + 1$</td>
</tr>
<tr>
<td>5</td>
<td>$v_f(0) = v_f(1) = v_f(2) + 1 = v_f(3)$</td>
<td>$e_f(0) = e_f(1) + 1 = e_f(2) + 1 = e_f(3)$</td>
</tr>
<tr>
<td>6</td>
<td>$v_f(0) = v_f(1) = v_f(2) = v_f(3)$</td>
<td>$e_f(0) + 1 = e_f(1) + 1 = e_f(2) + 1 = e_f(3)$</td>
</tr>
<tr>
<td>7</td>
<td>$v_f(0) + 1 = v_f(1) + 1 = v_f(2) + 1 = v_f(3)$</td>
<td>$e_f(0) = e_f(1) = e_f(2) = e_f(3)$</td>
</tr>
</tbody>
</table>

Table 4.1.7: 4-cordial labeling of double fan $DF_n$.

Illustration 4.1.7 The double fan $DF_9$ and its 4-cordial labeling is shown in Figure 4.1.7.

![Figure 4.1.7: 4-cordial labeling of the double fan $DF_9$.](image)

Theorem 4.1.8 The double wheel graph $DW_n$ is 4-cordial.

Proof: Let $G = DW_n$ be the Double wheel graph. Let $v_1, v_2, ..., v_n$ be the $n$ vertices of first wheel and $v'_1, v'_2, ..., v'_n$ be the vertices of other wheel. Let $v_0$ be the apex vertex, which
is common for both the wheels. It is noted that $|V(G)| = 2n + 1$ and $|E(G)| = 4n$.
To define 4-cordial labeling $f : V(G) \rightarrow \mathbb{Z}_4$ following cases are considered:

**Case 1: $n \equiv 0, 4(mod\ 8)$**

$f(v_0) = 0;$
$f(v_i) = 0; \quad i \equiv 0, 4(mod\ 8);$
$f(v_i) = 1; \quad i \equiv 1, 6(mod\ 8);$  
$f(v_i) = 2; \quad i \equiv 3, 7(mod\ 8);$  
$f(v_i) = 3; \quad i \equiv 2, 5(mod\ 8), \quad 1 \leq i \leq n,$
$f(v'_i) = 0; \quad i \equiv 3, 7(mod\ 8);$  
$f(v'_i) = 1; \quad i \equiv 1, 6(mod\ 8);$  
$f(v'_i) = 2; \quad i \equiv 4, 0(mod\ 8);$  
$f(v'_i) = 3; \quad i \equiv 2, 5(mod\ 8), \quad 1 \leq i \leq n.$

**Case 2: $n \equiv 1(mod\ 8)$**

$f(v_0) = 0;$
$f(v_i) = 0; \quad i \equiv 0, 4(mod\ 8);$  
$f(v_i) = 1; \quad i \equiv 1, 6(mod\ 8);$  
$f(v_i) = 2; \quad i \equiv 3, 7(mod\ 8);$  
$f(v_i) = 3; \quad i \equiv 2, 5(mod\ 8), \quad 1 \leq i \leq n,$
$f(v'_i) = 0; \quad i \equiv 3, 7(mod\ 8);$  
$f(v'_i) = 1; \quad i \equiv 1, 6(mod\ 8);$  
$f(v'_i) = 2; \quad i \equiv 4, 0(mod\ 8);$  
$f(v'_i) = 3; \quad i \equiv 2, 5(mod\ 8), \quad 1 \leq i \leq n - 3,$
$f(v'_{n-2}) = 2,$
$f(v'_{n-1}) = 2,$
$f(v'_n) = 3.$

**Case 3: $n \equiv 2(mod\ 8)$**

$f(v_0) = 0;$
$f(v_i) = 0; \quad i \equiv 0, 4(mod\ 8);$  
$f(v_i) = 1; \quad i \equiv 1, 6(mod\ 8);$  
$f(v_i) = 2; \quad i \equiv 3, 7(mod\ 8);$  
$f(v_i) = 3; \quad i \equiv 2, 5(mod\ 8), \quad 1 \leq i \leq n - 5,$
$f(v_{n-4}) = 2,$
$f(v_{n-3}) = 2.$
$f(v_{n-2}) = 1,$
$f(v_{n-1}) = 2,$
$f(v_n) = 0.$
$f(v'_i) = 0; \quad i \equiv 3, 7(mod\ 8);$  
$f(v'_i) = 1; \quad i \equiv 1, 6(mod\ 8);$
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\[ f(v'_i) = 2; \quad i \equiv 4, 0 \text{(mod 8)}; \]
\[ f(v'_i) = 3; \quad i \equiv 2, 5 \text{(mod 8), } \quad 1 \leq i \leq n. \]

**Case 4: \( n \equiv 3 \text{(mod 8)} \)**

\[ f(v_0) = 0; \]
\[ f(v_i) = 0; \quad i \equiv 0, 4 \text{(mod 8)}; \]
\[ f(v_i) = 1; \quad i \equiv 1, 6 \text{(mod 8)}; \]
\[ f(v_i) = 2; \quad i \equiv 3, 7 \text{(mod 8)}; \]
\[ f(v_i) = 3; \quad i \equiv 2, 5 \text{(mod 8), } \quad 1 \leq i \leq n - 1, \]
\[ f(v_n) = 3. \]
\[ f(v'_i) = 0; \quad i \equiv 3, 7 \text{(mod 8)}; \]
\[ f(v'_i) = 1; \quad i \equiv 1, 6 \text{(mod 8)}; \]
\[ f(v'_i) = 2; \quad i \equiv 4, 0 \text{(mod 8)}; \]
\[ f(v'_i) = 3; \quad i \equiv 2, 5 \text{(mod 8), } \quad 1 \leq i \leq n - 2, \]
\[ f(v'_{n-1}) = 2, \]
\[ f(v'_{n}) = 0. \]

**Case 5: \( n \equiv 5 \text{(mod 8)} \)**

\[ f(v_0) = 0; \]
\[ f(v_i) = 0; \quad i \equiv 0, 4 \text{(mod 8)}; \]
\[ f(v_i) = 1; \quad i \equiv 1, 6 \text{(mod 8)}; \]
\[ f(v_i) = 2; \quad i \equiv 3, 7 \text{(mod 8)}; \]
\[ f(v_i) = 3; \quad i \equiv 2, 5 \text{(mod 8), } \quad 1 \leq i \leq n, \]
\[ f(v'_i) = 0; \quad i \equiv 3, 7 \text{(mod 8)}; \]
\[ f(v'_i) = 1; \quad i \equiv 1, 6 \text{(mod 8)}; \]
\[ f(v'_i) = 2; \quad i \equiv 4, 0 \text{(mod 8)}; \]
\[ f(v'_i) = 3; \quad i \equiv 2, 5 \text{(mod 8), } \quad 1 \leq i \leq n - 3, \]
\[ f(v'_{n-2}) = 2, \]
\[ f(v'_{n-1}) = 2, \]
\[ f(v'_{n}) = 1. \]

**Case 6: \( n \equiv 6 \text{(mod 8)} \)**

\[ f(v_0) = 0; \]
\[ f(v_i) = 0; \quad i \equiv 0, 4 \text{(mod 8)}; \]
\[ f(v_i) = 1; \quad i \equiv 1, 6 \text{(mod 8)}; \]
\[ f(v_i) = 2; \quad i \equiv 3, 7 \text{(mod 8)}; \]
\[ f(v_i) = 3; \quad i \equiv 2, 5 \text{(mod 8), } \quad 1 \leq i \leq n - 2, \]
\[ f(v_{n-1}) = 1, \]
\[ f(v_n) = 2. \]
\[ f(v'_i) = 0; \quad i \equiv 3, 7 \text{(mod 8)}; \]
\[ f(v'_i) = 1; \quad i \equiv 1, 6 \text{(mod 8)}; \]
\[ f(v_i') = 2; \quad i \equiv 4, 0 (\text{mod } 8); \]
\[ f(v_i') = 3; \quad i \equiv 2, 5 (\text{mod } 8), \quad 1 \leq i \leq n - 2. \]
\[ f(v_{n-1}') = 2, \]
\[ f(v_n') = 3. \]

**Case 7:** \( n \equiv 7 (\text{mod } 8) \)
\[ f(v_0) = 0; \]
\[ f(v_i) = 0; \quad i \equiv 0, 4 (\text{mod } 8); \]
\[ f(v_i) = 1; \quad i \equiv 1, 6 (\text{mod } 8); \]
\[ f(v_i) = 2; \quad i \equiv 3, 7 (\text{mod } 8); \]
\[ f(v_i) = 3; \quad i \equiv 1, 6 (\text{mod } 8), \quad 1 \leq i \leq n - 3, \]
\[ f(v_{n-2}) = 2, \]
\[ f(v_{n-1}) = 1, \]
\[ f(v_n) = 3. \]
\[ f(v_i') = 0; \quad i \equiv 3, 7 (\text{mod } 8); \]
\[ f(v_i') = 1; \quad i \equiv 1, 6 (\text{mod } 8); \]
\[ f(v_i') = 2; \quad i \equiv 4, 0 (\text{mod } 8); \]
\[ f(v_i') = 3; \quad i \equiv 2, 5 (\text{mod } 8), \quad 1 \leq i \leq n - 2. \]
\[ f(v_{n-1}') = 0, \]
\[ f(v_n') = 1. \]

The labeling pattern defined above covers all possible arrangements of vertices. In each possibility the graph under consideration satisfies the vertex conditions and edge conditions of 4-cordial labeling as shown in table 4.1.8. That is \( G \) admits 4-cordial labeling.

Let \( n = 8a + b, \ a, b \in N \cup \{0\} \).

<table>
<thead>
<tr>
<th>( b )</th>
<th>Vertex conditions</th>
<th>Edge conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>0,4</td>
<td>( v_f(0) = v_f(1) + 1 = v_f(2) + 1 = v_f(3) + 1 )</td>
<td>( e_f(0) = e_f(1) = e_f(2) = e_f(3) )</td>
</tr>
<tr>
<td>1,5</td>
<td>( v_f(0) + 1 = v_f(1) = v_f(2) = v_f(3) )</td>
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</tr>
<tr>
<td>2,6</td>
<td>( v_f(0) + 1 = v_f(1) + 1 = v_f(2) = v_f(3) + 1 )</td>
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<tr>
<td>3,7</td>
<td>( v_f(0) = v_f(1) = v_f(2) + 1 = v_f(3) )</td>
<td>( e_f(0) = e_f(1) = e_f(2) = e_f(3) )</td>
</tr>
</tbody>
</table>

Table 4.1.8: 4-cordial labeling of double wheel graph \( DW_n \).
Illustration 4.1.8 The double wheel graph $D_{W_6}$ and its 4-cordial labeling is shown in Figure 4.1.8.

Figure 4.1.8: 4-cordial labeling of double wheel graph $D_{W_6}$.

Theorem 4.1.9 The triangular book graph $B(3, n)$ with $n$-pages is 4-cordial.

Proof: Let $G = B(3, n)$ be the triangular book graph. Let $v_1, v_2, ..., v_n$ be the vertices of $n$-pages of triangular book graph. Let $u$ and $v$ be the spine vertices. It is noted that $|V(G)| = n + 2$ and $|E(G)| = 2n + 1$.

Define 4-cordial labeling $f : V(G) \to Z_4$ as follows:

- $f(u) = 0$ and $f(v) = 3$.
- $f(v_i) = 0; \quad i \equiv 0(\text{mod } 4)$;
- $f(v_i) = 1; \quad i \equiv 1(\text{mod } 4)$;
- $f(v_i) = 2; \quad i \equiv 2(\text{mod } 4)$;
- $f(v_i) = 3; \quad i \equiv 3(\text{mod } 4), \quad 0 \leq i \leq n$.

The labeling pattern defined above covers all possible arrangements of vertices. In each possibility the graph under consideration satisfies the vertex conditions and edge conditions of 4-cordial labeling as shown in table 4.1.9. That is $G$ admits 4-cordial labeling.

Let $n = 4a + b, a, b \in N \cup \{0\}$.

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<td>$v_f(0) = v_f(1) + 1 = v_f(2) + 1 = v_f(3)$</td>
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<tr>
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</tr>
</tbody>
</table>

Table 4.1.9: 4-cordial labeling of triangular book graph $B(3, n)$.
Illustration 4.1.9 The triangular book graph $B(3, 7)$ and its 4-cordial labeling is shown in Figure 4.1.9.

Figure 4.1.9: 4-cordial labeling of triangular book graph $B(3, 7)$.

Theorem 4.1.10 The triangular snake $TS_n$ is 4-cordial.

Proof: Let $G = TS_n$ be the triangular snake obtained from the path $P_n$. Let $v_1, v_2, ..., v_n$ be the vertices of the path $P_n$ and $v_1', v_2', ..., v_{n-1}'$ be the newly added vertices to form $G$. It is noted that $|V(G)| = 2n - 1$ and $|E(G)| = 3n - 3$.

Define 4-cordial labeling $f : V(G) \to \mathbb{Z}_4$ as follows:

- $f(v_i) = 1; \quad i \equiv 3, 7\,(mod\,8)$
- $f(v_i) = 2; \quad i \equiv 1, 5\,(mod\,8)$
- $f(v_i) = 3; \quad i \equiv 0, 2, 4, 6\,(mod\,8); \quad 1 \leq i \leq n$
- $f(v'_i) = 0; \quad i \equiv 0, 1, 4, 5\,(mod\,8)$
- $f(v'_i) = 1; \quad i \equiv 2, 6\,(mod\,8)$
- $f(v'_i) = 2; \quad i \equiv 3, 7\,(mod\,8); \quad 1 \leq i \leq n - 1$

The labeling pattern defined above covers all possible arrangements of vertices. In each possibility the graph under consideration satisfies the vertex conditions and edge conditions of 4-cordial labeling as shown in table 4.1.10. That is $G$ admits 4-cordial labeling.

Let $n = 8a + b, a, b \in N \cup \{0\}$.

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</table>

Table 4.1.10: 4-cordial labeling of triangular snake $TS_n$. 

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Illustration 4.1.10 The triangular snake $TS_5$ obtained from path $P_5$ and its 4-cordial labeling is shown in Figure 4.1.10.

![4-cordial labeling of triangular snake $TS_5$ obtained from the path $P_5$.](image)

Figure 4.1.10: 4-cordial labeling of triangular snake $TS_5$ obtained from the path $P_5$.

Theorem 4.1.11 The irregular quadrilateral snake $IQ(S_n)$ is 4-cordial for all $n$.

Proof: Let $G = IQ(S_n)$ be the irregular quadrilateral snake obtained from the path $P_n$. Let $u_1, u_2, ..., u_n$ be the vertices of path $P_n$ and $v_1, v_2, ..., v_{n-2}$ & $w_1, w_2, ..., w_{n-2}$ be the newly added vertices. To find irregular quadrilateral snake join the vertices $u_i v_i, w_i u_{i+2}$ and $v_i w_i$ for all $1 \leq i \leq n - 2$. It is noted that $|V(G)| = 3n - 4$ and $|E(G)| = 4n - 7.$

Define 4-cordial labeling $f : V(G) \rightarrow \mathbb{Z}_4$ as follows:

- $f(u_i) = 0; \quad i \equiv 0 (mod 4)$;
- $f(u_i) = 1; \quad i \equiv 2 (mod 4)$;
- $f(u_i) = 2; \quad i \equiv 3 (mod 4)$;
- $f(u_i) = 3; \quad i \equiv 1 (mod 4)$; $1 \leq i \leq n$,
- $f(v_i) = 0; \quad i \equiv 3 (mod 4)$;
- $f(v_i) = 1; \quad i \equiv 1 (mod 4)$;
- $f(v_i) = 2; \quad i \equiv 0, 2 (mod 4)$; $1 \leq i \leq n - 2$,
- $f(w_i) = 0; \quad i \equiv 1 (mod 4)$;
- $f(w_i) = 1; \quad i \equiv 3 (mod 4)$;
- $f(w_i) = 3; \quad i \equiv 0, 2 (mod 4)$; $1 \leq i \leq n - 2$.

The labeling pattern defined above covers all possible arrangements of vertices. In each possibility the graph under consideration satisfies the vertex conditions and edge conditions of 4-cordial labeling as shown in table 4.1.11. That is $G$ admits 4-cordial labeling.

Let $n = 8a + b$, $a, b \in N \cup \{0\}$. 

Chapter 4. Results

NEHA B. RATHOD


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</tr>
</tbody>
</table>

Table 4.1.11: 4-cordial labeling of irregular quadrilateral snake \( IQ(S_n) \).

**Illustration 4.1.11** The irregular quadrilateral snake \( IQ(S_{11}) \) and its 4-cordial labeling is shown in Figure 4.1.11.

![Diagram](image)

Figure 4.1.11: 4-cordial labeling of irregular quadrilateral snake \( IQ(S_{11}) \).

**Theorem 4.1.12** The triangular belt \( TB_n \) is 4-cordial for all \( n \).

**Proof:** Let \( G = TB_n \) be the triangular belt obtained from the ladder \( L_n = P_n \times P_2 \) \((n \geq 2)\) by adding the edges \( u_i v_{i+1} \) for all \( 1 \leq i \leq n - 1 \), where the consecutive vertices of two copies of paths are \( v_1, v_2, \ldots, v_n \) and \( u_1, u_2, \ldots, u_n \) and the edges are \( u_i v_i \). It is noted that \(|V(G)| = 2n\) and \(|E(G)| = 4n - 3\).

Define 4-cordial labeling \( f : V(G) \rightarrow Z_4 \) as follows:

\[
\begin{align*}
f(u_i) &= 1; \quad i \equiv 0, 2 \pmod{4}; \\
f(u_i) &= 3; \quad i \equiv 1, 3 \pmod{4}; \\
f(v_i) &= 0; \quad i \equiv 0, 2 \pmod{4}; \\
f(v_i) &= 2; \quad i \equiv 1, 3 \pmod{4};
\end{align*}
\]

1 \leq i \leq n,

The labeling pattern defined above covers all possible arrangements of vertices. In each possibility the graph under consideration satisfies the vertex conditions and edge conditions of 4-cordial labeling as shown in Table 4.1.12. That is \( G \) admits 4-cordial labeling.

Let \( n = 2a + b, a, b \in N \cup \{0\} \).
Chapter 4. Results

Table 4.1.12: 4-cordial labeling of triangular belt $TB_n$.

\begin{tabular}{|c|c|c|}
\hline
b & Vertex conditions & Edge conditions \\
\hline
0 & $v_f(0) = v_f(1) = v_f(2) = v_f(3)$ & $e_f(0) + 1 = e_f(1) = e_f(2) + 1 = e_f(3) + 1$ \\
1 & $v_f(0) + 1 = v_f(1) + 1 = v_f(2) = v_f(3)$ & $e_f(0) + 1 = e_f(1) = e_f(2) + 1 = e_f(3) + 1$ \\
\hline
\end{tabular}

Illustration 4.1.12 The triangular belt $TB_6$ and its 4-cordial labeling is shown in Figure 4.1.12.

Figure 4.1.12: 4-cordial labeling of triangular belt $TB_6$.

Theorem 4.1.13 The graph $Z-P_n$ is 4-cordial for all $n$.

Proof: Let $G=Z-P_n$ be the graph obtained from the pair of paths $P_n'$ and $P_n''$. Let $v_1, v_2, ..., v_n$ be the vertices of path $P_n'$ and $u_1, u_2, ..., u_n$ are the vertices of path $P_n''$. To find $Z-P_n$ join $i^{th}$ vertex of path $P_n'$ with $(i+1)^{th}$ vertex of path $P_n''$ for all $1 \leq i \leq n-1$. It is noted that $|V(G)| = 2n$ and $|E(G)| = 3n - 3$.

Define 4-cordial labeling $f : V(G) \rightarrow Z_4$ as follows:

\begin{align*}
  f(v_i) &= 0; \quad i \equiv 1, 6\,(mod\,8); \\
  f(v_i) &= 1; \quad i \equiv 4, 7\,(mod\,8); \\
  f(v_i) &= 2; \quad i \equiv 2, 5\,(mod\,8); \\
  f(v_i) &= 3; \quad i \equiv 0, 3\,(mod\,8), \quad 1 \leq i \leq n, \\
  f(u_i) &= 0; \quad i \equiv 4, 7\,(mod\,8); \\
  f(u_i) &= 1; \quad i \equiv 2, 5\,(mod\,8); \\
  f(u_i) &= 2; \quad i \equiv 0, 3\,(mod\,8); \\
  f(u_i) &= 3; \quad i \equiv 1, 6\,(mod\,8), \quad 1 \leq i \leq n.
\end{align*}

The labeling pattern defined above covers all possible arrangements of vertices. In each possibility the graph under consideration satisfies the vertex conditions and edge conditions of 4-cordial labeling as shown in table 4.1.13. That is $G$ admits 4-cordial labeling.

Let $n = 8a + b, a, b \in N \cup \{0\}$. 

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Table 4.1.13: 4-cordial labeling of graph $Z-P_n$.

Illustration 4.1.13 The graph $Z-P_6$ and its 4-cordial labeling is shown in Figure 4.1.13.

Figure 4.1.13: 4-cordial labeling of $Z-P_6$.

Theorem 4.1.14 The braid graph $B(n)$ is 4-cordial for all $n$.

Proof: Let $G = B(n)$ be the braid graph obtained from a pair of paths $P_n'$ and $P_n''$. Let $u_1, u_2, ..., u_n$ be the vertices of path $P_n'$ and $v_1, v_2, ..., v_n$ are the vertices of path $P_n''$. To find braid graph $B(n)$ join $i^{th}$ vertex of path $P_n'$ with $(i+1)^{th}$ vertex of path $P_n''$ and $i^{th}$ vertex of path $P_n''$ with $(i+2)^{th}$ vertex of path $P_n'$ with the new edges for all $1 \leq i \leq n-2$.

It is noted that $|V(G)| = 2n$ and $|E(G)| = 4n - 5$.

To define 4-cordial labeling $f : V(G) \rightarrow Z_4$ following cases are considered:

Case 1: If $n \leq 3$.

- $f(u_1) = 1; \quad i \equiv 2, 3(\text{mod } 4)$;
- $f(u_i) = 3; \quad i \equiv 1(\text{mod } 4); \quad 1 \leq i \leq 3$,
- $f(v_i) = 0; \quad i \equiv 1(\text{mod } 4)$;
- $f(v_i) = 2; \quad i \equiv 2, 3(\text{mod } 4); \quad 1 \leq i \leq 3$.

Case 2: If $n \geq 4$.

- $f(u_1) = 3$;
The labeling pattern defined above covers all possible arrangements of vertices. In each possibility the graph under consideration satisfies the vertex conditions and edge conditions of 4-cordial labeling as shown in table 4.1.14. That is $G$ admits 4-cordial labeling.

Let $n = 4a + b$, $a, b \in N \cup \{0\}$.

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<tr>
<th>$b$</th>
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<tbody>
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<td>0,2</td>
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<tr>
<td>1,3</td>
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<td>$e_f(0) = e_f(1) = e_f(2) = e_f(3) + 1$</td>
</tr>
</tbody>
</table>

Table 4.1.14: 4-cordial labeling of braid graph $B(n)$.

**Illustratoin 4.1.14** The braid graph $B(3)$ and its 4-cordial labeling is shown in Figure 4.1.14.

![Image of 4-cordial labeling of braid graph $B(3)$](image-url)
Illustration 4.1.15 The braid graph \( B(7) \) and its 4-cordial labeling is shown in Figure 4.1.15.

![Figure 4.1.15: 4-cordial labeling of braid graph \( B(7) \).](image)

Theorem 4.1.15 The middle graph \( M(P_n) \) of the path \( P_n \) is 4-cordial.

Proof: Let \( G = M(P_n) \) be the middle graph of the path \( P_n \). Let \( v_1, v_2, \ldots, v_n \) be vertices of path \( P_n \) and \( v'_1, v'_2, \ldots, v'_{n-1} \) be the newly added vertices corresponding to the edges \( e_1, e_2, \ldots, e_{n-1} \) to form \( G \). It is noted that \( |V(G)| = 2n - 1 \) and \( |E(G)| = 3n - 4 \).

Define 4-cordial labeling \( f : V(G) \rightarrow \mathbb{Z}_4 \) as follows:

\[
\begin{align*}
 f(v_1) &= 3; \\
 f(v_i) &= 0; \quad i \equiv 1, 2, 5, 6 \pmod{8}; \\
 f(v_i) &= 1; \quad i \equiv 3, 7 \pmod{8}; \\
 f(v_i) &= 2; \quad i \equiv 0, 4 \pmod{8}; \quad 2 \leq i \leq n, \\
 f(v'_i) &= 1; \quad i \equiv 3, 7 \pmod{8}; \\
 f(v'_i) &= 2; \quad i \equiv 1, 5 \pmod{8}; \\
 f(v'_i) &= 3; \quad i \equiv 0, 2, 4, 6 \pmod{8}; \quad 1 \leq i \leq n - 1.
\end{align*}
\]

The labeling pattern defined above covers all possible arrangements of vertices. In each possibility the graph under consideration satisfies the vertex conditions and edge conditions of 4-cordial labeling as shown in Table 4.1.15. That is \( G \) admits 4-cordial labeling.

Let \( n = 8a + b, a, b \in N \cup \{0\} \).

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</tbody>
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Table 4.1.15: 4-cordial labeling of middle graph \( M(P_n) \) of the path \( P_n \).
Illustration 4.1.16 The middle graph \( M(P_3) \) and its 4-cordial labeling is shown in Figure 4.1.16.

![Illustration of middle graph and 4-cordial labeling](image)

**Figure 4.1.16:** 4-cordial labeling of middle graph \( M(P_3) \) of the path \( P_3 \).

Theorem 4.1.16 The total graph \( T(P_n) \) of the path \( P_n \) is 4-cordial.

**Proof:** Let \( G = T(P_n) \) be the total graph of the path \( P_n \). Let \( v_1, v_2, ..., v_n \) be the vertices of the path \( P_n \) and \( e_1, e_2, ..., e_{n-1} \) be the \( n-1 \) edges. Let \( v'_1, v'_2, ..., v'_{n-1} \) be the newly added vertices corresponding to the edges \( e_1, e_2, ..., e_{n-1} \) to form \( G \). It is noted that \( |V(G)| = 2n - 1 \) and \( |E(G)| = 4n - 5 \).

Define 4-cordial labeling \( f : V(G) \to \mathbb{Z}_4 \) as follows:

\[
\begin{align*}
f(v_i) &= 1; \quad i \equiv 3, 7 \pmod{8}; \\
f(v_i) &= 2; \quad i \equiv 1, 5 \pmod{8}; \\
f(v_i) &= 3; \quad i \equiv 0, 2, 4, 6 \pmod{8}; \quad 1 \leq i \leq n, \\
f(v'_i) &= 0; \quad i \equiv 0, 1, 4, 5 \pmod{8}; \\
f(v'_i) &= 1; \quad i \equiv 2, 6 \pmod{8}; \\
f(v'_i) &= 2; \quad i \equiv 3, 7 \pmod{8}; \quad 1 \leq i \leq n - 1.
\end{align*}
\]

The labeling pattern defined above covers all possible arrangements of vertices. In each possibility the graph under consideration satisfies the vertex conditions and edge conditions of 4-cordial labeling as shown in Table 4.1.16. That is \( G \) admits 4-cordial labeling.

Let \( n = 8a + b, \ a, b \in N \cup \{0\} \).

<table>
<thead>
<tr>
<th>( b )</th>
<th>Vertex conditions</th>
<th>Edge conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>0,4</td>
<td>( v_f(0) + 1 = v_f(1) = v_f(2) = v_f(3) )</td>
<td>( e_f(0) = e_f(1) = e_f(2) + 1 = e_f(3) )</td>
</tr>
<tr>
<td>1,5</td>
<td>( v_f(0) + 1 = v_f(1) + 1 = v_f(2) = v_f(3) + 1 )</td>
<td>( e_f(0) + 1 = e_f(1) = e_f(2) = e_f(3) )</td>
</tr>
<tr>
<td>2,6</td>
<td>( v_f(0) = v_f(1) + 1 = v_f(2) = v_f(3) )</td>
<td>( e_f(0) + 1 = e_f(1) = e_f(2) = e_f(3) )</td>
</tr>
<tr>
<td>3,7</td>
<td>( v_f(0) + 1 = v_f(1) = v_f(2) + 1 = v_f(3) + 1 )</td>
<td>( e_f(0) = e_f(1) = e_f(2) = e_f(3) + 1 )</td>
</tr>
</tbody>
</table>

**Table 4.1.16:** 4-cordial labeling of total graph \( T(P_n) \) of the path \( P_n \).
**Illustration 4.1.17** The total graph $T(P_4)$ of the path $P_4$ and its 4-cordial labeling is shown in Figure 4.1.17.

![Diagram of the total graph $T(P_4)$](image)

Figure 4.1.17: 4-cordial labeling of total graph $T(P_4)$ of the path $P_4$.

**Theorem 4.1.17** The splitting graph $S'(P_n)$ of the path $P_n$ is 4-cordial.

**Proof:** Let $G = S'(P_n)$ be the splitting graph of the path $P_n$. Let $v_1, v_2, ..., v_n$ be the vertices of path $P_n$ and $v'_1, v'_2, ..., v'_n$ be the newly added vertices corresponding to the vertices $v_1, v_2, ..., v_n$ to form $G$. It is noted that $|V(G)| = 2n$ and $|E(G)| = 3n - 3$.

Define 4-cordial labeling $f: V(G) \to \mathbb{Z}_4$ as follows:

\[
\begin{align*}
&f(v_i) = 0; \quad i \equiv 2, 3, 6, 7 \pmod{8}; \\
&f(v_i) = 2; \quad i \equiv 0, 4 \pmod{8}; \\
&f(v_i) = 3; \quad i \equiv 1, 5 \pmod{8}; \quad 1 \leq i \leq n, \\
&f(v'_i) = 1; \quad i \equiv 2, 3, 6, 7 \pmod{8}; \\
&f(v'_i) = 2; \quad i \equiv 1, 5 \pmod{8}; \\
&f(v'_i) = 3; \quad i \equiv 0, 4 \pmod{8}; \quad 1 \leq i \leq n.
\end{align*}
\]

The labeling pattern defined above covers all possible arrangements of vertices. In each possibility the graph under consideration satisfies the vertex conditions and edge conditions of 4-cordial labeling as shown in table 4.1.17. That is $G$ admits 4-cordial labeling.

Let $n = 8a + b$, $a, b \in \mathbb{N} \cup \{0\}$.

<table>
<thead>
<tr>
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<td>$v_f(0) = v_f(1) = v_f(2) = v_f(3)$</td>
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</tr>
<tr>
<td>1,5</td>
<td>$v_f(0) + 1 = v_f(1) + 1 = v_f(2) = v_f(3)$</td>
<td>$e_f(0) = e_f(1) = e_f(2) = e_f(3)$</td>
</tr>
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<td>2,6</td>
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</tr>
<tr>
<td>3,7</td>
<td>$v_f(0) = v_f(1) = v_f(2) + 1 = v_f(3) + 1$</td>
<td>$e_f(0) = e_f(1) = e_f(2) + 1 = e_f(3) + 1$</td>
</tr>
</tbody>
</table>

Table 4.1.17: 4-cordial labeling of splitting graph $S'(P_n)$ of the path $P_n$. 
Illustration 4.1.18: The splitting graph $S'(P_7)$ of the path $P_7$ and its 4-cordial labeling is shown in Figure 4.1.18.

Theorem 4.1.18: The square graph $P_n^2$ of the path $P_n$ is 4-cordial.

Proof: Let $G = P_n^2$ be the square of path $P_n$ with vertices $v_1, v_2, ..., v_n$. It is noted that $|V(G)| = n$ and $|E(G)| = 2n - 3$.

Define 4-cordial labeling $f : V(G) \rightarrow \mathbb{Z}_4$ as follows:

- $f(v_i) = 0; \quad i \equiv 1, 5 (mod\ 8)$
- $f(v_i) = 1; \quad i \equiv 2, 6 (mod\ 8)$
- $f(v_i) = 2; \quad i \equiv 3, 7 (mod\ 8)$
- $f(v_i) = 3; \quad i \equiv 0, 4 (mod\ 8); \quad 1 \leq i \leq n$

The labeling pattern defined above covers all possible arrangements of vertices. In each possibility the graph under consideration satisfies the vertex conditions and edge conditions of 4-cordial labeling as shown in Table 4.1.18. That is $G$ admits 4-cordial labeling.

Let $n = 8a + b$, $a, b \in \mathbb{N} \cup \{0\}$.

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<tr>
<td>0.4</td>
<td>$v_f(0) = v_f(1) = v_f(2) = v_f(3)$</td>
<td>$e_f(0) + 1 = e_f(1) = e_f(2) + 1 = e_f(3) + 1$</td>
</tr>
<tr>
<td>1.5</td>
<td>$v_f(0) = v_f(1) + 1 = v_f(2) + 1 = v_f(3) + 1$</td>
<td>$e_f(0) + 1 = e_f(1) = e_f(2) = e_f(3)$</td>
</tr>
<tr>
<td>2.6</td>
<td>$v_f(0) = v_f(1) + v_f(2) = v_f(3) + 1$</td>
<td>$e_f(0) + 1 = e_f(1) = e_f(2) + 1 = e_f(3) + 1$</td>
</tr>
<tr>
<td>3.7</td>
<td>$v_f(0) = v_f(1) = v_f(2) = v_f(3) + 1$</td>
<td>$e_f(0) + 1 = e_f(1) = e_f(2) = e_f(3)$</td>
</tr>
</tbody>
</table>

Table 4.1.18: 4-cordial labeling of square graph $P_n^2$ of the path $P_n$.
Illustration 4.1.19 The square graph $P_6^2$ of the path $P_6$ and its 4-cordial labeling is shown in Figure 4.1.19.

Figure 4.1.19: 4-cordial labeling of the square graph $P_6^2$ of the path $P_6$.

Theorem 4.1.19 The middle graph $M(C_n)$ of cycle $C_n$ is 4-cordial.

Proof: Let $G = M(C_n)$ be the middle graph of cycle $C_n$ and $v_1, v_2, ..., v_n$ be the vertices of $C_n$. Let $v'_1, v'_2, ..., v'_{n}$ be the newly added vertices corresponding to the edges $e_1, e_2, ..., e_n$ respectively to obtain $M(C_n)$. It is noted that $|V(G)| = 2n$ and $|E(G)| = 3n$.

To define 4-cordial labeling $f : V(G) \rightarrow Z_4$ following cases are considered:

Case 1: $n \equiv 0 (\text{mod } 8)$
- $f(v_i) = 0; \quad i \equiv 4, 7 (\text{mod } 8)$;
- $f(v_i) = 1; \quad i \equiv 2, 5 (\text{mod } 8)$;
- $f(v_i) = 2; \quad i \equiv 3, 0 (\text{mod } 8)$;
- $f(v_i) = 3; \quad i \equiv 1, 6 (\text{mod } 8), \quad 1 \leq i \leq n$;
- $f(v'_i) = 0; \quad i \equiv 2 (\text{mod } 4)$;
- $f(v'_i) = 1; \quad i \equiv 1 (\text{mod } 4)$;
- $f(v'_i) = 2; \quad i \equiv 0 (\text{mod } 4)$;
- $f(v'_i) = 3; \quad i \equiv 3 (\text{mod } 4), \quad 1 \leq i \leq n$.

Case 2: $n \equiv 1 (\text{mod } 8)$
- $f(v_i) = 0; \quad i \equiv 4, 7 (\text{mod } 8)$;
- $f(v_i) = 1; \quad i \equiv 2, 5 (\text{mod } 8)$;
- $f(v_i) = 2; \quad i \equiv 3, 0 (\text{mod } 8)$;
- $f(v_i) = 3; \quad i \equiv 1, 6 (\text{mod } 8), \quad 1 \leq i \leq n$;
- $f(v'_i) = 0; \quad i \equiv 2 (\text{mod } 4)$;
- $f(v'_i) = 1; \quad i \equiv 1 (\text{mod } 4)$;
- $f(v'_i) = 2; \quad i \equiv 0 (\text{mod } 4)$;
- $f(v'_i) = 3; \quad i \equiv 3 (\text{mod } 4), \quad 1 \leq i \leq n - 2$;
- $f(v'_{n-1}) = 1$. 

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Case 3: \( n \equiv 2 \pmod{8} \)
\[
\begin{align*}
f(v_i) &= 0; \quad i \equiv 4, 7 \pmod{8}; \\
f(v_i) &= 1; \quad i \equiv 2, 5 \pmod{8}; \\
f(v_i) &= 2; \quad i \equiv 3, 0 \pmod{8}; \\
f(v_i) &= 3; \quad i \equiv 1, 6 \pmod{8}, \quad 1 \leq i \leq n - 1, \\
f(v_n) &= 2.
\end{align*}
\]
\[
\begin{align*}
f(v'_i) &= 0; \quad i \equiv 2 \pmod{4}; \\
f(v'_i) &= 1; \quad i \equiv 1 \pmod{4}; \\
f(v'_i) &= 2; \quad i \equiv 0 \pmod{4}; \\
f(v'_i) &= 3; \quad i \equiv 3 \pmod{4}, \quad 1 \leq i \leq n,
\end{align*}
\]

Case 4: \( n \equiv 3, 5 \pmod{8} \)
\[
\begin{align*}
f(v_i) &= 0; \quad i \equiv 4, 7 \pmod{8}; \\
f(v_i) &= 1; \quad i \equiv 2, 5 \pmod{8}; \\
f(v_i) &= 2; \quad i \equiv 3, 0 \pmod{8}; \\
f(v_i) &= 3; \quad i \equiv 1, 6 \pmod{8}, \quad 1 \leq i \leq n, \\
f(v_n) &= 2.
\end{align*}
\]
\[
\begin{align*}
f(v'_i) &= 0; \quad i \equiv 2 \pmod{4}; \\
f(v'_i) &= 1; \quad i \equiv 1 \pmod{4}; \\
f(v'_i) &= 2; \quad i \equiv 0 \pmod{4}; \\
f(v'_i) &= 3; \quad i \equiv 3 \pmod{4}, \quad 1 \leq i \leq n - 1,
\end{align*}
\]

Case 5: \( n \equiv 4 \pmod{8} \)
\[
\begin{align*}
f(v_i) &= 0; \quad i \equiv 4, 7 \pmod{8}; \\
f(v_i) &= 1; \quad i \equiv 2, 5 \pmod{8}; \\
f(v_i) &= 2; \quad i \equiv 3, 0 \pmod{8}; \\
f(v_i) &= 3; \quad i \equiv 1, 6 \pmod{8}, \quad 1 \leq i \leq n - 2, \\
f(v_{n-1}) &= 3, \\
f(v_n) &= 0. \\
f(v'_i) &= 0; \quad i \equiv 2 \pmod{4}; \\
f(v'_i) &= 1; \quad i \equiv 1 \pmod{4}; \\
f(v'_i) &= 2; \quad i \equiv 0 \pmod{4}; \\
f(v'_i) &= 3; \quad i \equiv 3 \pmod{4}, \quad 1 \leq i \leq n - 2.
\end{align*}
\]
\[
\begin{align*}
f(v'_{n-1}) &= 2, \\
f(v'_n) &= 2.
\end{align*}
\]

Case 6: \( n \equiv 6 \pmod{8} \)
\[
\begin{align*}
f(v_i) &= 0; \quad i \equiv 4, 7 \pmod{8};
\end{align*}
\]
\[ f(v_i) = 1; \quad i \equiv 2, 5 \mod 8; \]
\[ f(v_i) = 2; \quad i \equiv 3, 0 \mod 8; \]
\[ f(v_i) = 3; \quad i \equiv 1, 6 \mod 8, \quad 1 \leq i \leq n - 2, \]
\[ f(v_{n-1}) = 3, \]
\[ f(v_n) = 1. \]
\[ f(v'_i) = 0; \quad i \equiv 2 \mod 4; \]
\[ f(v'_i) = 1; \quad i \equiv 1 \mod 4; \]
\[ f(v'_i) = 2; \quad i \equiv 0 \mod 4; \]
\[ f(v'_i) = 3; \quad i \equiv 3 \mod 4, \quad 1 \leq i \leq n - 2. \]
\[ f(v'_{n-1}) = 0, \]
\[ f(v'_n) = 2. \]

Case 7: \( n \equiv 7 \mod 8 \)
\[ f(v_i) = 0; \quad i \equiv 4, 7 \mod 8; \]
\[ f(v_i) = 1; \quad i \equiv 2, 5 \mod 8; \]
\[ f(v_i) = 2; \quad i \equiv 3, 0 \mod 8; \]
\[ f(v_i) = 3; \quad i \equiv 1, 6 \mod 8, \quad 1 \leq i \leq n - 1, \]
\[ f(v_n) = 1. \]
\[ f(v'_i) = 0; \quad i \equiv 2 \mod 4; \]
\[ f(v'_i) = 1; \quad i \equiv 1 \mod 4; \]
\[ f(v'_i) = 2; \quad i \equiv 0 \mod 4; \]
\[ f(v'_i) = 3; \quad i \equiv 3 \mod 4, \quad 1 \leq i \leq n - 3. \]
\[ f(v'_{n-2}) = 2, \]
\[ f(v'_{n-1}) = 0, \]
\[ f(v'_n) = 3. \]

The labeling pattern defined above covers all possible arrangements of vertices. In each possibility the graph under consideration satisfies the vertex conditions and edge conditions of 4-cordial labeling as shown in Table 4.1.19. That is \( G \) admits 4-cordial labeling.

Let \( n = 8a + b, a, b \in N \cup \{0\} \).

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<td>( e_f(0) = e_f(1) = e_f(2) = e_f(3) )</td>
</tr>
<tr>
<td>1</td>
<td>( v_f(0) + 1 = v_f(1) = v_f(2) + 1 = v_f(3) )</td>
<td>( e_f(0) + 1 = e_f(1) = e_f(2) = e_f(3) )</td>
</tr>
<tr>
<td>2,6</td>
<td>( v_f(0) = v_f(1) = v_f(2) = v_f(3) )</td>
<td>( e_f(0) + 1 = e_f(1) = e_f(2) + 1 = e_f(3) )</td>
</tr>
<tr>
<td>3</td>
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</tr>
<tr>
<td>5</td>
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<td>( e_f(0) + 1 = e_f(1) = e_f(2) = e_f(3) )</td>
</tr>
<tr>
<td>7</td>
<td>( v_f(0) + 1 = v_f(1) = v_f(2) + 1 = v_f(3) )</td>
<td>( e_f(0) = e_f(1) + 1 = e_f(2) + 1 = e_f(3) + 1 )</td>
</tr>
</tbody>
</table>

Table 4.1.19: 4-cordial labeling of middle graph \( M(C_n) \) of cycle \( C_n \).
Illustration 4.1.20 The middle graph $M(C_5)$ and its 4-cordial labeling is shown in Figure 4.1.20.

Figure 4.1.20: 4-cordial labeling of middle graph $M(C_5)$ of cycle $C_5$.

Theorem 4.1.20 The splitting graph $S'(C_n)$ of cycle $C_n$ is 4-cordial.

Proof: Let $G = S'(C_n)$ be the splitting graph of cycle $C_n$ and $v_1, v_2, ..., v_n$ be the $n$ vertices of $C_n$. Let $G$ be the graph obtained by adding to each vertex $v$ a new vertex $v'$ such that $v'$ is adjacent to every vertex that is adjacent to $v$ in $G$. It is noted that $|V(G)| = 2n$ and $|E(G)| = 3n$.

To define 4-cordial labeling $f : V(G) \rightarrow \mathbb{Z}_4$ following cases are considered:

Case 1: $n \equiv 0(\text{mod } 8)$
\[
\begin{align*}
  f(v_i) &= 0; & i &\equiv 4, 7(\text{mod } 8); \\
  f(v_i) &= 1; & i &\equiv 2, 5(\text{mod } 8); \\
  f(v_i) &= 2; & i &\equiv 3, 0(\text{mod } 8); \\
  f(v_i) &= 3; & i &\equiv 1, 6(\text{mod } 8), & 1 \leq i \leq n, \\
  f(v'_i) &= 0; & i &\equiv 4, 7(\text{mod } 8); \\
  f(v'_i) &= 1; & i &\equiv 2, 5(\text{mod } 8); \\
  f(v'_i) &= 2; & i &\equiv 3, 0(\text{mod } 8); \\
  f(v'_i) &= 3; & i &\equiv 1, 6(\text{mod } 8), & 1 \leq i \leq n.
\end{align*}
\]

Case 2: $n \equiv 1(\text{mod } 8)$
\[
\begin{align*}
  f(v_i) &= 0; & i &\equiv 4, 7(\text{mod } 8); \\
  f(v_i) &= 1; & i &\equiv 2, 5(\text{mod } 8); \\
  f(v_i) &= 2; & i &\equiv 3, 0(\text{mod } 8); \\
  f(v_i) &= 3; & i &\equiv 1, 6(\text{mod } 8), & 1 \leq i \leq n - 1, \\
  f(v_n) &= 2.
\end{align*}
\]
Chapter 4. Results

\[ f(v_i) = 0; \quad i \equiv 4, 7 \pmod{8}; \]
\[ f(v_i) = 1; \quad i \equiv 2, 5 \pmod{8}; \]
\[ f(v_i) = 2; \quad i \equiv 3, 0 \pmod{8}; \]
\[ f(v_i) = 3; \quad i \equiv 1, 6 \pmod{8}, \]
\[ f(v_{n-1}) = 1, \]
\[ f(v_n) = 3. \]

**Case 3:** \( n \equiv 2 \pmod{8} \)

\[ f(v_i) = 0; \quad i \equiv 4, 7 \pmod{8}; \]
\[ f(v_i) = 1; \quad i \equiv 2, 5 \pmod{8}; \]
\[ f(v_i) = 2; \quad i \equiv 3, 0 \pmod{8}; \]
\[ f(v_i) = 3; \quad i \equiv 1, 6 \pmod{8}, \]
\[ f(v_{n-1}) = 1, \]
\[ f(v_n) = 2. \]

**Case 4:** \( n \equiv 3 \pmod{8} \)

\[ f(v_i) = 0; \quad i \equiv 4, 7 \pmod{8}; \]
\[ f(v_i) = 1; \quad i \equiv 2, 5 \pmod{8}; \]
\[ f(v_i) = 2; \quad i \equiv 3, 0 \pmod{8}; \]
\[ f(v_i) = 3; \quad i \equiv 1, 6 \pmod{8}, \]
\[ f(v_{n-1}) = 0, \]
\[ f(v_n) = 1. \]

**Case 5:** \( n \equiv 4 \pmod{8} \)

\[ f(v_i) = 0; \quad i \equiv 4, 7 \pmod{8}; \]
\[ f(v_i) = 1; \quad i \equiv 2, 5 \pmod{8}; \]
\[ f(v_i) = 2; \quad i \equiv 3, 0 \pmod{8}; \]
\[ f(v_i) = 3; \quad i \equiv 1, 6 \pmod{8}, \]
\[ f(v_{n-1}) = 3. \]
Case 6: \( n \equiv 5(\text{mod } 8) \)
\[ f(v_i) = \begin{cases} 
0, & \text{if } i \equiv 4, 7 \text{ (mod } 8) \\
1, & \text{if } i \equiv 2, 5 \text{ (mod } 8) \\
2, & \text{if } i \equiv 3, 0 \text{ (mod } 8) \\
3, & \text{if } i \equiv 1, 6 \text{ (mod } 8), \quad 1 \leq i \leq n - 1,
\end{cases} \]
\[ f(v_n) = 2. \]
\[ f(v_i') = \begin{cases} 
0, & \text{if } i \equiv 4, 7 \text{ (mod } 8) \\
1, & \text{if } i \equiv 2, 5 \text{ (mod } 8) \\
2, & \text{if } i \equiv 3, 0 \text{ (mod } 8) \\
3, & \text{if } i \equiv 1, 6 \text{ (mod } 8), \quad 1 \leq i \leq n - 2,
\end{cases} \]
\[ f(v_{n-1}') = 3, \quad f(v_n') = 0. \]

Case 7: \( n \equiv 6(\text{mod } 8) \)
\[ f(v_i) = \begin{cases} 
0, & \text{if } i \equiv 4, 7 \text{ (mod } 8) \\
1, & \text{if } i \equiv 2, 5 \text{ (mod } 8) \\
2, & \text{if } i \equiv 3, 0 \text{ (mod } 8) \\
3, & \text{if } i \equiv 1, 6 \text{ (mod } 8), \quad 1 \leq i \leq n - 1,
\end{cases} \]
\[ f(v_n) = 2. \]
\[ f(v_i') = \begin{cases} 
0, & \text{if } i \equiv 4, 7 \text{ (mod } 8) \\
1, & \text{if } i \equiv 2, 5 \text{ (mod } 8) \\
2, & \text{if } i \equiv 3, 0 \text{ (mod } 8) \\
3, & \text{if } i \equiv 1, 6 \text{ (mod } 8), \quad 1 \leq i \leq n - 2,
\end{cases} \]
\[ f(v_{n-1}') = 0, \quad f(v_n') = 3. \]

Case 8: \( n \equiv 7(\text{mod } 8) \)
\[ f(v_i) = \begin{cases} 
0, & \text{if } i \equiv 4, 7 \text{ (mod } 8) \\
1, & \text{if } i \equiv 2, 5 \text{ (mod } 8) \\
2, & \text{if } i \equiv 3, 0 \text{ (mod } 8) \\
3, & \text{if } i \equiv 1, 6 \text{ (mod } 8), \quad 1 \leq i \leq n - 1,
\end{cases} \]
\[ f(v_n) = 2. \]
\[ f(v_i') = \begin{cases} 
0, & \text{if } i \equiv 4, 7 \text{ (mod } 8) \\
1, & \text{if } i \equiv 2, 5 \text{ (mod } 8) \\
2, & \text{if } i \equiv 3, 0 \text{ (mod } 8) \\
3, & \text{if } i \equiv 1, 6 \text{ (mod } 8), \quad 1 \leq i \leq n - 2,
\end{cases} \]
\[ f(v_{n-1}') = 0, \quad f(v_n') = 3. \]
\( f(v_i') = 1; \ i \equiv 2,5(\text{mod } 8); \)
\( f(v_i') = 2; \ i \equiv 3,0(\text{mod } 8); \)
\( f(v_i') = 3; \ i \equiv 1,6(\text{mod } 8), \ 1 \leq i \leq n - 3, \)
\( f(v_{n-2}') = 0, \)
\( f(v_{n-1}') = 1, \)
\( f(v_n') = 3. \)

The labeling pattern defined above covers all possible arrangements of vertices. In each possibility the graph under consideration satisfies the vertex conditions and edge conditions of 4-cordial labeling as shown in Table 4.1.20. That is \( G \) admits 4-cordial labeling.

Let \( n = 8a + b, \ a, b \in N \cup \{ 0 \}. \)

<table>
<thead>
<tr>
<th>( b )</th>
<th>Vertex conditions</th>
<th>Edge conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>0,4</td>
<td>( v_f(0) = v_f(1) = v_f(2) = v_f(3) )</td>
<td>( e_f(0) = e_f(1) = e_f(2) = e_f(3) )</td>
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<tr>
<td>1</td>
<td>( v_f(0) + 1 = v_f(1) = v_f(2) + 1 = v_f(3) )</td>
<td>( e_f(0) = e_f(1) = e_f(2) + 1 = e_f(3) )</td>
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<tr>
<td>2</td>
<td>( v_f(0) = v_f(1) = v_f(2) = v_f(3) )</td>
<td>( e_f(0) = e_f(1) + 1 = e_f(2) = e_f(3) + 1 )</td>
</tr>
<tr>
<td>3</td>
<td>( v_f(0) + 1 = v_f(1) = v_f(2) + 1 = v_f(3) )</td>
<td>( e_f(0) = e_f(1) + 1 = e_f(2) + 1 = e_f(3) + 1 )</td>
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<tr>
<td>5</td>
<td>( v_f(0) + 1 = v_f(1) + 1 = v_f(2) = v_f(3) )</td>
<td>( e_f(0) = e_f(1) + 1 = e_f(2) + 1 = e_f(3) )</td>
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<tr>
<td>6</td>
<td>( v_f(0) = v_f(1) = v_f(2) = v_f(3) )</td>
<td>( e_f(0) = e_f(1) + 1 = e_f(2) = e_f(3) + 1 )</td>
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<tr>
<td>7</td>
<td>( v_f(0) + 1 = v_f(1) = v_f(2) + 1 = v_f(3) )</td>
<td>( e_f(0) + 1 = e_f(1) + 1 = e_f(2) = e_f(3) + 1 )</td>
</tr>
</tbody>
</table>

Table 4.1.20: 4-cordial labeling of splitting graph \( S'(C_n) \) of cycle \( C_n \).

**Illustration 4.1.21** The splitting graph \( S'(C_8) \) of cycle \( C_8 \) and its 4-cordial labeling is shown in Figure 4.1.21.

---

Figure 4.1.21: 4-cordial labeling of splitting graph \( S'(C_8) \) of cycle \( C_8 \).
Chapter 4. Results

Theorem 4.1.21 The splitting graph $S'(K_{1,n})$ of star $K_{1,n}$ is 4-cordial.

**Proof:** Let $G = S'(K_{1,n})$ be the splitting graph of star $K_{1,n}$. Let $v_1, v_2, ..., v_n$ be the pendant vertices and $v_0$ be the apex vertex of $K_{1,n}$ and $u_0, u_1, u_2, ..., u_n$ are newly added vertices corresponding to $v_0, v_1, v_2, ..., v_n$ to obtain $S'(K_{1,n})$. It is noted that $|V(G)| = 2n + 2$ and $|E(G)| = 3n$.

To define 4-cordial labeling $f : V(G) \to \mathbb{Z}_4$ following cases are considered:

1. **Case 1:** $n \equiv 0, 2, 4, 5, 6 \pmod{8}$
   - $f(v_0) = 0$ and $f(u_0) = 1$,
   - $f(v_i) = 0; \quad i \equiv 3, 7 \pmod{8}$,
   - $f(v_i) = 1; \quad i \equiv 1, 6 \pmod{8}$,
   - $f(v_i) = 2; \quad i \equiv 4, 0 \pmod{8}$,
   - $f(v_i) = 3; \quad i \equiv 2, 5 \pmod{8}; \quad 1 \leq i \leq n$.

2. **Case 2:** $n \equiv 1 \pmod{8}$
   - $f(v_i) = 0; \quad i \equiv 3, 7 \pmod{8}$,
   - $f(v_i) = 1; \quad i \equiv 1, 6 \pmod{8}$,
   - $f(v_i) = 2; \quad i \equiv 4, 0 \pmod{8}$,
   - $f(v_i) = 3; \quad i \equiv 2, 5 \pmod{8}; \quad 1 \leq i \leq n - 1$,
   - $f(v_n) = 3$.

3. **Case 3:** $n \equiv 3, 7 \pmod{8}$
   - $f(v_i) = 0; \quad i \equiv 3, 7 \pmod{8}$,
   - $f(v_i) = 1; \quad i \equiv 1, 6 \pmod{8}$,
   - $f(v_i) = 2; \quad i \equiv 4, 0 \pmod{8}$,
   - $f(v_i) = 3; \quad i \equiv 2, 5 \pmod{8}; \quad 1 \leq i \leq n$.

   - $f(u_i) = 0; \quad i \equiv 2 \pmod{4}$,
   - $f(u_i) = 1; \quad i \equiv 0 \pmod{4}$,
   - $f(u_i) = 2; \quad i \equiv 1 \pmod{4}$,
   - $f(u_i) = 3; \quad i \equiv 3 \pmod{4}; \quad 1 \leq i \leq n$.  

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$f(u_i) = 2; \; i \equiv 1(\mod 4)$;
$f(u_i) = 3; \; i \equiv 3(\mod 4); \; 1 \leq i \leq n - 2$,
$f(u_{n-1}) = 2$,
$f(u_n) = 3$.

The labeling pattern defined above covers all possible arrangements of vertices. In each possibility the graph under consideration satisfies the vertex conditions and edge conditions of 4-cordial labeling as shown in table 4.1.21. That is $G$ admits 4-cordial labeling.

Let $n = 8a + b, a, b \in N \cup \{0\}$.

<table>
<thead>
<tr>
<th>$b$</th>
<th>Vertex conditions</th>
<th>Edge conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.4</td>
<td>$v_f(0) = v_f(1) = v_f(2) + 1 = v_f(3) + 1$</td>
<td>$e_f(0) = e_f(1) = e_f(2) = e_f(3)$</td>
</tr>
<tr>
<td>1.5</td>
<td>$v_f(0) = v_f(1) = v_f(2) = v_f(3)$</td>
<td>$e_f(0) = e_f(1) + 1 = e_f(2) = e_f(3)$</td>
</tr>
<tr>
<td>2.6</td>
<td>$v_f(0) = v_f(1) = v_f(2) + 1 = v_f(3) + 1$</td>
<td>$e_f(0) = e_f(1) + 1 = e_f(2) = e_f(3) + 1$</td>
</tr>
<tr>
<td>3.7</td>
<td>$v_f(0) = v_f(1) = v_f(2) = v_f(3)$</td>
<td>$e_f(0) + 1 = e_f(1) + 1 = e_f(2) = e_f(3) + 1$</td>
</tr>
</tbody>
</table>

Table 4.1.21: 4-cordial labeling of splitting graph $S'(K_{1,n})$ of star $K_{1,n}$.

Illustration 4.1.22 The splitting graph $S'(K_{1,5})$ of star $K_{1,5}$ and its 4-cordial labeling is shown in Figure 4.1.22.

Theorem 4.1.22 The one point union $f_3^{(n)}$ of $n$ copies of a fan $f_3$ is 4-cordial for all $n$.

Proof: Let $G = f_3^{(n)}$ be the one point union of $n$ copies of fan graph $f_3$. Let $v_1, v_2, ..., v_{3n}$ be the $3n$ vertices of $n$ copies of fan $f_3$. It is noted that $|V(G)| = 3n + 1$ and $|E(G)| = 5n$. 

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Define 4-cordial labeling $f : V(G) \rightarrow Z_4$ as follows:

\[
\begin{align*}
    f(v) &= 0, \\
    f(v_i) &= 0; \quad i \equiv 0, 4, 9 \mod 12; \\
    f(v_i) &= 1; \quad i \equiv 1, 7, 10 \mod 12; \\
    f(v_i) &= 2; \quad i \equiv 3, 5, 11 \mod 12; \\
    f(v_i) &= 3; \quad i \equiv 2, 6, 8 \mod 12, \quad 0 \leq i \leq 3n.
\end{align*}
\]

The labeling pattern defined above covers all possible arrangements of vertices. In each possibility the graph under consideration satisfies the vertex conditions and edge conditions of 4-cordial labeling as shown in table 4.1.22. That is $G$ admits 4-cordial labeling. Let $n = 8a + b$, $a, b \in N \cup \{0\}$.

<table>
<thead>
<tr>
<th>$b$</th>
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<th>Edge conditions</th>
</tr>
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<tbody>
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<td>$v_f(0) = v_f(1) + 1 = v_f(2) + 1 = v_f(3) + 1$</td>
<td>$e_f(0) = e_f(1) = e_f(2) = e_f(3)$</td>
</tr>
<tr>
<td>1, 5</td>
<td>$v_f(0) = v_f(1) = v_f(2) = v_f(3)$</td>
<td>$e_f(0) + 1 = e_f(1) = e_f(2) + 1 = e_f(3) + 1$</td>
</tr>
<tr>
<td>2, 6</td>
<td>$v_f(0) = v_f(1) + 1 = v_f(2) = v_f(3)$</td>
<td>$e_f(0) + 1 = e_f(1) = e_f(2) = e_f(3) + 1$</td>
</tr>
<tr>
<td>3, 7</td>
<td>$v_f(0) = v_f(1) + 1 = v_f(2) + 1 = v_f(3)$</td>
<td>$e_f(0) = e_f(1) = e_f(2) + 1 = e_f(3)$</td>
</tr>
</tbody>
</table>

Table 4.1.22: 4-cordial labeling of the one point union $f_3^{(n)}$ of fan $f_3$.

Illustration 4.1.23 The one point union $f_3^{(6)}$ of fan $f_3$ and its 4-cordial labeling is shown in Figure 4.1.23.

Figure 4.1.23: 4-cordial labeling of the one point union $f_3^{(6)}$ of fan $f_3$. 
4.2 \textit{k}-cordial Labeling

\textit{k}-cordial Labeling

Here, \textit{k}-cordial labeling of standard graphs was discussed and derived new results. Let \( < A, \ast > \) be any Abelian group. A graph \( G = (V(G), E(G)) \) is said to be \textit{A-cordial} if there is a mapping \( f : V(G) \rightarrow A \) which satisfies the following two conditions when the edge \( e = uv \) is labelled as \( f(u) \ast f(v) \)

(i) \(|v_f(a) - v_f(b)| \leq 1; \) for all \( a, b \in A \),
(ii) \(|e_f(a) - e_f(b)| \leq 1; \) for all \( a, b \in A \).

Where

\( v_f(a) \) = the number of vertices with label \( a \); \\
\( v_f(b) \) = the number of vertices with label \( b \); \\
\( e_f(a) \) = the number of edges with label \( a \); \\
\( e_f(b) \) = the number of edges with label \( b \).

If \( A = < \mathbb{Z}_k, +_k > \), that is additive group of modulo \( k \) then the labeling is known as \textit{k-cordial labeling}.

Some Existing Results of \textit{k}-cordial Labeling

The concept of \textit{A-cordial labeling} was introduced by Hovey[26] and he proved the following results:

- Caterpillars are \textit{k}-cordial for all \( k \).
- All trees are \textit{k}-cordial for all \( k = 2, 3, 4, 5 \).
- Cycles are \textit{k}-cordial for all odd \( k \).
- Cycles with pendant edges attached are \textit{k}-cordial for all \( k \).
- Complete graph \( K_n \) is 3-cordial for all \( n \).
- Complete Bipartite graph \( K_{n,k} \) is \textit{k}-cordial if and only if \( n = 1 \).

In [57] and [59] Yousef proved the following results.

- \( C_{2k}^n \) is not \( (2k + 1) \)-cordial for \( k > 1 \).
- Let \( G \) and \( H \) be \((p_1, q_1)\) and \((p_2, q_2)\) are \textit{k}-cordial graphs. If \((p_1 \text{ or } p_2 \equiv 0(\text{mod} k))\) and \((q_1 \text{ or } q_2 \equiv 0(\text{mod} k))\), then \( G + H \) is \textit{k}-cordial.

Here are new results of \textit{k}-cordial labeling.
**Theorem 4.2.1** The pan graph $C_n^{+1}$ is $k$-cordial for all even $k$ and $n = k + j$, $0 \leq j \leq k - 1$.

**Proof:** Let $G = C_n^{+1}$ be the pan graph obtained by attaching a pendant edge to any one vertex of the cycle $C_n$. Let $n = k + j$, where $0 \leq j \leq k - 1$. Divide the $n$ vertices of cycle of the pan $C_n^{+1}$ into two blocks of $k$ and $j$ vertices namely $v_1, v_2, ..., v_k$ and $v'_1, v'_2, ..., v'_j$. Let $v'$ be the pendant vertex. It is noted that $|V(G)| = n + 1$ and $|E(G)| = n + 1$.

To define $k-$cordial labeling $f : V(G) \rightarrow \mathbb{Z}_k$ following cases are considered:

**Case 1:** $j = 0$

For the first block of $k$ vertices,
\[
f(v_i) = k - \frac{p_i + 1}{2}; \quad i \equiv p_i \pmod{k}, \quad p_i \text{ is odd},
\]
\[
= \frac{k}{2} - \frac{p_i}{2}; \quad i \equiv p_i \pmod{k}, \quad p_i \text{ is even}, \quad 1 \leq i \leq k,
\]
\[
f(v') = \frac{k}{2} - 1.
\]

**Case 2:** $j = 1$

For the first block of $k$ vertices,
\[
f(v_i) = k - \frac{p_i + 1}{2}; \quad i \equiv p_i \pmod{k}, \quad p_i \text{ is odd},
\]
\[
= \frac{k}{2} - \frac{p_i}{2}; \quad i \equiv p_i \pmod{k}, \quad p_i \text{ is even}, \quad 1 \leq i \leq k.
\]

The labeling pattern of second block of $j$ vertex is
\[
f(v'_1) = \frac{k}{2} - 1,
\]
\[
f(v') = \frac{k}{2} - 2.
\]

**Case 3:** $2 \leq j \leq k - 2$

For the first block of $k$ vertices,
\[
f(v_i) = k - \frac{p_i + 1}{2}; \quad i \equiv p_i \pmod{k}, \quad p_i \text{ is odd},
\]
\[
= \frac{k}{2} - \frac{p_i}{2}; \quad i \equiv p_i \pmod{k}, \quad p_i \text{ is even}, \quad 1 \leq i \leq k.
\]

The labeling pattern of second block of $j$ vertices, where $2 \leq j \leq k - 2$ is divided into following subcases.

**Subcase 1:** If $j$ is odd.

**Sub-subcase 1:** If $\frac{j + 1}{2}$ is odd.

\[
f(v'_i) = \frac{k}{2} - \frac{i + 1}{2}; \quad i \text{ is odd},
\]
\[
= k - \frac{i}{2}; \quad i \text{ is even}, \quad 1 \leq i < \frac{j + 1}{2},
\]
\( f(v'_i) = k - \frac{i+1}{2}; \quad i \text{ is odd,} \)
\[ = \frac{k}{2} - \frac{i}{2}; \quad i \text{ is even,} \quad \frac{i+1}{2} \leq i \leq k - 2, \]

\( f(v') = \frac{k}{2} - \frac{i+1}{2}. \)

**Sub-case 2:** If \( \frac{i+1}{2} \) is even.

\( f(v'_i) = k - \frac{i+1}{2}; \quad i \text{ is odd,} \)
\[ = k - \frac{i}{2}; \quad i \text{ is even,} \quad 1 \leq i < \frac{i+1}{2}, \]

\( f(v'_i) = k - \frac{i-1}{2}; \quad i \text{ is odd,} \)
\[ = \frac{k}{2} - \frac{i+2}{2}; \quad i \text{ is even,} \quad \frac{i+1}{2} \leq i \leq k - 2, \]

\( f(v') = \frac{k}{2} - \frac{i+3}{2}. \)

**Sub-case 2:** If \( j \) is even.

\( f(v'_i) = k - \frac{i+1}{2}; \quad i \text{ is odd,} \)
\[ = k - \frac{i}{2}; \quad i \text{ is even,} \quad 1 \leq i \leq j, \]

\( f(v') = \frac{k}{2} - \frac{i+2}{2}. \)

**Case 4:** \( j = k - 1 \)

In this case divide the \( 2k - 1 \) vertices \( v_1, v_2, \ldots, v_k, v'_1, v'_2, \ldots, v'_{k-1} \) of the cycle \( C_n \) into \( k - 1 \) blocks of two vertices and one block of one vertex. Next label the \( i^{th} \) block of the cycle with \( i \) for \( 1 \leq i \leq k - 1 \) and last block with one vertex by 0 and the vertex \( v' \) by 0.

The labeling pattern defined above covers all possible arrangements of vertices. In each possibility the graph under consideration satisfies the vertex conditions and edge conditions of \( k \)-cordial labeling. That is \( G \) admits \( k \)-cordial labeling.
Illustration 4.2.1 The pan graph $C_{25}^{+1}$ and its 20-cordial labeling is shown in Figure 4.2.1.

![Figure 4.2.1: 20-cordial labeling of pan graph $C_{25}^{+1}$](image)

Theorem 4.2.2 The pan graph $C_n^{+1}$ is $k$-cordial for all even $k$ and $n = 2tk + j$, where $t \in \mathbb{N} \cup \{0\}$ and $0 \leq j \leq k - 1$.

Proof: Let $G = C_n^{+1}$ be the pan graph obtained by attaching a pendant edge to any one vertex of the cycle $C_n$. Let $n = 2tk + j$, where $0 \leq j \leq k - 1$ and $t \geq 0$. Divide the $n$ cycle vertices of the pan $C_n^{+1}$ into two blocks of $2tk$ and $j$ vertices namely $v_1, v_2, ..., v_{2tk}$ and $v'_1, v'_2, ..., v'_j$. Let $v'$ be the pendant vertex. It is noted that $|V(G)| = n + 1$ and $|E(G)| = n + 1$.

To define $k$-cordial labeling $f : V(G) \rightarrow \mathbb{Z}_k$ following cases are considered:

Case 1: $j = 0$.

For the first block of $2tk$ vertices, where $t \geq 0$,

$f(v_i) = k - \frac{n_i + 1}{2}$; \quad $i \equiv p_i \pmod{k}$, $p_i$ is odd,

$= \frac{k}{2} - \frac{n_i}{2}$; \quad $i \equiv p_i \pmod{k}$, $p_i$ is even, $(m - 1)k + 1 \leq i \leq mk$, $m$ is odd, $1 \leq m \leq 2t$,

$f(v_i) = \frac{k}{2} - \frac{n_i + 1}{2}$; \quad $i \equiv p_i \pmod{k}$, $p_i$ is odd,

$= k - \frac{n_i}{2}$; \quad $i \equiv p_i \pmod{k}$, $p_i$ is even, $(m - 1)k + 1 \leq i \leq mk$, $m$ is even, $1 \leq m \leq 2t$,
$f(v') = k - 1$.

**Case 2: $j = 1$.**

For the first block of $2tk$ vertices, where $t \geq 1$,

\[
f(v_i) = k - \frac{p_i + 1}{2}; \quad i \equiv p_i \pmod{k}, p_i \text{ is odd},
\]

\[
= \frac{k}{2} - \frac{p_i}{2}; \quad i \equiv p_i \pmod{k}, p_i \text{ is even}, (m - 1)k + 1 \leq i \leq mk,
\]

$m$ is odd, $1 \leq m \leq 2t$,

\[
f(v_i) = \frac{k}{2} - \frac{p_i + 1}{2}; \quad i \equiv p_i \pmod{k}, p_i \text{ is odd},
\]

\[
= k - \frac{p_i}{2}; \quad i \equiv p_i \pmod{k}, p_i \text{ is even}, (m - 1)k + 1 \leq i \leq mk,
\]

$m$ is even, $1 \leq m \leq 2t$,

The labeling pattern of second block of $j$ vertex is

\[
f(v'_1) = k - 1,
\]

\[
f(v'_2) = \frac{k}{2} - 1.
\]

**Case 3: $j = 2$.**

For the first block of $2tk$ vertices, where $t \geq 1$,

\[
f(v_i) = k - \frac{p_i + 1}{2}; \quad i \equiv p_i \pmod{k}, p_i \text{ is odd},
\]

\[
= \frac{k}{2} - \frac{p_i}{2}; \quad i \equiv p_i \pmod{k}, p_i \text{ is even}, (m - 1)k + 1 \leq i \leq mk,
\]

$m$ is odd, $1 \leq m \leq 2t$,

\[
f(v_i) = \frac{k}{2} - \frac{p_i + 1}{2}; \quad i \equiv p_i \pmod{k}, p_i \text{ is odd},
\]

\[
= k - \frac{p_i}{2}; \quad i \equiv p_i \pmod{k}, p_i \text{ is even}, (m - 1)k + 1 \leq i \leq mk,
\]

$m$ is even, $1 \leq m \leq 2t$,

The labeling pattern of second block of $j$ vertex is

\[
f(v'_1) = \frac{k}{2} - 1,
\]

\[
f(v'_2) = k - 1,
\]

\[
f(v'_3) = k - \frac{k}{2}.
\]

**Case 4: $3 \leq j \leq k - 1$.**

For the first block of $2tk$ vertices, where $t \geq 1$,

\[
f(v_i) = k - \frac{p_i + 1}{2}; \quad i \equiv p_i \pmod{k}, p_i \text{ is odd},
\]

\[
= \frac{k}{2} - \frac{p_i}{2}; \quad i \equiv p_i \pmod{k}, p_i \text{ is even}, (m - 1)k + 1 \leq i \leq mk,
\]

$m$ is odd, $1 \leq m \leq 2t$,

\[
f(v_i) = \frac{k}{2} - \frac{p_i + 1}{2}; \quad i \equiv p_i \pmod{k}, p_i \text{ is odd},
\]

\[
= k - \frac{p_i}{2}; \quad i \equiv p_i \pmod{k}, p_i \text{ is even}, (m - 1)k + 1 \leq i \leq mk,
\]

$m$ is even, $1 \leq m \leq 2t$,

The labeling pattern of second block of $j$ vertices, where $3 \leq j \leq k - 1$ is divided into following two subcases.

**Subcase 1: If $j$ is odd.**

\[
f(v'_i) = k - \frac{i + 1}{2}; \quad i \text{ is odd},
\]

\[
= \frac{k}{2} - \frac{i}{2}; \quad i \text{ is even}, \quad 1 \leq i \leq k - 1,
\]

\[
f(v'_i) = \frac{k}{2} - \frac{i + 1}{2}.
\]
Subcase 2: If \( j \) is even.
For \( 3 \leq j \leq \frac{k}{2} \),
\[
\begin{align*}
f(v'_i) &= \frac{k}{2} - \frac{i+1}{2}; & i & \text{ is odd,} \\
&= \frac{k}{2} - \frac{i}{2}; & i & \text{ is even, } 1 \leq i < j, \\
f(v'_i) &= k - \frac{i+2}{2}; & \text{ if } i = j \\
f(v') &= \frac{k}{2} - \frac{j}{2}.
\end{align*}
\]
For \( \frac{k}{2} < j \leq k - 1 \).
The labeling pattern of remaining block of \( j \) vertices is divided into following two sub-subcases.

**Sub-subcase 1:** If \( \frac{j}{2} \) is odd.
\[
\begin{align*}
f(v'_i) &= k - \frac{i+1}{2}; & i & \text{ is odd,} \\
&= \frac{k}{2} - \frac{i}{2}; & i & \text{ is even, } 1 \leq i < \frac{j}{2}, \\
f(v'_i) &= \frac{k}{2} - \frac{i+1}{2}; & i & \text{ is odd,} \\
&= k - \frac{i}{2}; & i & \text{ is even, } \frac{j}{2} \leq i \leq k - 1, \\
f(v') &= \frac{k}{2} - \frac{i+2}{2}.
\end{align*}
\]

**Sub-subcase 2:** If \( \frac{j}{2} \) is even.
\[
\begin{align*}
f(v'_i) &= k - \frac{i+1}{2}; & i & \text{ is odd,} \\
&= \frac{k}{2} - \frac{i}{2}; & i & \text{ is even, } 1 \leq i < \frac{j}{2}, \\
f(v'_i) &= \frac{k}{2} - \frac{i-1}{2}; & i & \text{ is odd,} \\
&= k - \frac{i+2}{2}; & i & \text{ is even, } \frac{j}{2} \leq i \leq k - 1, \\
f(v') &= \frac{k}{2} - \frac{j}{2}.
\end{align*}
\]

The labeling pattern defined above covers all possible arrangements of vertices. In each possibility the graph under consideration satisfies the vertex conditions and edge conditions of \( k \)-cordial labeling. That is \( G \) admits \( k \)-cordial labeling.
Illustration 4.2.2 The pan graph $C_{25}^{+1}$ and its 12-cordial labeling is shown in Figure 4.2.2.

![Figure 4.2.2: 12-cordial labeling of pan graph $C_{25}^{+1}$](image)

Theorem 4.2.3 The pan graph $C_n^{+1}$ is $k$-cordial for all even $k$ and $n = 2tk + k + j$, where $t \in \mathbb{N}$ and $0 \leq j \leq k - 1$.

Proof: Let $G = C_n^{+1}$ be the pan graph obtained by attaching a pendant edge to any one vertex of the cycle $C_n$. Let $n = 2tk + k + j$, where $0 \leq j \leq k - 1$ and $t \geq 1$. Divide the $n$ cycle vertices of the pan $C_n^{+1}$ into three blocks of $2tk$, $k$ and $j$ vertices namely $v_1, v_2, ..., v_{2tk}, v'_1, v'_2, ..., v'_k$ and $v''_1, v''_2, ..., v''_j$. Let $v'$ be the pendant vertex. It is noted that $|V(G)| = n + 1$ and $|E(G)| = n + 1$.

Define $k$-cordial labeling $f : V(G) \rightarrow Z_k$ as follows:

For the first block of $2tk$ vertices, where $t \geq 1$,

\[
 f(v_i) = k - \frac{p_i^{+1}}{2}; \quad i \equiv p_i (mod k), p_i \text{ is odd}, \\
= \frac{k}{2} - \frac{p_i}{2}^2; \quad i \equiv p_i (mod k), p_i \text{ is even, } (m - 1)k + 1 \leq i \leq mk, \\
\quad m \text{ is odd}, 1 \leq m \leq 2t,
\]

\[
 f(v_i) = \frac{k}{2} - \frac{p_i^{+1}}{2}; \quad i \equiv p_i (mod k), p_i \text{ is odd}, \\
= k - \frac{p_i}{2}^2; \quad i \equiv p_i (mod k), p_i \text{ is even, } (m - 1)k + 1 \leq i \leq mk, \\
\quad m \text{ is even}, 1 \leq m \leq 2t,
\]

\[
 f(v') = k - 1.
\]
For the second block of \( k \) vertices,
\[
f(v_i') = k - \frac{p_i' + 1}{2} ; \quad i \equiv p_i'(mod \ k), \ p_i' \ is \ odd,
\]
\[
= \frac{k}{2} - \frac{p_i'}{2} ; \quad i \equiv p_i'(mod \ k), \ p_i' \ is \ even, \ 1 \leq p_i' \leq k,
\]

\[
f(v') = \frac{k}{2} - 1.
\]

The labeling pattern for the third block of \( j \) vertices is divided into following cases.

**Case 1:** \( j = 0 \)

For the first block of \( 2tk \) vertices, where \( t \geq 1 \),
\[
f(v_i) = k - \frac{p_i + 1}{2} ; \quad i \equiv p_i(mod \ k), \ p_i \ is \ odd,
\]
\[
= \frac{k}{2} - \frac{p_i}{2} ; \quad i \equiv p_i(mod \ k), \ p_i \ is \ even, \ (m - 1)k + 1 \leq i \leq mk,
\]

\[
m is \ odd, \ 1 \leq m \leq 2t,
\]

\[
f(v_i) = \frac{k}{2} - \frac{p_i + 1}{2} ; \quad i \equiv p_i(mod \ k), \ p_i \ is \ odd,
\]
\[
= k - \frac{p_i}{2} ; \quad i \equiv p_i(mod \ k), \ p_i \ is \ even, \ (m - 1)k + 1 \leq i \leq mk,
\]

\[
m is \ even, \ 1 \leq m \leq 2t,
\]

For the second block of \( k \) vertices,
\[
f(v_i') = k - \frac{p_i' + 1}{2} ; \quad i \equiv p_i'(mod \ k), \ p_i' \ is \ odd,
\]
\[
= \frac{k}{2} - \frac{p_i'}{2} ; \quad i \equiv p_i'(mod \ k), \ p_i' \ is \ even, \ 1 \leq p_i' \leq k,
\]

\[
f(v') = \frac{k}{2} - 1.
\]

**Case 2:** \( j = 1 \)

For the first block of \( 2tk \) vertices, where \( t \geq 1 \),
\[
f(v_i) = k - \frac{p_i + 1}{2} ; \quad i \equiv p_i(mod \ k), \ p_i \ is \ odd,
\]
\[
= \frac{k}{2} - \frac{p_i}{2} ; \quad i \equiv p_i(mod \ k), \ p_i \ is \ even, \ (m - 1)k + 1 \leq i \leq mk,
\]

\[
m is \ odd, \ 1 \leq m \leq 2t,
\]

\[
f(v_i) = \frac{k}{2} - \frac{p_i + 1}{2} ; \quad i \equiv p_i(mod \ k), \ p_i \ is \ odd,
\]
\[
= k - \frac{p_i}{2} ; \quad i \equiv p_i(mod \ k), \ p_i \ is \ even, \ (m - 1)k + 1 \leq i \leq mk,
\]

\[
m is \ even, \ 1 \leq m \leq 2t,
\]

For the second block of \( k \) vertices,
\[
f(v_i') = k - \frac{p_i' + 1}{2} ; \quad i \equiv p_i'(mod \ k), \ p_i' \ is \ odd,
\]
\[
= \frac{k}{2} - \frac{p_i'}{2} ; \quad i \equiv p_i'(mod \ k), \ p_i' \ is \ even, \ 1 \leq p_i' \leq k,
\]

The labeling pattern of third block of \( j \) vertex is
\[
f(v_i'') = \frac{k}{2} - 1,
\]
\[
f(v') = \frac{k}{2} - 2.
\]

**Case 3:** \( 2 \leq j \leq k - 2 \)

For the first block of \( 2tk \) vertices, where \( t \geq 1 \),
\[
f(v_i) = k - \frac{p_i + 1}{2} ; \quad i \equiv p_i(mod \ k), \ p_i \ is \ odd,
\]
\[ k^2 - \frac{p_i}{2} \quad i \equiv p_i (\text{mod } k), p_i \text{ is even}, \quad (m - 1)k + 1 \leq i \leq mk, \]
\[ m \text{ is odd}, \quad 1 \leq m \leq 2t, \]
\[ f(v_i) = k - \frac{p_i}{2}; \quad i \equiv p_i (\text{mod } k), p_i \text{ is odd}, \]
\[ = k - \frac{p_i}{2}; \quad i \equiv p_i (\text{mod } k), p_i \text{ is even}, \quad (m - 1)k + 1 \leq i \leq mk, \]
\[ m \text{ is even}, \quad 1 \leq m \leq 2t, \]

For the second block of \( k \) vertices,
\[ f(v'_i) = k - \frac{p'_i}{2}; \quad i \equiv p'_i (\text{mod } k), p'_i \text{ is even}, \quad 1 \leq p'_i \leq k, \]
\[ f(v'_i) = k - \frac{p'_i}{2}; \quad i \equiv p'_i (\text{mod } k), p'_i \text{ is odd}, \]

The labeling pattern of third block of \( j \) vertices, where \( 2 \leq j \leq k - 2 \) is divided into following subcases.

Subcase 1: If \( j \) is odd.

Sub-subcase 1: If \( \frac{j+1}{2} \) is odd.
\[ f(v''_i) = \frac{k}{2} - \frac{i+1}{2}; \quad i \text{ is odd}, \]
\[ = k - \frac{i}{2}; \quad i \text{ is even}, \quad 1 \leq i < \frac{j+1}{2}, \]
\[ f(v''_i) = k - \frac{i}{2}; \quad i \text{ is even}, \quad \frac{j+1}{2} \leq i \leq k - 2, \]
\[ f(v') = \frac{k}{2} - \frac{i+1}{2}. \]

Sub-subcase 2: If \( \frac{j+1}{2} \) is even.
\[ f(v''_i) = \frac{k}{2} - \frac{i+1}{2}; \quad i \text{ is odd}, \]
\[ = k - \frac{i}{2}; \quad i \text{ is even}, \quad 1 \leq i < \frac{j+1}{2}, \]
\[ f(v''_i) = k - \frac{i}{2}; \quad i \text{ is even}, \quad \frac{j+1}{2} \leq i \leq k - 2, \]
\[ f(v') = \frac{k}{2} - \frac{i+3}{2}. \]

Subcase 2: If \( j \) is even.
\[ f(v''_i) = \frac{k}{2} - \frac{i+1}{2}; \quad i \text{ is odd}, \]
\[ = k - \frac{i}{2}; \quad i \text{ is even}, \quad 1 \leq i \leq j, \]
\[ f(v') = \frac{k}{2} - \frac{i+2}{2}. \]

Case 4: \( j = k - 1 \)

In this case divide the \( 2tk + 2k - 1 \) vertices \( v_1, v_2, \ldots, v_{2tk}, v'_1, v'_2, \ldots, v'_k, v''_1, v''_2, \ldots, v''_{k-1} \) of the cycle \( C_n \) into \((tk + k) - 1\) blocks of two vertices and one block of one vertex. Next
label the $i^{th}$ block of the cycle with $i$ for $1 \leq i \leq k - 1$ and last block with one vertex by 0 and the vertex $v'$ by 0.

The labeling pattern defined above covers all possible arrangements of vertices. In each possibility the graph under consideration satisfies the vertex conditions and edge conditions of $k$-cordial labeling. That is $G$ admits $k$-cordial labeling.

**Illustration 4.2.3** The pan graph $C_3^{+1}_{25}$ and its 8-cordial labeling is shown in Figure 4.2.3.

![Figure 4.2.3: 8-cordial labeling of pan graph $C_3^{+1}_{25}$.](image)

**Theorem 4.2.4** The triangular book $B(3, n)$ with $n$-pages is $k$-cordial.

**Proof:** Let $G = B(3, n)$ be the triangular book graph. Let $n = mk + j$, where $m \geq 0$ and $1 \leq j \leq k - 1$. Divide $n$ vertices into two blocks of $mk$ and $j$ vertices, which are denoted by $v_1, v_2, ..., v_{mk}$ and $v'_1, v'_2, ..., v'_j$. Let $u$ and $v$ be the spine vertices. It is noted that $|V(G)| = n + 2$ and $|E(G)| = 2n + 1$.

To define $k$-cordial labeling $f : V(G) \to Z_k$ following cases are considered:

**Case 1:** $m > 0$ and $j = 0$.

The labeling pattern of $mk$ vertices $v_1, v_2, ..., v_{mk}$ is defined as follows:
**Subcase 1**: $k \equiv 0 \pmod{2}$.

- $f(u) = 0$ and $f(v) = k - 1$,
- $f(v_i) = 2p_i$; $i \equiv p_i \pmod{k}$; $1 \leq i \leq \frac{k}{2} - 1$,
- $f(v_i) = 2p_i - (k - 1)$; $i \equiv p_i \pmod{k}$; $\frac{k}{2} \leq i \leq k - 1$,
- $f(v_{mk}) = 0, \forall m > 0$.

**Subcase 2**: $k \equiv 1 \pmod{2}$.

- $f(u) = 0$ and $f(v) = k - 1$,
- $f(v_i) = 2p_i - 1$; $i \equiv p_i \pmod{k}$; $1 \leq i \leq \frac{k-1}{2}$,
- $f(v_i) = 2p_i - (k - 1)$; $i \equiv p_i \pmod{k}$; $\frac{k-1}{2} + 1 \leq i \leq k - 1$,
- $f(v_{mk}) = 0, \forall m > 0$.

**Case 2**: $m \geq 0$ and $1 \leq j \leq k - 1$.

**Subcase 1**: $k \equiv 0 \pmod{2}$.

The labeling pattern of first block of $mk$ vertices $v_1, v_2, \ldots, v_{mk}$ is defined as follows:

- $f(u) = 0$ and $f(v) = k - 1$,
- $f(v_i) = 2p_i$; $i \equiv p_i \pmod{k}$; $1 \leq i \leq \frac{k}{2} - 1$,
- $f(v_i) = 2p_i - (k - 1)$; $i \equiv p_i \pmod{k}$; $\frac{k}{2} \leq i \leq k - 1$,
- $f(v_{mk}) = 0, \forall m > 0$.

The labeling pattern of second block of $j$ vertices $v'_1, v'_2, \ldots, v'_j$ is defined as follows:

- $f(v'_j) = 2j$; $1 \leq j \leq \frac{k}{2} - 1$,
- $f(v'_j) = 2j - (k - 1)$; $\frac{k}{2} \leq j \leq k - 1$.

**Subcase 2**: $k \equiv 1 \pmod{2}$.

The labeling pattern of first block of $mk$ vertices $v_1, v_2, \ldots, v_{mk}$ is defined as follows:

- $f(u) = 0$ and $f(v) = k - 1$,
- $f(v_i) = 2p_i - 1$; $i \equiv p_i \pmod{k}$; $1 \leq i \leq \frac{k-1}{2}$,
- $f(v_i) = 2p_i - (k - 1)$; $i \equiv p_i \pmod{k}$; $\frac{k-1}{2} + 1 \leq i \leq k - 1$,
- $f(v_{mk}) = 0, \forall m > 0$.

The labeling pattern of second block of $j$ vertices $v'_1, v'_2, \ldots, v'_j$ is defined as follows:

- $f(v'_j) = 2j - 1$; $1 \leq j \leq \frac{k-1}{2}$,
- $f(v'_j) = 2j - (k - 1)$; $\frac{k-1}{2} + 1 \leq j \leq k - 1$.

The labeling pattern defined above covers all possible arrangements of vertices. In each possibility the graph under consideration satisfies the vertex conditions and edge conditions of $k$-cordial labeling. That is $G$ admits $k$-cordial labeling.
Illustration 4.2.4 The triangular book $B(3, 10)$ and its 5-cordial labeling is shown in Figure 4.2.4.

![Figure 4.2.4: 5-cordial labeling of triangular book $B(3, 10)$](image)

Illustration 4.2.5 The triangular book $B(3, 12)$ and its 9-cordial labeling is shown in Figure 4.2.5.

![Figure 4.2.5: 9-cordial labeling of triangular book $B(3, 12)$](image)

Theorem 4.2.5 The triangular book with book mark $B(3, n)(u, v)(v, w)$ is $k$-cordial.

Proof: Let $G = B(3, n)(u, v)(v, w)$ be the triangular book with book mark obtained from triangular book $B(3, n)$ by attaching one pendant edge at any one of the spine vertices. Let $u$ and $v$ are vertices of spine. Let $w$ be the pendant vertex and $vw$ be the pendant edge. Let $n = mk + j$, where $m \geq 0$ and $1 \leq j \leq k - 1$. Divide $n$ vertices into two blocks $mk$ and $j$ vertices, which are denoted by $v_1, v_2, ..., v_{mk}$ and $v'_1, v'_2, ..., v'_j$. It is noted that $|V(G)| = n + 3$ and $|E(G)| = 2(n + 1)$. 
To define $k$-cordial labeling $f : V(G) \rightarrow Z_k$ following cases are considered:

**Case 1:** $m > 0$ and $j = 0$.
The labeling pattern of $mk$ vertices $v_1, v_2, ..., v_{mk}$ is defined as follows:

**Subcase 1:** $k \equiv 0 \text{(mod 2)}$.
$$f(u) = 0, \ f(v) = k - 1 \text{ and } f(w) = 1,$$
$$f(v_i) = 2p_i; \quad i \equiv p_i(\text{mod } k); \quad 1 \leq i \leq \frac{k}{2} - 1,$$
$$f(v_i) = 2(k - p_i) - 3; \quad i \equiv p_i(\text{mod } k); \quad \frac{k}{2} \leq i \leq k - 2,$$
$$f(v_{mk-1}) = k - 1,$$
$$f(v_{mk}) = 0, \forall m > 0.$$

**Subcase 2:** $k \equiv 1 \text{(mod 2)}$.
$$f(u) = 0, \ f(v) = k - 1 \text{ and } f(w) = 1,$$
$$f(v_i) = 2p_i; \quad i \equiv p_i(\text{mod } k); \quad 1 \leq i \leq \frac{k-1}{2} - 1,$$
$$f(v_i) = 2(k - p_i) - 3; \quad i \equiv p_i(\text{mod } k); \quad \frac{k-1}{2} \leq i \leq k - 2,$$
$$f(v_{mk-1}) = k - 1,$$
$$f(v_{mk}) = 0, \forall m > 0.$$

**Case 2:** $m \geq 0$ and $1 \leq j \leq k - 1$.

**Subcase 1:** $k \equiv 0 \text{(mod 2)}$.
The labeling pattern of first block of $mk$ vertices $v_1, v_2, ..., v_{mk}$ is defined as follows:
$$f(u) = 0, \ f(v) = k - 1 \text{ and } f(w) = 1,$$
$$f(v_i) = 2p_i; \quad i \equiv p_i(\text{mod } k); \quad 1 \leq i \leq \frac{k}{2} - 1,$$
$$f(v_i) = 2(k - p_i) - 3; \quad i \equiv p_i(\text{mod } k); \quad \frac{k}{2} \leq i \leq k - 2,$$
$$f(v_{mk-1}) = k - 1,$$
$$f(v_{mk}) = 0, \forall m > 0.$$

The labeling pattern of second block of $j$ vertices $v'_1, v'_2, ..., v'_j$ is defined as follows:
$$f(v'_j) = 2j; \quad 1 \leq j \leq \frac{k-1}{2} - 1,$$
$$f(v'_j) = 2(k - j) - 3; \quad \frac{k}{2} \leq j \leq k - 2,$$
$$f(v'_{k-1}) = k - 1.$$

**Subcase 2:** $k \equiv 1 \text{(mod 2)}$.
The labeling pattern of first block of $mk$ vertices $v_1, v_2, ..., v_{mk}$ is defined as follows:
$$f(u) = 0, \ f(v) = k - 1 \text{ and } f(w) = 1,$$
$$f(v_i) = 2p_i; \quad i \equiv p_i(\text{mod } k); \quad 1 \leq i \leq \frac{k-1}{2} - 1,$$
$$f(v_i) = 2(k - p_i) - 3; \quad i \equiv p_i(\text{mod } k); \quad \frac{k-1}{2} \leq i \leq k - 2,$$
$$f(v_{mk-1}) = k - 1,$$
$f(v_{mk}) = 0, \forall m > 0.$

The labeling pattern of second block of $j$ vertices $v'_1, v'_2, ..., v'_j$ is defined as follows:

- $f(v'_j) = 2j$, \[1 \leq j \leq \frac{k-1}{2},\]
- $f(v'_j) = 2(k - j) - 3$, \[\frac{k-1}{2} + 1 \leq j \leq k - 2,\]
- $f(v'_{k-1}) = k - 1$.

The labeling pattern defined above covers all possible arrangements of vertices. In each possibility the graph under consideration satisfies the vertex conditions and edge conditions of $k$-cordial labeling. That is $G$ admits $k$-cordial labeling.

**Illustration 4.2.6** The triangular book with book mark $B(3, 6)(u, v)(v, w)$ and its 6-cordial labeling is shown in Figure 4.2.6.

![Figure 4.2.6: 6-cordial labeling of triangular book with book mark B(3, 6)(u, v)(v, w).](image)

**Illustration 4.2.7** The triangular book with book mark $B(3, 15)(u, v)(v, w)$ and its 7-cordial labeling is shown in Figure 4.2.7.

![Figure 4.2.7: 7-cordial labeling of triangular book with book mark B(3, 15)(u, v)(v, w).](image)
Theorem 4.2.6 The graph $B(3, n)(P_{2m})$ is $k$-cordial for all $n$ and $2m = 4$ and $k$ is even.

Proof: Let $B(3, n)(P_{2m})$ be the graph with the vertex set $V(G) = \{u_i, v_j, 1 \leq i \leq 2m, 1 \leq j \leq n\}$ and the edge set $E(G) = \{u_iu_{i+1}, u_iv_j, u_{2m}v_j : 1 \leq i \leq (2m - 1), 1 \leq j \leq n\}$. Let $u_1, u_2, \ldots, u_{2m}$ be the vertices of path $P_{2m}$ and $v_1, v_2, \ldots, v_n$ be the vertices of pages. It is noted that $|V(G)| = 2m + n$ and $|E(G)| = 2n + 2m - 1$.

Define $k$-cordial labeling $f : V(G) \to \mathbb{Z}_k$ as follows:

After defining the labeling pattern in the path vertices, define the labeling pattern for the vertices of the pages.

Let $2m = 4$.

The labeling pattern of the path vertices $u_1, u_2, u_3, u_4$ is defined as follows:

$f(u_1) = 0, f(u_2) = k - 2, f(u_3) = 1, f(u_4) = k - 1$.

The labeling pattern of the vertices $v_1, v_2, \ldots, v_n$ is defined as follows.

Let $n = rk + t$. Divide the $n$ vertices into two blocks say $rk$ and $t$.

The labeling pattern of the first block of $rk$ vertices $v_1, v_2, \ldots, v_{rk}$ is defined in the following steps.

Here, $1 \leq j \leq rk$ and $r \geq 0$.

1. First, label the $j$ vertices by the even numbers in ascending order from $\{2, 3, 4, \ldots, k - 3\}$.

2. Label the next $j$ vertices by the odd numbers in descending order from $\{2, 3, 4, \ldots, k - 3\}$.

3. Label the next four vertices fix by $0, k - 2, 1$ and $k - 1$.

Repeat above steps $r$ times.

The labeling pattern of second block of $t$ vertices $v_1', v_2', \ldots, v_t'$ is defined in the following steps.

Here, $1 \leq t \leq k - 1$.

1. First, label the $t$ vertices by the even numbers in ascending order from $\{2, 3, 4, \ldots, k - 3\}$.

2. Label the next $t$ vertices by the odd numbers in descending order from $\{2, 3, 4, \ldots, k - 3\}$.

3. Label the next three vertices fix by $0, k - 2$ and $1$.

The labeling pattern defined above covers all possible arrangements of vertices. In each possibility the graph under consideration satisfies the vertex conditions and edge conditions of $k$-cordial labeling. That is $G$ admits $k$-cordial labeling.
**Illustration 4.2.8** The graph \( B(3, 8)(P_4) \) and its 16-cordial labeling is shown in Figure 4.2.8.

![Figure 4.2.8: 16-cordial labeling of \( B(3, 8)(P_4) \)](image)

**Theorem 4.2.7** The graph \( B(3, n)(P_{2m}) \) is \( k \)-cordial for all \( n, 2m = lk \), where \( l \in \mathbb{N} \) and \( k \) is even.

**Proof:** Let \( B(3, n)(P_{2m}) \) be the graph with the vertex set \( V(G) = \{u_i, v_j, 1 \leq i \leq 2m, 1 \leq j \leq n\} \) and the edge set \( E(G) = \{u_iu_{i+1}, u_iv_j, u_{2m}v_j : 1 \leq i \leq (2m - 1), 1 \leq j \leq n\} \). Let \( u_1, u_2, ..., u_{2m} \) be the vertices of path \( P_{2m} \) and \( v_1, v_2, ..., v_n \) be the vertices of pages. It is noted that \( |V(G)| = 2m + n \) and \( |E(G)| = 2n + 2m - 1 \).

Define \( k \)-cordial labeling \( f : V(G) \to \mathbb{Z}_k \) as follows:

After defining the labeling pattern in the path vertices, define the labeling pattern for the vertices of the pages.

The labeling pattern of the path vertices \( u_1, u_2, ..., u_{2m} \) is defined as follows:

Here, \( 1 \leq i \leq 2m \).

- \( f(u_1) = 0 \) and \( f(u_{2m}) = k - 1 \).
- \( f(u_{2i+1}) = p_i ; \quad i \equiv p_i \ (mod\ k) ; 1 \leq i \leq 2m - 1, \)
- \( f(u_{2i+1}) = k - (m - p_i) ; \quad i \equiv p_i \ (mod\ k) ; 0 \leq i \leq 2m - 1. \)

The labeling pattern of the page vertices \( v_1, v_2, ..., v_n \) is defined as follows.

Let \( n = rk + t \). Divide the \( n \) vertices into two blocks say \( rk \) and \( t \).

The labeling pattern of the first block of \( rk \) vertices \( v_1, v_2, ..., v_{rk} \) is defined in the following cases.

Here, \( 1 \leq j \leq rk \) and \( r \geq 0 \).

Case 1: \( l \) is even.

1. First label the \( j \) vertices by the remaining odd numbers in descending order from \{0, 1, 2, ..., \( k - 1 \)\} in such a way that vertex condition will be satisfied.
2. Then label remaining \( j \) vertices by the even numbers in ascending order from \( \{0, 1, 2, \ldots, k-1\} \) in such a way that vertex condition will be satisfied.

Repeat above steps \( r \) times.

**Case 2: \( l \) is odd.**

**Subcase 1:** \( \frac{k}{2} \) is even.

1. \( f(v_j) = \frac{k}{2}; \quad j \equiv 1 \pmod{k}; \quad 1 \leq j \leq rk. \)

2. Label all the vertices from \( v_2 \) to \( v_{\frac{k}{2}} \) with remaining even numbers in ascending order from \( \{0, 1, 2, \ldots, k-1\} \) except \( \frac{k}{2} \) in such a way that vertex condition will be satisfied.

3. \( f(v_j) = \frac{k}{2} - 1; \quad j \equiv \frac{k}{2} + 1 \pmod{k}; \quad 1 \leq j \leq rk. \)

4. Label all the vertices from \( v_{\frac{k}{2}+2} \) to \( v_k \) with remaining odd numbers in ascending order from \( \{0, 1, 2, \ldots, k-1\} \) except \( \frac{k}{2} - 1 \) in such a way that vertex condition will be satisfied.

Repeat above steps \( r \) times.

**Subcase 2:** \( \frac{k}{2} \) is odd.

1. \( f(v_j) = \frac{k}{2}; \quad j \equiv 1 \pmod{k}; \quad 1 \leq j \leq rk. \)

2. Label all the vertices from \( v_2 \) to \( v_{\frac{k}{2}} \) with remaining odd numbers in ascending order from \( \{0, 1, 2, \ldots, k-1\} \) except \( \frac{k}{2} \) in such a way that vertex condition will be satisfied.

3. \( f(v_j) = \frac{k}{2} - 1; \quad j \equiv \frac{k}{2} + 1 \pmod{k}; \quad 1 \leq j \leq rk. \)

4. Label all the vertices from \( v_{\frac{k}{2}+2} \) to \( v_k \) with remaining even numbers in ascending order from \( \{0, 1, 2, \ldots, k-1\} \) except \( \frac{k}{2} - 1 \) in such a way that vertex condition will be satisfied.

Repeat above steps \( r \) times.

The labeling pattern of the second block of \( t \) vertices \( v_1', v_2', \ldots, v_t' \) is defined in the following steps.

Here, \( 1 \leq t \leq k - 1. \)

**Case 1:** \( l \) is even.

1. First label the \( t \) vertices by the remaining odd numbers in descending order from \( \{0, 1, 2, \ldots, k-1\} \) in such a way that vertex condition will be satisfied.

2. Then label remaining \( t \) vertices by the remaining even numbers in ascending order from \( \{0, 1, 2, \ldots, k-1\} \) in such a way that vertex condition will be satisfied.

**Case 2:** \( l \) is odd.

**Subcase 1:** \( \frac{k}{2} \) is even.

1. \( f(v'_t) = \frac{k}{2}; \quad t \equiv 1 \pmod{k}; \quad 1 \leq t \leq k-1. \)
2. Label all the vertices from $v'_2$ to $v'_{\frac{k}{2}}$ with remaining even numbers in ascending order from $\{0, 1, 2, \ldots, k - 1\}$ except $\frac{k}{2}$ in such a way that vertex condition will be satisfied.

3. $f(v'_t) = \frac{k}{2} - 1; \quad t \equiv \frac{k}{2} + 1 \pmod{k}; \quad 1 \leq t \leq k - 1.$

4. Label all the vertices from $v'_{\frac{k}{2}+2}$ to $v'_k$ with remaining odd numbers in ascending order from $\{0, 1, 2, \ldots, k - 1\}$ except $\frac{k}{2} - 1$ in such a way that vertex condition will be satisfied.

Subcase 2: $\frac{k}{2}$ is odd.

1. $f(v'_t) = \frac{k}{2}; \quad t \equiv 1 \pmod{k}; \quad 1 \leq t \leq k - 1.$

2. Label all the vertices from $v'_2$ to $v'_{\frac{k}{2}}$ with remaining odd numbers in ascending order from $\{0, 1, 2, \ldots, k - 1\}$ except $\frac{k}{2}$ in such a way that vertex condition will be satisfied.

3. $f(v'_t) = \frac{k}{2} - 1; \quad t \equiv \frac{k}{2} \pmod{k}; \quad 1 \leq t \leq k - 1.$

4. Label all the vertices from $v'_{\frac{k}{2}+2}$ to $v'_k$ with remaining even numbers in ascending order from $\{0, 1, 2, \ldots, k - 1\}$ except $\frac{k}{2} - 1$ in such a way that vertex condition will be satisfied.

The labeling pattern defined above covers all possible arrangements of vertices. In each possibility the graph under consideration satisfies the vertex conditions and edge conditions of $k$-cordial labeling. That is $G$ admits $k$-cordial labeling.

Illustration 4.2.9 The graph $B(3, 8)(P_8)$ and its 4-cordial labeling is shown in Figure 4.2.9.

![Figure 4.2.9: 4-cordial labeling of $B(3, 8)(P_8)$](image-url)
**Theorem 4.2.8** The graph $B(3, n)(P_{2m})$ is $k$-cordial for all $n$ and $4 < 2m < lk$, where $k$ is even and $l$ is the smallest positive odd integer such that $2m < lk$.

**Proof:** Let $B(3, n)(P_{2m})$ be the graph with the vertex set $V(G) = \{u_i, v_j, 1 \leq i \leq 2m, 1 \leq j \leq n\}$ and the edge set $E(G) = \{u_iu_{i+1}, u_iv_j, u_{2m}v_j : 1 \leq i \leq (2m - 1), 1 \leq j \leq n\}$. Let $u_1, u_2, ..., u_{2m}$ be the vertices of path $P_{2m}$ and $v_1, v_2, ..., v_n$ be the vertices of pages. It is noted that $|V(G)| = 2m + n$ and $|E(G)| = 2n + 2m - 1$.

Define $k$-cordial labeling $f : V(G) \to \mathbb{Z}_k$ as follows:

After defining the labeling pattern in the path vertices, define the labeling pattern for the vertices of the pages.

The labeling pattern of the path vertices $u_1, u_2, ..., u_{2m}$ is defined as follows:

Here, $1 \leq i \leq 2m$.

1. Let $n = rk + t$. Divide the $n$ vertices into two blocks say $rk$ and $t$.

2. The labeling pattern of the first block of $rk$ vertices $v_1, v_2, ..., v_{rk}$ is defined in the following cases.

   **Case 1:** If in path vertices $u_{2m-1}$th vertex labelled by odd number.

   1. First label the $j$ vertices by remaining odd numbers in ascending order from $\{0, 1, 2, ..., k-1\}$ in such a way that vertex condition will be satisfied for $k$-cordial labeling.

   2. Label the next $j$ vertices by remaining even numbers in descending order from $\{0, 1, 2, ..., k-1\}$ in such a way that vertex condition will be satisfied for $k$-cordial labeling.

   3. Label the remaining $j$ vertices by the even numbers in ascending order from $\{0, 1, 2, ..., k-1\}$ but except the numbers which are labelled in step 2 and also labelled in such a way that vertex condition will be satisfied for $k$-cordial labeling.

   4. Now label the next one vertex fix by the same labeling which is labelled on $u_2$ vertex of path.

   5. Label the remaining $j$ vertices by the odd numbers in ascending order from $\{0, 1, 2, ..., k-1\}$ but except the numbers which are labelled in step 1 and 4 and also labelled in such a way that vertex condition will be satisfied for $k$-cordial labeling.

Repeat above steps $r$ times.

**Case 2:** If in path vertices $u_{2m-1}$th vertex labelled by even number.

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1. First label the \( j \) vertices by remaining even numbers in ascending order from \( \{0, 1, 2, ..., k-1\} \) in such a way that vertex condition will be satisfied for \( k \)-cordial labeling.

2. Label the next \( j \) vertices by remaining odd numbers in descending order from \( \{0, 1, 2, ..., k-1\} \) in such a way that vertex condition will be satisfied for \( k \)-cordial labeling.

3. Label the remaining \( j \) vertices by the even numbers in ascending order from \( \{0, 1, 2, ..., k-1\} \) but except the numbers which are labelled in step 1 and also labelled in such a way that vertex condition will be satisfied for \( k \)-cordial labeling.

4. Now label next one vertex fix by the same labeling which is labelled on \( u_{2m-1} \) vertex of path.

5. Label the remaining \( j \) vertices by the odd numbers in ascending order from \( \{0, 1, 2, ..., k-1\} \) but except the numbers which are labelled in step 2 and 4 and also labelled in such a way that vertex condition will be satisfied for \( k \)-cordial labeling.

Repeat above steps \( r \) times.

The labeling pattern of second block of \( t \) vertices \( v_1', v_2', ..., v_t' \) is defined in the following steps.

Here, \( 1 \leq t \leq k-1 \).

Case 1: If in path vertices \( u_{2m-1}^{th} \) vertex labelled by even number.

1. First label the \( t \) vertices by remaining odd numbers in ascending order from \( \{0, 1, 2, ..., k-1\} \) in such a way that vertex condition will be satisfied for \( k \)-cordial labeling.

2. Label the next \( t \) vertices by remaining even numbers in descending order from \( \{0, 1, 2, ..., k-1\} \) in such a way that vertex condition will be satisfied for \( k \)-cordial labeling.

3. Label the remaining \( t \) vertices by the even numbers in ascending order from \( \{0, 1, 2, ..., k-1\} \) but except the numbers which are labelled in step 2 and also labelled in such a way that vertex condition will be satisfied for \( k \)-cordial labeling.

4. Now label the next one vertex fix by the same labeling which is labelled on \( u_2 \) vertex of path.

5. Label the remaining \( t \) vertices by the odd numbers in ascending order from \( \{0, 1, 2, ..., k-1\} \) but except the numbers which are labelled in step 1 and 4 and also labelled in such a way that vertex condition will be satisfied for \( k \)-cordial labeling.

Case 2: If in path vertices \( u_{2m-1}^{th} \) vertex labelled by odd number.

1. First label the \( t \) vertices by remaining even numbers in ascending order from \( \{0, 1, 2, ..., k-1\} \) in such a way that vertex condition will be satisfied for \( k \)-cordial labeling.

2. Label the next \( t \) vertices by remaining odd numbers in descending order from \( \{0, 1, 2, ..., k-1\} \) in such a way that vertex condition will be satisfied for \( k \)-cordial labeling.

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3. Label the remaining \( t \) vertices by the even numbers in ascending order from \( \{0, 1, 2, \ldots, k-1\} \) but except the numbers which are labelled in step 1 and also labelled in such a way that vertex condition will be satisfied for \( k \)-cordial labeling.

4. Now label the next one vertex fix by the same labeling which is labelled on \( u_{2m-1} \) vertex of path.

5. Label the remaining \( t \) vertices by the odd numbers in ascending order from \( \{0, 1, 2, \ldots, k-1\} \) but except the numbers which are labelled in step 2 and 4 and also labelled in such a way that vertex condition will be satisfied for \( k \)-cordial labeling.

The labeling pattern defined above covers all possible arrangements of vertices. In each possibility the graph under consideration satisfies the vertex conditions and edge conditions of \( k \)-cordial labeling. That is \( G \) admits \( k \)-cordial labeling.

**Illustration 4.2.10** The graph \( B(3, 8)(P_{16}) \) and its 6-cordial labeling is shown in Figure 4.2.10.

![Figure 4.2.10: 6-cordial labeling of the graph \( B(3, 8)(P_{16}) \).](image)

**Theorem 4.2.9** The graph \( B(3, n)(P_{2m}) \) is \( k \)-cordial for all \( n \) and \( 4 < 2m < lk \), where \( k \) is even and \( l \) is the smallest positive even integer such that \( 2m < lk \).

**Proof:** Let \( B(3, n)(P_{2m}) \) be the graph with the vertex set \( V(G) = \{u_i, v_j, 1 \leq i \leq 2m, 1 \leq j \leq n\} \) and the edge set \( E(G) = \{u_iu_{i+1}, u_iv_j, u_mv_j : 1 \leq i \leq (2m - 1), 1 \leq j \leq n\} \). Let \( u_1, u_2, \ldots, u_{2m} \) be the vertices of path \( P_{2m} \) and \( v_1, v_2, \ldots, v_n \) be the vertices of pages. It is noted that \( |V(G)| = 2m + n \) and \( |E(G)| = 2n + 2m - 1 \).

Define \( k \)-cordial labeling \( f : V(G) \rightarrow Z_k \) as follows:
After defining the labeling pattern in the path vertices, define the labeling pattern for the vertices of the pages.

The labeling pattern of the path vertices $u_1, u_2, \ldots, u_{2m}$ is defined as follows:

Here, $1 \leq i \leq 2m$.

- $f(u_1) = 0$ and $f(u_{2m}) = k - 1$.
- $f(u_{2i+1}) = p_i$; $i \equiv p_i \pmod{k}$; $1 \leq i \leq 2m - 1$.
- $f(u_{2i(m-1)}) = k - (m - p_i)$; $i \equiv p_i \pmod{k}$; $0 \leq i \leq 2m - 1$.

The labeling pattern of the vertices $v_1, v_2, \ldots, v_n$ is defined as follows.

Let $n = rk + t$. Divide the $n$ vertices into two blocks say $rk$ and $t$.

The labeling pattern of the first block of $rk$ vertices $v_1, v_2, \ldots, v_{rk}$ is defined in the following cases. Here, $1 \leq j \leq rk$ and $r \geq 0$.

**Case 1:** If in path vertices $u_{2m-1}^{th}$ vertex labelled by even number.

1. First label the $j$ vertices by remaining odd numbers in ascending order from $\{0, 1, 2, \ldots, k-1\}$ in such a way that vertex condition will be satisfied for $k$-cordial labeling.

2. Now label next one vertex fix by the previous number from the number which is labelled on $u_2$ vertex of path.

3. Label the next $j$ vertices by remaining even numbers in ascending order from $\{0, 1, 2, \ldots, k-1\}$ but except the numbers which are labelled in step 2 and also labelled in such a way that vertex condition will be satisfied for $k$-cordial labeling.

4. Label the remaining $j$ vertices by the odd numbers in descending order from $\{0, 1, 2, \ldots, k-1\}$ but except the numbers which are labelled in step 2 and 3 and also labelled in such a way that vertex condition will be satisfied for $k$-cordial labeling.

5. Label the remaining $j$ vertices by the even numbers in descending order from $\{0, 1, 2, \ldots, k-1\}$ but except the numbers which are labelled in step 1 and 2 and also labelled in such a way that vertex condition will be satisfied for $k$-cordial labeling.

Repeat above steps $r$ times.

**Case 2:** If in path vertices $u_{2m-1}^{th}$ vertex labelled by odd number.

1. First label the $j$ vertices by remaining even numbers in ascending order from $\{0, 1, 2, \ldots, k-1\}$ in such a way that vertex condition will be satisfied for $k$-cordial labeling.

2. Now label next one vertex fix by the previous number from the number which is labelled on $u_2$ vertex of path.

3. Labelled the next $j$ vertices by remaining odd numbers in ascending order from $\{0, 1, 2, \ldots, k-1\}$ but except the numbers which are labelled in step 2 and also labelled in such a way that vertex condition will be satisfied for $k$-cordial labeling.

4. Label the remaining $j$ vertices by the even numbers in descending order from $\{0, 1, 2, \ldots, k-1\}$ but except the numbers which are labelled in step 1 and also labelled in such a way that vertex condition will be satisfied for $k$-cordial labeling.
5. Label the remaining \( j \) vertices by the odd numbers in descending order from \( \{0, 1, 2, \ldots, k-1\} \) but except the numbers which are labelled in step 2 and 3 and also labelled in such a way that vertex condition will be satisfied for \( k \)-cordial labeling.

Repeat above steps \( r \) times.

The labeling pattern of second block of \( t \) vertices \( v'_1, v'_2, \ldots, v'_t \) is defined in the following steps. Here, \( 1 \leq t \leq k - 1 \).

**Case 1**: If in path vertices \( u_{2m-1} \)th vertex labelled by even number.

1. First label the \( t \) vertices by remaining odd numbers in ascending order from \( \{0, 1, 2, \ldots, k-1\} \) in such a way that vertex condition will be satisfied for \( k \)-cordial labeling.

2. Now label next one vertex fix by the previous number from the number which is labelled on \( u_2 \) vertex of path.

3. Label the next \( t \) vertices by remaining even numbers in ascending order from \( \{0, 1, 2, \ldots, k-1\} \) but except the numbers which are labelled in step 2 and also labelled in such a way that vertex condition will be satisfied for \( k \)-cordial labeling.

4. Label the remaining \( t \) vertices by the odd numbers in descending order from \( \{0, 1, 2, \ldots, k-1\} \) but except the numbers which are labelled in step 1 and also labelled in such a way that vertex condition will be satisfied for \( k \)-cordial labeling.

5. Label the remaining \( t \) vertices by the even numbers in descending order from \( \{0, 1, 2, \ldots, k-1\} \) but except the numbers which are labelled in step 2 and 3 and also labelled in such a way that vertex condition will be satisfied for \( k \)-cordial labeling.

**Case 2**: If in path vertices \( u_{2m-1} \)th vertex labelled by odd number.

1. First label the \( t \) vertices by remaining even numbers in ascending order from \( \{0, 1, 2, \ldots, k-1\} \) in such a way that vertex condition will be satisfied for \( k \)-cordial labeling.

2. Now label next one vertex fix by the previous number from the number which is labelled on \( u_2 \) vertex of path.

3. Label the next \( t \) vertices by remaining odd numbers in ascending order from \( \{0, 1, 2, \ldots, k-1\} \) but except the numbers which are labelled in step 2 and also labelled in such a way that vertex condition will be satisfied for \( k \)-cordial labeling.

4. Label the remaining \( t \) vertices by the even numbers in descending order from \( \{0, 1, 2, \ldots, k-1\} \) but except the numbers which are labelled in step 1 and also labelled in such a way that vertex condition will be satisfied for \( k \)-cordial labeling.

5. Label the remaining \( t \) vertices by the odd numbers in descending order from \( \{0, 1, 2, \ldots, k-1\} \) but except the numbers which are labelled in step 2 and 3 and also labelled in such a way that vertex condition will be satisfied for \( k \)-cordial labeling.
The labeling pattern defined above covers all possible arrangements of vertices. In each possibility the graph under consideration satisfies the vertex conditions and edge conditions of $k$-cordial labeling. That is $G$ admits $k$-cordial labeling.

Illustration 4.2.11 The graph $B(3, 8)(P_{10})$ and its 8-cordial labeling is shown in Figure 4.2.11.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure4.2.11.png}
\caption{8-cordial labeling of $B(3, 8)(P_{10})$.}
\end{figure}

Theorem 4.2.10 The graph $B(3, n)(P_{2m})$ is $k$-cordial for all $n$, $4 \leq 2m$ and odd $k$.

Proof: Let $B(3, n)(P_{2m})$ be the graph with the vertex set $V(G) = \{u_i, v_j, 1 \leq i \leq 2m, 1 \leq j \leq n\}$ and the edge set $E(G) = \{u_iu_{i+1}, u_iv_j, u_mv_j : 1 \leq i \leq (2m-1), 1 \leq j \leq n\}$. Let $u_1, u_2, ..., u_{2m}$ be the vertices of path $P_{2m}$ and $v_1, v_2, ..., v_n$ be the vertices of pages. It is noted that $|V(G)| = 2m + n$ and $|E(G)| = 2n + 2m - 1$.

Define $k$-cordial labeling $f : V(G) \to Z_k$ as follows:

After defining the labeling pattern in the path vertices, define the labeling pattern for the vertices of the pages. The labeling pattern of the path vertices $u_1, u_2, ..., u_{2m}$ is defined as follows:

Here, $1 \leq i \leq 2m$.

$f(u_1) = 0$ and $f(u_{2m}) = k - 1$.

$f(u_{2i+1}) = p_i$; \hspace{0.5cm} $i \equiv p_i \text{ (mod} k\text{)}$; \hspace{0.5cm} $1 \leq i \leq 2m - 1$,

$f(u_{2(i+1)}) = k - (m - p_i)$; \hspace{0.5cm} $i \equiv p_i \text{ (mod} k\text{)}$; \hspace{0.5cm} $0 \leq i \leq 2m - 1$.

The labeling pattern of the vertices $v_1, v_2, ..., v_n$ is defined as follows.

Let $n = rk + t$. Divide the $n$ vertices into two blocks say $rk$ and $t$.

The labeling pattern of the first block of $rk$ vertices $v_1, v_2, ..., v_{rk}$ is defined in the following cases.

Here, $1 \leq j \leq rk$ and $r \geq 0$.

Case 1: If in path vertices $u_{2m-1}^{th}$ vertex labelled by even number.
1. First label the $j$ vertices by remaining odd numbers in ascending order from $\{0, 1, 2, \ldots, k-1\}$ in such a way that vertex condition will be satisfied for $k$-cordial labeling.

2. Label the next $j$ vertices by remaining even numbers in ascending order from $\{0, 1, 2, \ldots, k-1\}$ in such a way that vertex condition will be satisfied for $k$-cordial labeling.

3. Label the remaining $j$ vertices by the even numbers in ascending order from $\{0, 1, 2, \ldots, k-1\}$ but except the numbers which are labelled in step 2 and also labelled in such a way that vertex condition will be satisfied for $k$-cordial labeling.

4. Label the remaining $j$ vertices by the odd numbers in ascending order from $\{0, 1, 2, \ldots, k-1\}$ but except the numbers which are labelled in step 1 and also labelled in such a way that vertex condition will be satisfied for $k$-cordial labeling.

Repeat above steps $r$ times.

Case 2: If in path vertices $u_{2m-1}$th vertex labelled by odd number.

1. First label the $j$ vertices by remaining even numbers in ascending order from $\{0, 1, 2, \ldots, k-1\}$ in such a way that vertex condition will be satisfied for $k$-cordial labeling.

2. Label the next $j$ vertices by remaining odd numbers in ascending order from $\{0, 1, 2, \ldots, k-1\}$ in such a way that vertex condition will be satisfied for $k$-cordial labeling.

3. Label the remaining $j$ vertices by the odd numbers in ascending order from $\{0, 1, 2, \ldots, k-1\}$ but except the numbers which are labelled in step 2 and also labelled in such a way that vertex condition will be satisfied for $k$-cordial labeling.

4. Label the remaining $j$ vertices by the even numbers in ascending order from $\{0, 1, 2, \ldots, k-1\}$ but except the numbers which are labelled in step 1 and also labelled in such a way that vertex condition will be satisfied for $k$-cordial labeling.

Repeat above steps $r$ times.

The labeling pattern of second block of $t$ vertices $v'_1, v'_2, \ldots, v'_t$ is defined in the following steps.

Here, $1 \leq t \leq k - 1$.

Case 1: If in path vertices $u_{2m-1}$th vertex labelled by even number.

1. First label the $t$ vertices by remaining odd numbers in ascending order from $\{0, 1, 2, \ldots, k-1\}$ in such a way that vertex condition will be satisfied for $k$-cordial labeling.

2. Label the next $t$ vertices by remaining even numbers in ascending order from $\{0, 1, 2, \ldots, k-1\}$ in such a way that vertex condition will be satisfied for $k$-cordial labeling.

3. Label the remaining $t$ vertices by the even numbers in ascending order from $\{0, 1, 2, \ldots, k-1\}$ but except the numbers which are labelled in step 2 and also labelled in such a way that vertex condition will be satisfied for $k$-cordial labeling.
4. Label the remaining \( t \) vertices by the odd numbers in ascending order from \( \{0, 1, 2, \ldots, k-1\} \) but except the numbers which are labelled in step 1 and also labelled in such a way that vertex condition will be satisfied for \( k \)-cordial labeling.

**Case 2:** If in path vertices \( u_{2m-1}^{th} \) vertex labelled by odd number.

1. First label the \( t \) vertices by remaining even numbers in ascending order from \( \{0, 1, 2, \ldots, k-1\} \) in such a way that vertex condition will be satisfied for \( k \)-cordial labeling.

2. Label the next \( t \) vertices by remaining odd numbers in ascending order from \( \{0, 1, 2, \ldots, k-1\} \) in such a way that vertex condition will be satisfied for \( k \)-cordial labeling.

3. Label the remaining \( t \) vertices by the odd numbers in ascending order from \( \{0, 1, 2, \ldots, k-1\} \) but except the numbers which are labelled in step 2 and also labelled in such a way that vertex condition will be satisfied for \( k \)-cordial labeling.

4. Label the remaining \( t \) vertices by the even numbers in ascending order from \( \{0, 1, 2, \ldots, k-1\} \) but except the numbers which are labelled in step 1 and also labelled in such a way that vertex condition will be satisfied for \( k \)-cordial labeling.

The labeling pattern defined above covers all possible arrangements of vertices. In each possibility the graph under consideration satisfies the vertex conditions and edge conditions of \( k \)-cordial labeling. That is \( G \) admits \( k \)-cordial labeling.

**Illustration 4.2.12** The graph \( B(3, 8)(P_8) \) and its 9-cordial labeling is shown in Figure 4.2.12.

![Figure 4.2.12: 9-cordial labeling of the graph \( B(3, 8)(P_8) \).](image-url)
The labeling pattern of first block of $mk$ vertices $v_1, v_2, ..., v_{mk}$ is defined as follows:

**Subcase 1:** $k \equiv 0 \pmod{2}$.

\[
\begin{align*}
  f(u) &= 0, \quad f(v) = k - 1, \quad f(x) = 1, \quad f(y) = k - 2, \\
  f(v_i) &= 2p_i + 1; \quad i \equiv p_i \pmod{k}; \quad 1 \leq i \leq \frac{k}{2} - 2, \\
  f(v_i) &= 2(k - p_i) - 6; \quad i \equiv p_i \pmod{k}; \quad \frac{k}{2} - 1 \leq i \leq k - 4, \\
  f(v_{mk-3}) &= k - 1, \\
  f(v_{mk-2}) &= 1, \\
  f(v_{mk-1}) &= k - 2, \\
  f(v_{mk}) &= 0, \quad \forall m > 0.
\end{align*}
\]

**Subcase 2:** $k \equiv 1 \pmod{2}$.

\[
\begin{align*}
  f(u) &= 0, \quad f(v) = k - 1, \quad f(x) = 1, \quad f(y) = k - 2, \\
  f(v_i) &= 2p_i + 1; \quad i \equiv p_i \pmod{k}; \quad 1 \leq i \leq \frac{k-1}{2} - 2, \\
  f(v_i) &= 2(k - p_i) - 6; \quad i \equiv p_i \pmod{k}; \quad \frac{k-1}{2} - 1 \leq i \leq k - 4, \\
  f(v_{mk-3}) &= k - 1, \\
  f(v_{mk-2}) &= 1, \\
  f(v_{mk-1}) &= k - 2, \\
  f(v_{mk}) &= 0, \quad \forall m > 0.
\end{align*}
\]

**Case 2:** $m \geq 0$ and $1 \leq j \leq k - 1$.

**Subcase 1:** $k \equiv 0 \pmod{2}$.

The labeling pattern of first block of $mk$ vertices $v_1, v_2, ..., v_{mk}$ is defined as follows:

\[
\begin{align*}
  f(u) &= 0, \quad f(v) = k - 1, \quad f(x) = 1, \quad f(y) = k - 2, \\
  f(v_i) &= 2p_i + 1; \quad i \equiv p_i \pmod{k}; \quad 1 \leq i \leq \frac{k}{2} - 2, \\
  f(v_i) &= 2(k - p_i) - 6; \quad i \equiv p_i \pmod{k}; \quad \frac{k}{2} - 1 \leq i \leq k - 4, \\
  f(v_{mk-3}) &= k - 1, \\
  f(v_{mk-2}) &= 1, \\
  f(v_{mk-1}) &= k - 2, \\
  f(v_{mk}) &= 0, \quad \forall m > 0.
\end{align*}
\]

The labeling pattern of second block of $j$ vertices $v'_1, v'_2, ..., v'_j$ is defined as follows:
Subcase 2: $k \equiv 1 (mod \ 2)$.

The labeling pattern of first block of $mk$ vertices $v_1, v_2, ..., v_{mk}$ is defined as follows:

- $f(u) = 0$, $f(v) = k - 1$, $f(x) = 1$, $f(y) = k - 2$,
- $f(v_i) = 2p_i + 1$; \quad $i \equiv p_i (mod \ k)$; \quad $1 \leq i \leq \frac{k-1}{2} - 2$,
- $f(v_i) = 2(k - p_i) - 6$; \quad $i \equiv p_i (mod \ k)$; \quad $\frac{k-1}{2} - 1 \leq i \leq k - 4$,
- $f(v_{mk-3}) = k - 1$,
- $f(v_{mk-2}) = 1$,
- $f(v_{mk-1}) = k - 2$,
- $f(v_{mk}) = 0$, $\forall m > 0$.

The labeling pattern of second block of $j$ vertices $v'_1, v'_2, ..., v'_j$ is defined as follows:

- $f(v'_j) = 2j + 1$; \quad $1 \leq j \leq \frac{k}{2} - 2$,
- $f(v'_j) = 2(k - j) - 6$; \quad $\frac{k}{2} - 1 \leq j \leq k - 4$,
- $f(v'_{k-3}) = k - 1$,
- $f(v'_{k-2}) = 1$,
- $f(v'_{k-1}) = k - 2$.

The labeling pattern defined above covers all possible arrangements of vertices. In each possibility the graph under consideration satisfies the vertex conditions and edge conditions of $k$-cordial labeling. That is $G$ admits $k$-cordial labeling.

**Illustration 4.2.13** The jewel graph $J_8$ and its 8-cordial labeling is shown in Figure 4.2.13.

Figure 4.2.13: 8-cordial labeling of jewel graph $J_8$. 
Theorem 4.2.12 The triangular belt $TB(n)(\mathbb{Z}^n)$ is $k$-cordial for all $n$.

Proof: Let $G = TB(n)$ be the triangular belt. Let $L_n = P_n \times P_2$ $(n \geq 2)$ be the ladder graph with vertex set $u_i$ and $v_i$, $i = 1, 2, ..., n$. The triangular belt is obtained from the ladder by adding the edges $u_i u_{i+1}$ for all $1 \leq i \leq n - 1$. Let $n = mk + j$, where $m \geq 0$ and $1 \leq j \leq k - 1$. Divide $n$ vertices into two blocks of $mk$ and $j$, which are denoted by $u_1, u_2, ..., u_{mk}$ and $u'_1, u'_2, ..., u'_j$ also $v_1, v_2, ..., v_{mk}$ and $v'_1, v'_2, ..., v'_j$. It is noted that $|V(G)| = 2n$ and $|E(G)| = 4n - 3$.

To define $k$-cordial labeling $f : V(G) \rightarrow \mathbb{Z}_k$ following cases are considered:

Case 1: $m > 0$ and $j = 0$.

The labeling pattern of $mk$ vertices is defined as follows:

Subcase 1: $k \equiv 0(\text{mod } 2)$, $1 \leq i \leq mk$.

\begin{align*}
  f(u_i) & = 2p_i - 1; \quad i \equiv p_i(\text{mod } k); \quad 1 \leq i \leq k, \\
  f(v_i) & = 2p_i - 2; \quad i \equiv p_i(\text{mod } k); \quad 1 \leq i \leq k.
\end{align*}

Subcase 2: $k \equiv 1(\text{mod } 2)$, $1 \leq i \leq mk$.

\begin{align*}
  f(u_i) & = 2p_i - 1; \quad i \equiv p_i(\text{mod } k); \quad 1 \leq i \leq \frac{k-1}{2}, \\
  f(u_i) & = 2p_i; \quad i \equiv p_i(\text{mod } k + 1); \quad \frac{k-1}{2} + 1 \leq i \leq k, \\
  f(v_i) & = 2p_i - 2; \quad i \equiv p_i(\text{mod } k); \quad 1 \leq i \leq \frac{k-1}{2} + 1, \\
  f(v_i) & = 2p_i - 1; \quad i \equiv p_i(\text{mod } k + 1); \quad \frac{k-1}{2} + 2 \leq i \leq k.
\end{align*}

Case 2: $m \geq 0$ and $1 \leq j \leq k - 1$.

Subcase 1: $k \equiv 0(\text{mod } 2)$.

The labeling pattern of first block of $mk$ vertices is defined as follows:

\begin{align*}
  f(u_i) & = 2p_i - 1; \quad i \equiv p_i(\text{mod } k); \quad 1 \leq i \leq k, \\
  f(v_i) & = 2p_i - 2; \quad i \equiv p_i(\text{mod } k); \quad 1 \leq i \leq k.
\end{align*}

The labeling pattern of second block of $j$ vertices is defined as follows:

\begin{align*}
  f(u'_j) & = 2j - 1; \quad 1 \leq j \leq k - 1, \\
  f(v'_j) & = 2j - 2; \quad 1 \leq j \leq k - 1.
\end{align*}

Subcase 2: $k \equiv 1(\text{mod } 2)$.

The labeling pattern of first block of $mk$ vertices is defined as follows:

\begin{align*}
  f(u_i) & = 2p_i - 1; \quad i \equiv p_i(\text{mod } k); \quad 1 \leq i \leq \frac{k-1}{2}, \\
  f(u_i) & = 2p_i; \quad i \equiv p_i(\text{mod } k + 1); \quad \frac{k-1}{2} + 1 \leq i \leq k, \\
  f(v_i) & = 2p_i - 2; \quad i \equiv p_i(\text{mod } k); \quad 1 \leq i \leq \frac{k-1}{2} + 1, \\
  f(v_i) & = 2p_i - 1; \quad i \equiv p_i(\text{mod } k + 1); \quad \frac{k-1}{2} + 2 \leq i \leq k.
\end{align*}

The labeling pattern of second block of $j$ vertices is defined as follows:

\begin{align*}
  f(u'_j) & = 2j - 1; \quad 1 \leq j \leq \frac{k-1}{2}, \\
  f(u'_j) & = 2j; \quad \frac{k-1}{2} + 1 \leq j \leq k - 1,
\end{align*}
\[ f(v'_j) = 2j - 2; \quad 1 \leq j \leq \frac{k-1}{2} + 1, \]
\[ f(v'_j) = 2j - 1; \quad \frac{k-1}{2} + 2 \leq j \leq k - 1. \]

The labeling pattern defined above covers all possible arrangements of vertices. In each possibility the graph under consideration satisfies the vertex conditions and edge conditions of \( k \)-cordial labeling. That is \( G \) admits \( k \)-cordial labeling.

**Illustration 4.2.14** The triangular belt \( TB(9)(\downarrow^9) \) and its 8-cordial labeling is shown in Figure 4.2.14.

![Figure 4.2.14: 8-cordial labeling of triangular belt \( TB(9)(\downarrow^9) \).](image)

**Illustration 4.2.15** The triangular belt \( TB(10)(\downarrow^{10}) \) and its 7-cordial labeling is shown in Figure 4.2.15.

![Figure 4.2.15: 7-cordial labeling of triangular belt \( TB(10)(\downarrow^{10}) \).](image)

**Theorem 4.2.13** The alternate triangular belt \( ATB(n)(\downarrow\uparrow\downarrow\uparrow\ldots) \) is \( k \)-cordial for all \( n \).

**Proof:** Let \( G = ATB(n) \) be the alternate triangular belt. Let \( L_n = P_n \times P_2 \) (\( n \geq 2 \)) be the ladder graph with vertex set \( u_i \) and \( v_i, i = 1, 2, \ldots, n \). The alternate triangular belt is obtained from the ladder by adding the edges \( u_{2i+1}v_{2i+2} \) for all \( i = 0, 1, 2, \ldots, n - 1 \) and...
The labeling pattern of second block of vertices is defined as follows:

Subcase 1: $k \equiv 0 (mod 2)$, $1 \leq i \leq mk$.

- $f(u_i) = 2p_i - 1$; \hspace{1cm} $i \equiv p_i (mod k)$; \hspace{1cm} $1 \leq i \leq k$,
- $f(v_i) = 2p_i - 2$; \hspace{1cm} $i \equiv p_i (mod k)$; \hspace{1cm} $1 \leq i \leq k$.

Subcase 2: $k \equiv 1 (mod 2)$, $1 \leq i \leq mk$.

- $f(u_i) = 2p_i - 1$; \hspace{1cm} $i \equiv p_i (mod k)$; \hspace{1cm} $1 \leq i \leq \frac{k-1}{2}$,
- $f(u_i) = 2p_i$; \hspace{1cm} $i \equiv p_i (mod k + 1)$; \hspace{1cm} $\frac{k-1}{2} + 1 \leq i \leq k$,
- $f(v_i) = 2p_i - 2$; \hspace{1cm} $i \equiv p_i (mod k)$; \hspace{1cm} $1 \leq i \leq \frac{k-1}{2} + 1$,
- $f(v_i) = 2p_i - 1$; \hspace{1cm} $i \equiv p_i (mod k + 1)$; \hspace{1cm} $\frac{k-1}{2} + 2 \leq i \leq k$.

Case 2: $m \geq 0$ and $1 \leq j \leq k - 1$.

Subcase 1: $k \equiv 0 (mod 2)$.

The labeling pattern of first block of $mk$ vertices is defined as follows:

- $f(u_i) = 2p_i - 1$; \hspace{1cm} $i \equiv p_i (mod k)$; \hspace{1cm} $1 \leq i \leq k$,
- $f(v_i) = 2p_i - 2$; \hspace{1cm} $i \equiv p_i (mod k)$; \hspace{1cm} $1 \leq i \leq k$.

The labeling pattern of second block of $j$ vertices is defined as follows:

- $f(u'_j) = 2j - 1$; \hspace{1cm} $1 \leq j \leq k - 1$,
- $f(v'_j) = 2j - 2$; \hspace{1cm} $1 \leq j \leq k - 1$.

Subcase 2: $k \equiv 1 (mod 2)$.

The labeling pattern of first block of $mk$ vertices is defined as follows:

- $f(u_i) = 2p_i - 1$; \hspace{1cm} $i \equiv p_i (mod k)$; \hspace{1cm} $1 \leq i \leq \frac{k-1}{2}$,
- $f(u_i) = 2p_i$; \hspace{1cm} $i \equiv p_i (mod k + 1)$; \hspace{1cm} $\frac{k-1}{2} + 1 \leq i \leq k$,
- $f(v_i) = 2p_i - 2$; \hspace{1cm} $i \equiv p_i (mod k)$; \hspace{1cm} $1 \leq i \leq \frac{k-1}{2} + 1$,
- $f(v_i) = 2p_i - 1$; \hspace{1cm} $i \equiv p_i (mod k + 1)$; \hspace{1cm} $\frac{k-1}{2} + 2 \leq i \leq k$.

The labeling pattern of second block of $j$ vertices is defined as follows:

- $f(u'_j) = 2j - 1$; \hspace{1cm} $1 \leq j \leq \frac{k-1}{2}$,
- $f(u'_j) = 2j$; \hspace{1cm} $\frac{k-1}{2} + 1 \leq j \leq k - 1$,
- $f(v'_j) = 2j - 2$; \hspace{1cm} $1 \leq j \leq \frac{k-1}{2} + 1$,
- $f(v'_j) = 2j - 1$; \hspace{1cm} $\frac{k-1}{2} + 2 \leq j \leq k - 1$.

The labeling pattern defined above covers all possible arrangements of vertices. In each possibility the graph under consideration satisfies the vertex conditions and edge condi-
Illustration 4.2.16 The alternate triangular belt $ATB(6)$ and its 8-cordial labeling is shown in Figure 4.2.16.

![Figure 4.2.16: 8-cordial labeling of an alternate triangular belt $ATB(6)$.](image)

Illustration 4.2.17 The alternate triangular belt $ATB(8)$ and its 7-cordial labeling is shown in Figure 4.2.17.

![Figure 4.2.17: 7-cordial labeling of an alternate triangular belt $ATB(8)$.](image)

Theorem 4.2.14 The graph $Z-P_n$ is $k$-cordial for all odd $k$ and for all $n$.

Proof: Let $G = Z-P_n$ be the graph obtained from a pair of paths $P_n'$ and $P_n''$. Let $v_1, v_2, ..., v_n$ be the vertices of path $P_n'$ and $u_1, u_2, ..., u_n$ are the vertices of path $P_n''$. To find $Z-P_n$ join $i^{th}$ vertex of path $P_n'$ with $(i+1)^{th}$ vertex of path $P_n''$ for all $1 \leq i \leq n-1$. Let $n = mk + j$, where $m \geq 0$ and $1 \leq j \leq k - 1$. Divide $n$ vertices into two blocks of $mk$ and $j$, which are denoted by $u_1, u_2, ..., u_{mk}$ and $u_1', u_2', ..., u_j'$ also $v_1, v_2, ..., v_{mk}$ and $v_1', v_2', ..., v_j'$. It is noted that $|V(G)| = 2n$ and $|E(G)| = 3n - 3$.

Define $k$-cordial labeling $f : V(G) \rightarrow Z_k$ as follows:
Chapter 4. Results

The labeling pattern of first block of $mk$ vertices is defined as follows.

$$f(u_i) = 2p_i - 1; \quad i \equiv p_i \pmod{k}; \quad 1 \leq i \leq \frac{k-1}{2},$$

$$f(u_i) = 2p_i; \quad i \equiv p_i \pmod{k+1}; \quad \frac{k-1}{2} + 1 \leq i \leq k,$$

$$f(v_i) = 2p_i - 2; \quad i \equiv p_i \pmod{k}; \quad 1 \leq i \leq \frac{k-1}{2} + 1,$$

$$f(v_i) = 2p_i - 1; \quad i \equiv p_i \pmod{k+1}; \quad \frac{k-1}{2} + 2 \leq i \leq k.$$

The labeling pattern of second block of $j$ vertices is defined as follows:

$$f(u'_j) = 2j - 1; \quad 1 \leq j \leq \frac{k-1}{2},$$

$$f(u'_j) = 2j; \quad \frac{k-1}{2} + 1 \leq j \leq k - 1,$$

$$f(v'_j) = 2j - 2; \quad 1 \leq j \leq \frac{k-1}{2} + 1,$$

$$f(v'_j) = 2j - 1; \quad \frac{k-1}{2} + 2 \leq j \leq k - 1.$$

The labeling pattern defined above covers all possible arrangements of vertices. In each possibility the graph under consideration satisfies the vertex conditions and edge conditions of $k$-cordial labeling. That is $G$ admits $k$-cordial labeling.

**Illustration 4.2.18** The graph $Z \cdot P_{10}$ and its $5$-cordial labeling is shown in Figure 4.2.18.

![Figure 4.2.18: 5-cordial labeling of $Z \cdot P_{10}$](image)

**Theorem 4.2.15** The braid graph $B(n)$ is $k$-cordial for all $n$.

**Proof:** Let $G = B(n)$ be the braid graph. The braid graph is obtained from a pair of paths $P_n'$ and $P_n''$. Let $u_1, u_2, ..., u_n$ be the vertices of path $P_n'$ and $v_1, v_2, ..., v_n$ are the vertices of path $P_n''$. To find braid graph join $i^{th}$ vertex of path $P_n'$ with $(i + 1)^{th}$ vertex of path $P_n''$ and $i^{th}$ vertex of path $P_n''$ with $(i + 2)^{th}$ vertex of path $P_n'$ with the new edges for all $1 \leq i \leq n - 2$. Let $n = mk + j$, where $m \geq 0$ and $1 \leq j \leq k - 1$. Divide $n$ vertices into two blocks of $mk$ and $j$, which are denoted by $u_1, u_2, ..., u_{mk}$ and $u'_1, u'_2, ..., u'_j$ also $v_1, v_2, ..., v_{mk}$ and $v'_1, v'_2, ..., v'_j$. It is noted that $|V(G)| = 2n$ and $|E(G)| = 4n - 5$.

To define $k$-cordial labeling $f : V(G) \to Z_k$ following cases are considered:
Case 1: \( m > 0 \) and \( j = 0 \).
The labeling pattern of \( mk \) vertices is defined as follows:

Subcase 1: \( k \equiv 0(\text{mod } 2) \), \( 1 \leq i \leq mk \).
\[
\begin{align*}
f(u_i) &= 2p_i - 1; & i &\equiv p_i(\text{mod } k); & 1 \leq i \leq k, \\
f(v_i) &= 2p_i - 2; & i &\equiv p_i(\text{mod } k); & 1 \leq i \leq k.
\end{align*}
\]

Subcase 2: \( k \equiv 1(\text{mod } 2) \), \( 1 \leq i \leq mk \).
\[
\begin{align*}
f(u_i) &= 2p_i - 1; & i &\equiv p_i(\text{mod } k); & 1 \leq i \leq \frac{k-1}{2}, \\
f(u_i) &= 2p_i; & i &\equiv p_i(\text{mod } k + 1); & \frac{k-1}{2} + 1 \leq i \leq k, \\
f(v_i) &= 2p_i - 2; & i &\equiv p_i(\text{mod } k); & 1 \leq i \leq \frac{k-1}{2} + 1, \\
f(v_i) &= 2p_i - 1; & i &\equiv p_i(\text{mod } k + 1); & \frac{k-1}{2} + 2 \leq i \leq k.
\end{align*}
\]

Case 2: \( m \geq 0 \) and \( 1 \leq j \leq k - 1 \).

Subcase 1: \( k \equiv 0(\text{mod } 2) \).
The labeling pattern of first block of \( mk \) vertices is defined as follows:
\[
\begin{align*}
f(u_i) &= 2p_i - 1; & i &\equiv p_i(\text{mod } k); & 1 \leq i \leq k, \\
f(v_i) &= 2p_i - 2; & i &\equiv p_i(\text{mod } k); & 1 \leq i \leq k.
\end{align*}
\]
The labeling pattern of second block of \( j \) vertices is defined as follows:
\[
\begin{align*}
f(u'_j) &= 2j - 1; & 1 \leq j \leq k - 1, \\
f(v'_j) &= 2j - 2; & 1 \leq j \leq k - 1.
\end{align*}
\]

Subcase 2: \( k \equiv 1(\text{mod } 2) \).
The labeling pattern of first block of \( mk \) vertices is defined as follows:
\[
\begin{align*}
f(u_i) &= 2p_i - 1; & i &\equiv p_i(\text{mod } k); & 1 \leq i \leq \frac{k-1}{2}, \\
f(u_i) &= 2p_i; & i &\equiv p_i(\text{mod } k + 1); & \frac{k-1}{2} + 1 \leq i \leq k, \\
f(v_i) &= 2p_i - 2; & i &\equiv p_i(\text{mod } k); & 1 \leq i \leq \frac{k-1}{2} + 1, \\
f(v_i) &= 2p_i - 1; & i &\equiv p_i(\text{mod } k + 1); & \frac{k-1}{2} + 2 \leq i \leq k.
\end{align*}
\]
The labeling pattern of second block of \( j \) vertices is defined as follows:
\[
\begin{align*}
f(u'_j) &= 2j - 1; & 1 \leq j \leq \frac{k-1}{2}, \\
f(u'_j) &= 2j; & \frac{k-1}{2} + 1 \leq j \leq k - 1, \\
f(v'_j) &= 2j - 2; & 1 \leq j \leq \frac{k-1}{2} + 1, \\
f(v'_j) &= 2j - 1; & \frac{k-1}{2} + 2 \leq j \leq k - 1.
\end{align*}
\]
The labeling pattern defined above covers all possible arrangements of vertices. In each possibility the graph under consideration satisfies the vertex conditions and edge conditions of \( k \)-cordial labeling. That is \( G \) admits \( k \)-cordial labeling.
Illustration 4.2.19 The braid graph $B(9)$ and its 10-cordial labeling is shown in Figure 4.2.19.

![Figure 4.2.19: 10-cordial labeling of braid graph $B(9)$](image)

Illustration 4.2.20 The braid graph $B(10)$ and its 9-cordial labeling is shown in Figure 4.2.20.

![Figure 4.2.20: 9-cordial labeling of braid graph $B(10)$](image)

Theorem 4.2.16 The square graph $P_n^2$ of path $P_n$ is $k$-cordial.

Proof: Let $G = P_n^2$ be the square graph of path $P_n$ with vertices $v_1, v_2, ..., v_n$. It is noted that $|V(G)| = n$ and $|E(G)| = 2n - 3$.

Define $k$-cordial labeling $f : V(G) \rightarrow Z_k$ as follows:

$$f(v_i) = p_i - 1; \quad i \equiv p_i (mod k); \quad 1 \leq i \leq n.$$ 

The labeling pattern defined above covers all possible arrangements of vertices. In each possibility the graph under consideration satisfies the vertex conditions and edge conditions of $k$-cordial labeling. That is $G$ admits $k$-cordial labeling.
Illustration 4.2.21 The square graph $P_{12}^2$ of the path $P_{12}$ and its 8-cordial labeling is shown in Figure 4.2.21.

Figure 4.2.21: 8-cordial labeling of square graph $P_{12}^2$ of the path $P_{12}$. 
4.3 $V_4$-cordial Labeling

$V_4$-cordial Labeling

Let $V_4 = Z_2 \times Z_2 = \{0 =< 0, 0 >, a =< 1, 0 >, b =< 0, 1 >, c =< 1, 1 >\}$ be the Klein-four group in which the operation $\ast$ is defined as follows:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>a</td>
<td>b</td>
<td>c</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>0</td>
<td>0</td>
<td>c</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>c</td>
<td>0</td>
<td>a</td>
</tr>
<tr>
<td>c</td>
<td>c</td>
<td>b</td>
<td>a</td>
<td>0</td>
</tr>
</tbody>
</table>

The graph $G = (V(G), E(G))$, with vertex set $V(G)$ and edge set $E(G)$, is said to be $V_4$-cordial if there exist a mapping $f : V(G) \to V_4$ which satisfies the following two conditions when the edge $e = uv$ is labelled as $f(u) \ast f(v)$

(i) $|v_f(p) - v_f(q)| \leq 1$; for all $p, q \in V_4$.
(ii) $|e_f(p) - e_f(q)| \leq 1$; for all $p, q \in V_4$.

Where

$v_f(p)$=the number of vertices with label $p$;
$v_f(q)$=the number of vertices with label $q$;
$e_f(p)$=the number of edges with label $p$;
$e_f(q)$=the number of edges with label $q$.

Some Existing Results of $V_4$-cordial Labeling

The concept of $V_4$-cordiality was introduced by Riskin [31] and proved the following results:

- Complete graph $K_n$ is $V_4$-cordial if and only if $n < 4$.
- The star graph $K_{1,n}$ is $V_4$-cordial.

Seenivasan and Lourdusamy [33] proved the following results:

- If $G$ is an Eulerian graph with $q$ edges, where $q \equiv 2 \,(mod\,\,4)$, then $G$ has no $V_4$-cordial labeling.
- If $f$ be a $V_4$-cordial labeling of a graph $G$ with $p \geq 4$ and $uv$ be an edge of $G$ such that $f(u) = 0$ and $f(u) \neq f(v)$. Then the graph $G'$ obtained from $G$ by replacing the edge $uv$ by a path of length five is $V_4$-cordial.
- The path $P_4$ and $P_5$ are not $V_4$-cordial.
- All trees except $P_4$ and $P_5$ are $V_4$-cordial.
- The cycle $C_n$ is $V_4$-cordial if and only if $n \neq 4$ or $5$ or $n \neq 2 \,(mod\,\,4)$. 
Chapter 4. Results

Pechenik and Wise [30] proved the following results:

- The complete bipartite graph $K_{m,n}$ is $V_4$-cordial if and only if $m$ and $n$ are not both congruent to $2 \pmod{4}$.
- The path $P_n$ is $V_4$-cordial unless $n \in \{4, 5\}$.
- The cycle $C_n$ is $V_4$-cordial if and only if $n \notin \{4, 5\}$ and $n \not\equiv 2 \pmod{4}$.
- All ladders $P_2 \times P_k$ are $V_4$-cordial, except $P_2 \times P_2$.
- The prism $P_2 \times C_k$ is $V_4$-cordial if and only if $k \not\equiv 2 \pmod{4}$.
- The $d$-dimensional hypercube $Q_d$ is $V_4$-cordial, unless $d = 2$.

Here are new results of $V_4$-cordial labeling.

$V_4$-cordial Labeling of Standard Graphs

**Theorem 4.3.1** The pan graph $C_n^{+1}$ is $V_4$-cordial for all $n$.

**Proof:** Let $G = C_n^{+1}$ be the cycle with one pendant edge known as pan graph. Let $v_1, v_2, ..., v_n$ be the cycle vertices and $v'$ be the pendant vertex. It is noted that $|V(G)| = n + 1$ and $|E(G)| = n + 1$.

To define $V_4$-cordial labeling $f : V(G) \to V_4$ following cases are considered:

**Case 1:** $n \equiv 0 \pmod{8}$.

- $f(v_i) = 0; \quad i \equiv 0, 7 \pmod{8}$;
- $f(v_i) = a; \quad i \equiv 1, 5 \pmod{8}$;
- $f(v_i) = b; \quad i \equiv 4, 6 \pmod{8}$;
- $f(v_i) = c; \quad i \equiv 2, 3 \pmod{8}; \quad 1 \leq i \leq n$,
- $f(v') = a$.

**Case 2:** $n \equiv 1 \pmod{8}$.

- $f(v_i) = 0; \quad i \equiv 0, 7 \pmod{8}$;
- $f(v_i) = a; \quad i \equiv 1, 5 \pmod{8}$;
- $f(v_i) = b; \quad i \equiv 4, 6 \pmod{8}$;
- $f(v_i) = c; \quad i \equiv 2, 3 \pmod{8}; \quad 1 \leq i \leq n$,
- $f(v') = c$.

**Case 3:** $n \equiv 2 \pmod{8}$.

- $f(v_i) = 0; \quad i \equiv 0, 7 \pmod{8}$;
- $f(v_i) = a; \quad i \equiv 1, 5 \pmod{8}$;
- $f(v_i) = b; \quad i \equiv 4, 6 \pmod{8}$;
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\[ f(v_i) = c; \quad i \equiv 2, 3(\text{mod } 8); \quad 1 \leq i \leq n - 3, \]
\[ f(v_{n-2}) = a; \]
\[ f(v_{n-1}) = b; \]
\[ f(v_n) = 0; \]
\[ f(v') = 0. \]

**Case 4:** \( n \equiv 3(\text{mod } 8). \)
\[ f(v_i) = 0; \quad i \equiv 0, 7(\text{mod } 8); \]
\[ f(v_i) = a; \quad i \equiv 1, 5(\text{mod } 8); \]
\[ f(v_i) = b; \quad i \equiv 4, 6(\text{mod } 8); \]
\[ f(v_i) = c; \quad i \equiv 2, 3(\text{mod } 8); \quad 1 \leq i \leq n - 5, \]
\[ f(v_{n-4}) = a; \]
\[ f(v_{n-3}) = c; \]
\[ f(v_{n-2}) = b; \]
\[ f(v_{n-1}) = 0; \]
\[ f(v_n) = 0; \]
\[ f(v') = 0. \]

**Case 5:** \( n \equiv 4, 5, 7(\text{mod } 8). \)
\[ f(v_i) = 0; \quad i \equiv 0, 7(\text{mod } 8); \]
\[ f(v_i) = a; \quad i \equiv 1, 5(\text{mod } 8); \]
\[ f(v_i) = b; \quad i \equiv 4, 6(\text{mod } 8); \]
\[ f(v_i) = c; \quad i \equiv 2, 3(\text{mod } 8); \quad 1 \leq i \leq n, \]
\[ f(v') = 0. \]

**Case 6:** \( n \equiv 6(\text{mod } 8). \)
\[ f(v_i) = 0; \quad i \equiv 0, 7(\text{mod } 8); \]
\[ f(v_i) = a; \quad i \equiv 1, 5(\text{mod } 8); \]
\[ f(v_i) = b; \quad i \equiv 4, 6(\text{mod } 8); \]
\[ f(v_i) = c; \quad i \equiv 2, 3(\text{mod } 8); \quad 1 \leq i \leq n - 3, \]
\[ f(v_{n-2}) = 0; \]
\[ f(v_{n-1}) = b; \]
\[ f(v_n) = a; \]
\[ f(v') = 0. \]

The labeling pattern defined above covers all possible arrangements of vertices. In each possibility the graph under consideration satisfies the vertex conditions and edge conditions of \( V_4 \)-cordial labeling as shown in table 4.3.1. That is \( G \) admits \( V_4 \)-cordial labeling.

Let \( n = 8a + b \), where \( a, b \in N \cup \{0\} \).
<table>
<thead>
<tr>
<th>b</th>
<th>Vertex conditions</th>
<th>Edge conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( v_f(0) + 1 = v_f(a) = v_f(b) + 1 = v_f(c) + 1 )</td>
<td>( e_f(0) + 1 = e_f(a) = e_f(b) + 1 = e_f(c) + 1 )</td>
</tr>
<tr>
<td>1</td>
<td>( v_f(0) + 1 = v_f(a) = v_f(b) + 1 = v_f(c) )</td>
<td>( e_f(0) = e_f(a) + 1 = e_f(b) = e_f(c) + 1 )</td>
</tr>
<tr>
<td>2</td>
<td>( v_f(0) = v_f(a) = v_f(b) = v_f(c) + 1 )</td>
<td>( e_f(0) + 1 = e_f(a) = e_f(b) = e_f(c) )</td>
</tr>
<tr>
<td>3,7</td>
<td>( v_f(0) = v_f(a) = v_f(b) = v_f(c) )</td>
<td>( e_f(0) = e_f(a) = e_f(b) = e_f(c) )</td>
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<tr>
<td>4</td>
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<td>( e_f(0) + 1 = e_f(a) + 1 = e_f(b) = e_f(c) + 1 )</td>
</tr>
<tr>
<td>5</td>
<td>( v_f(0) + 1 = v_f(a) = v_f(b) + 1 = v_f(c) )</td>
<td>( e_f(0) = e_f(a) = e_f(b) + 1 = e_f(c) + 1 )</td>
</tr>
<tr>
<td>6</td>
<td>( v_f(0) = v_f(a) = v_f(b) + 1 = v_f(c) )</td>
<td>( e_f(0) = e_f(a) + 1 = e_f(b) = e_f(c) )</td>
</tr>
</tbody>
</table>

Table 4.3.1: \( V_4 \)-cordial labeling of pan graph \( C_{n+1} \).

**Illustration 4.3.1** The pan graph \( C_{5+1} \) and its \( V_4 \)-cordial labeling is shown in Figure 4.3.1.

![Figure 4.3.1: \( V_4 \)-cordial labeling of pan graph \( C_{5+1} \).](image)

**Theorem 4.3.2** The triangular book \( B(3, n) \) with \( n \)-pages is \( V_4 \)-cordial except \( n \equiv 2 (mod 4) \).

**Proof:** Let \( G = B(3, n) \) be the triangular book graph. Let \( v_1, v_2, ..., v_n \) be the vertices of \( n \)-pages of triangular book graph. Let \( u \) and \( v \) be the spine vertices. It is noted that \( |V(G)| = n + 2 \) and \( |E(G)| = 2n + 1 \).

To define \( V_4 \)-cordial labeling \( f : V(G) \rightarrow V_4 \) following cases are considered:

**Case 1:** \( n \equiv 0, 3 (mod 4) \).
\[
\begin{align*}
f(u) &= 0 \text{ and } f(v) = a, \\
f(v_i) &= 0; \quad i \equiv 0 (mod 4), \quad f(v_i) = a; \quad i \equiv 1 (mod 4), \\
f(v_i) &= b; \quad i \equiv 2 (mod 4), \quad f(v_i) = c; \quad i \equiv 3 (mod 4), \quad 1 \leq i \leq n.
\end{align*}
\]

**Case 2:** \( n \equiv 1 (mod 4) \).
\[
\begin{align*}
f(u) &= 0 \text{ and } f(v) = a, \\
f(v_i) &= 0; \quad i \equiv 0 (mod 4), \quad f(v_i) = b; \quad i \equiv 1 (mod 4), \\
f(v_i) &= a; \quad i \equiv 2 (mod 4), \quad f(v_i) = c; \quad i \equiv 3 (mod 4), \quad 1 \leq i \leq n.
\end{align*}
\]
Case 3: $n \equiv 2 \pmod{4}$.

Subcase 1: Base vertices are different.
Label all the vertices of pages from $0, a, b$ and $c$ such that the vertex condition of $V_4$-cordiality will be satisfied. But if we labelled the vertices in such a way that the vertex condition will satisfy then the edge condition will not satisfy because one edge labelled from $0, a, b$ and $c$ will appear $n/2 - 1$ times, one will appear $n/2$ times and two will appear $n/2 + 1$ times. Hence, the edge condition will not satisfy. Here, $|E(G)| = 2n + 1$ so three edge labelled from $0, a, b$ and $c$ must appear $n/2$ times and one must appear $n/2 + 1$ times.

Subcase 2: Base vertices are same.
Label all the vertices of pages from $0, a, b$ and $c$ such that the vertex condition of $V_4$-cordiality will be satisfied. But if we labelled the vertices in such a way that the vertex condition will satisfy then the edge condition will not satisfy because three edges labelled from $0, a, b$ and $c$ will appear $n/2 + 1$ times and one will appear $n/2 - 2$ times. Hence, the edge condition will not satisfy. Here, $|E(G)| = 2n + 1$ so three edge labelled from $0, a, b$ and $c$ must appear at least $n/2$ times and one must appear $n/2 + 1$ times.

Hence, for $n \equiv 2 \pmod{4}$ the triangular book graph $B(3, n)$ is not $V_4$-cordial.

The labeling pattern defined above covers all possible arrangements of vertices. In each possibility the graph under consideration satisfies the vertex conditions and edge conditions of $V_4$-cordial labeling as shown in table 4.3.2. That is $G$ admits $V_4$-cordial labeling.

Let $n = 4a + b$, $a, b \in N \cup \{0\}$.

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<tr>
<th>$b$</th>
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<td>$v_f(0) + 1 = v_f(a) = v_f(b) + 1 = v_f(c) + 1$</td>
<td>$e_f(0) + 1 = e_f(a) = e_f(b) = e_f(c)$</td>
</tr>
</tbody>
</table>

Table 4.3.2: $V_4$-cordial labeling of triangular book graph $B(3, n)$. 
Illustration 4.3.2 The $V_4$-cordial labeling of triangular book $B(3, 5)$ is shown in Figure 4.3.2.

Figure 4.3.2: $V_4$-cordial labeling of triangular book $B(3, 5)$.

Theorem 4.3.3 The triangular book with book mark $B(3, n)(u, v)(v, w)$ is $V_4$-cordial except $n \equiv 1 \pmod{4}$.

Proof: Let $G = B(3, n)(u, v)(v, w)$ be the triangular book with book mark obtained from triangular book graph $B(3, n)$ by attaching one pendant edge at any one of the spine vertices. Let $u$ and $v$ are vertices of spine. Let $w$ be the pendant vertex and $vw$ be the pendant edge. The vertices of pages of book are denoted by $v_1, v_2, ..., v_n$. It is noted that $|V(G)| = n + 3$ and $|E(G)| = 2(n + 1)$.

To define $V_4$-cordial labeling $f : V(G) \rightarrow V_4$ following cases are considered:

Case 1: $n \equiv 0, 2 \pmod{4}$.
$f(u) = 0, f(v) = a$ and $f(w) = c$, $f(v_i) = c$; $i \equiv 0 \pmod{4}$, $f(v_i) = b$; $i \equiv 1 \pmod{4}$, $f(v_i) = 0$; $i \equiv 2 \pmod{4}$, $f(v_i) = a$; $i \equiv 3 \pmod{4}$, $1 \leq i \leq n$.

Case 2: $n \equiv 3 \pmod{4}$.
$f(u) = 0, f(v) = a$ and $f(w) = a$, $f(v_1) = b$, $f(v_2) = c$, $f(v_i) = b$; $i \equiv 0 \pmod{4}$, $f(v_i) = c$; $i \equiv 1 \pmod{4}$, $f(v_i) = a$; $i \equiv 2 \pmod{4}$, $f(v_i) = 0$; $i \equiv 3 \pmod{4}$, $3 \leq i \leq n$.

Case 3: $n \equiv 1 \pmod{4}$.
Subcase 1: Base vertices and pendent vertex are distinct.
Label all the vertices of pages from 0, a, b and c such that the vertex condition of $V_4$-cordiality will be satisfied. But if we labelled the vertices in such a way that the vertex condition will satisfy then the edge condition will not satisfy because two edges labelled from 0, a, b and c will appear $(n+1)/2$ times, one will appear $(n+1)/2 + 1$ times and one will appear $(n+1)/2 - 1$ times. Hence, the edge condition will not satisfy. Here, $|E(G)| = 2n + 2$ so each edge labelled with 0, a, b and c must appear exactly $(n+1)/2$ times.

Subcase 2: Base vertices are same and pendent vertex is different and $n > 1$. Label all the vertices of pages from 0, a, b and c such that the vertex condition of $V_4$-cordiality will be satisfied. But if we labelled the vertices in such a way that the vertex condition will satisfy then the edge condition will not satisfy because two edges labelled from 0, a, b and c will appear $(n+1)/2 + 1$ times, one will appear $(n+1)/2$ times and one will appear $(n+1)/2 - 2$ times. Hence, the edge condition will not satisfy. Here, $|E(G)| = 2n + 2$ so each edge labelled with 0, a, b and c must appear exactly $(n+1)/2$ times.

Subcase 3: Base vertices are different but one base vertex with which pendent vertex is attached are same and $n > 1$. Label all the vertices of pages from 0, a, b and c such that the vertex condition of $V_4$-cordiality will be satisfied. But if we labelled the vertices in such a way that the vertex condition will satisfy then the edge condition will not satisfy because two edges labelled from 0, a, b and c will appear $(n+1)/2 - 1$ times and two will appear $(n+1)/2 + 1$ times. Hence, the edge condition will not satisfy. Here, $|E(G)| = 2n + 2$ so each edge labelled with 0, a, b and c must appear exactly $(n+1)/2$ times.

Subcase 4: Base vertices are different but one base vertex which is not joined with pendent vertex and pendent vertex are same and $n > 1$. Label all the vertices of pages from 0, a, b and c such that the vertex condition of $V_4$-cordiality will be satisfied. But if we labelled the vertices in such a way that the vertex condition will satisfy then the edge condition will not satisfy because two edges labelled from 0, a, b and c will appear $n - 1$ times, one will appear $n - 2$ times and one will appear $(n+1)/2 - 2$ times. Hence, the edge condition will not satisfy. Here, $|E(G)| = 2n + 2$ so each edge labelled with 0, a, b and c must appear exactly $(n+1)/2$ times.

Subcase 5: Base vertices and pendent vertex are same and $n > 8$. Label all the vertices of pages from 0, a, b and c such that the vertex condition of $V_4$-cordiality will be satisfied. But if we labelled the vertices in such a way that the vertex condition will satisfy then the edge condition will not satisfy because three edges labelled from 0, a, b and c will appear $(n+1)/2 + 1$ times and one will appear $(n+1)/2 - 3$ times. Hence, the edge condition will not satisfy. Here, $|E(G)| = 2n + 2$ so each edge labelled...
from 0, a, b and c must appear exactly \((n + 1)/2\) times. In this case the Triangular Book with Book Mark \(B(3, n)(u, v)(v, w)\) is not \(V_4\)-cordial.

Hence, for \(n \equiv 1(\text{mod} \ 4)\) the Triangular Book with Book Mark \(B(3, n)(u, v)(v, w)\) is not \(V_4\)-cordial.

The labeling pattern defined above covers all possible arrangements of vertices. In each possibility the graph under consideration satisfies the vertex conditions and edge conditions of \(V_4\)-cordial labeling as shown in table 4.3.3. That is \(G\) admits \(V_4\)-cordial labeling.

Let \(n = 4a + b, a, b \in \mathbb{N} \cup \{0\}\).

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<thead>
<tr>
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</tr>
<tr>
<td>3</td>
<td>(v_f(0) = v_f(a) = v_f(b) + 1 = v_f(c) + 1)</td>
<td>(e_f(0) = e_f(a) = e_f(b) = e_f(c))</td>
</tr>
</tbody>
</table>

Table 4.3.3: \(V_4\)-cordial labeling of triangular book with book mark \(B(3, n)(u, v)(v, w)\).

Illustration 4.3.3 The \(V_4\)-cordial labeling of triangular book with book mark \(B(3, 4)(u, v)(v, w)\) is shown in Figure 4.3.3.

![Figure 4.3.3: \(V_4\)-cordial labeling of triangular book with book mark \(B(3, 4)(u, v)(v, w)\).](image)

Theorem 4.3.4 The crown \(C_n \odot K_1\) is \(V_4\)-cordial for all \(n\).

Proof: Let \(G = C_n \odot K_1\) be the crown graph. Let \(v_1, v_2, ..., v_n\) be the vertices of cycle \(C_n\) and \(v'_1, v'_2, ..., v'_n\) are the pendant vertices of crown \(C_n \odot K_1\). It is noted that \(|V(G)| = 2n\) and \(|E(G)| = 2n\).
To define $V_4$-cordial labeling $f : V(G) \to V_4$ following cases are considered:

**Case 1:** $n \equiv 0, 3(\text{mod } 4)$
\[ f(v_i) = 0; \quad i \equiv 0(\text{mod } 4); \]
\[ f(v_i) = a; \quad i \equiv 1(\text{mod } 4); \]
\[ f(v_i) = b; \quad i \equiv 2(\text{mod } 4); \]
\[ f(v_i) = c; \quad i \equiv 3(\text{mod } 4); \quad 1 \leq i \leq n, \]
\[ f(v'_i) = 0; \quad i \equiv 2(\text{mod } 4); \]
\[ f(v'_i) = a; \quad i \equiv 1(\text{mod } 4); \]
\[ f(v'_i) = b; \quad i \equiv 0(\text{mod } 4); \]
\[ f(v'_i) = c; \quad i \equiv 3(\text{mod } 4); \quad 1 \leq i \leq n. \]

**Case 2:** $n \equiv 1(\text{mod } 4)$
\[ f(v_i) = 0; \quad i \equiv 0(\text{mod } 4); \]
\[ f(v_i) = a; \quad i \equiv 1(\text{mod } 4); \]
\[ f(v_i) = b; \quad i \equiv 2(\text{mod } 4); \]
\[ f(v_i) = c; \quad i \equiv 3(\text{mod } 4); \quad 1 \leq i \leq n - 1, \]
\[ f(v_n) = 0; \]
\[ f(v'_i) = 0; \quad i \equiv 2(\text{mod } 4); \]
\[ f(v'_i) = a; \quad i \equiv 1(\text{mod } 4); \]
\[ f(v'_i) = b; \quad i \equiv 0(\text{mod } 4); \]
\[ f(v'_i) = c; \quad i \equiv 3(\text{mod } 4); \quad 1 \leq i \leq n - 1, \]
\[ f(v'_n) = b. \]

**Case 3:** $n \equiv 2(\text{mod } 4)$
\[ f(v_i) = 0; \quad i \equiv 0(\text{mod } 4); \]
\[ f(v_i) = a; \quad i \equiv 1(\text{mod } 4); \]
\[ f(v_i) = b; \quad i \equiv 2(\text{mod } 4); \]
\[ f(v_i) = c; \quad i \equiv 3(\text{mod } 4); \quad 1 \leq i \leq n - 2, \]
\[ f(v_{n-1}) = 0; \]
\[ f(v_n) = b; \]
\[ f(v'_i) = 0; \quad i \equiv 2(\text{mod } 4); \]
\[ f(v'_i) = a; \quad i \equiv 1(\text{mod } 4); \]
\[ f(v'_i) = b; \quad i \equiv 0(\text{mod } 4); \]
\[ f(v'_i) = c; \quad i \equiv 3(\text{mod } 4); \quad 1 \leq i \leq n - 1, \]
\[ f(v'_n) = c. \]

The labeling pattern defined above covers all possible arrangements of vertices. In each possibility the graph under consideration satisfies the vertex conditions and edge conditions of $V_4$-cordial labeling as shown in table 4.3.4. That is $G$ admits $V_4$-cordial labeling.
Let \( n = 4a + b \), where \( a, b \in N \cup \{0\} \).

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</tr>
<tr>
<td>1</td>
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</tr>
<tr>
<td>3</td>
<td>( v_f(0) + 1 = v_f(a) = v_f(b) + 1 = v_f(c) )</td>
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</tr>
</tbody>
</table>

Table 4.3.4: \( V_4 \)-cordial labeling of crown \( C_n \bowtie K_1 \).

**Illustration 4.3.4** The crown \( C_5 \bowtie K_1 \) and its \( V_4 \)-cordial labeling is shown in Figure 4.3.4.

![Figure 4.3.4: \( V_4 \)-cordial labeling of crown \( C_5 \bowtie K_1 \).](image)

**Theorem 4.3.5** The armed crown \( AC_n \) is \( V_4 \)-cordial for all \( n \).

**Proof:** Let \( G = AC_n \) be the armed crown. Let \( v_1, v_2, ..., v_n \) be the vertices of cycle \( C_n \). Let \( v_i, v_i' \) and \( v_i'' \) be the vertices of path where \( 1 \leq i \leq n \). Here, each \( v_i' \) is adjacent to both \( v_i \) and \( v_i'' \) for all \( i \). It is noted that \( |V(G)| = 3n \) and \( |E(G)| = 3n \).

To define \( V_4 \)-cordial labeling \( f : V(G) \rightarrow V_4 \) following cases are considered:

**Case 1:** \( n \equiv 0(\text{mod } 4) \)
- \( f(v_i) = 0; \; i \equiv 0(\text{mod } 4) \);
- \( f(v_i) = a; \; i \equiv 1(\text{mod } 4) \);
- \( f(v_i) = b; \; i \equiv 2(\text{mod } 4) \);
- \( f(v_i) = c; \; i \equiv 3(\text{mod } 4); \; 1 \leq i \leq n \),
- \( f(v_i') = 0; \; i \equiv 2(\text{mod } 4) \);
Case 3: \( n \equiv 2 (\text{mod} \ 4) \)
\[
\begin{align*}
f(v_i) &= 0; \quad i \equiv 0 (\text{mod} \ 4); \\
f(v_i) &= a; \quad i \equiv 1 (\text{mod} \ 4); \\
f(v_i) &= b; \quad i \equiv 2 (\text{mod} \ 4); \\
f(v_i) &= c; \quad i \equiv 3 (\text{mod} \ 4); \quad 1 \leq i \leq n, \\
f(v_i') &= 0; \quad i \equiv 3 (\text{mod} \ 4); \\
f(v_i') &= a; \quad i \equiv 1 (\text{mod} \ 4); \\
f(v_i') &= b; \quad i \equiv 2 (\text{mod} \ 4); \\
f(v_i') &= c; \quad i \equiv 0 (\text{mod} \ 4); \quad 1 \leq i \leq n - 2, \\
f(v_{n-1}') &= 0; \\
f(v_n') &= c; \\
f(v_n') &= 0; \quad i \equiv 3 (\text{mod} \ 4); \\
f(v_n') &= a; \quad i \equiv 1 (\text{mod} \ 4); \\
f(v_n') &= b; \quad i \equiv 2 (\text{mod} \ 4); \\
f(v_n') &= c; \quad i \equiv 0 (\text{mod} \ 4); \quad 1 \leq i \leq n - 2, \\
f(v_{n-1}') &= b; \\
f(v_n') &= c. 
\end{align*}
\]
Case 4: $n \equiv 3 (\text{mod } 4)$

$f(v_i) = 0; \ i \equiv 0 (\text{mod } 4)$;
$f(v_i) = a; \ i \equiv 1 (\text{mod } 4)$;
$f(v_i) = b; \ i \equiv 2 (\text{mod } 4)$;
$f(v_i) = c; \ i \equiv 3 (\text{mod } 4); \ 1 \leq i \leq n,$
$f(v'_i) = 0; \ i \equiv 2 (\text{mod } 4)$;
$f(v'_i) = a; \ i \equiv 1 (\text{mod } 4)$;
$f(v'_i) = b; \ i \equiv 0 (\text{mod } 4)$;
$f(v'_i) = c; \ i \equiv 3 (\text{mod } 4); \ 1 \leq i \leq n,$
$f(v''_i) = 0; \ i \equiv 3 (\text{mod } 4)$;
$f(v''_i) = a; \ i \equiv 1 (\text{mod } 4)$;
$f(v''_i) = b; \ i \equiv 2 (\text{mod } 4)$;
$f(v''_i) = c; \ i \equiv 0 (\text{mod } 4); \ 1 \leq i \leq n - 3,$
$f(v''_{n-2}) = 0$;
$f(v''_{n-1}) = c$;
$f(v''_n) = b$.

The labeling pattern defined above covers all possible arrangements of vertices. In each possibility the graph under consideration satisfies the vertex conditions and edge conditions of $V_4$-cordial labeling as shown in table 4.3.5. That is $G$ admits $V_4$-cordial labeling.

Let $n = 4a + b$, where $a, b \in N \cup \{0\}$.

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</tr>
</tbody>
</table>

Table 4.3.5: $V_4$-cordial labeling of an armed crown $AC_n$.  

NEHA B. RATHOD
**Illustration 4.3.5** The armed crown $AC_5$ and its $V_4$-cordial labeling is shown in Figure 4.3.5.

**Theorem 4.3.6** The corona $C_n \odot mK_1$ of cycle $C_n$ and $m$ complete graph $K_1$ is $V_4$-cordial for all $n$ and $m$.

**Proof:** Let $G = C_n \odot mK_1$ be the corona of cycle $C_n$ and $m$ complete graph $K_1$. Let $v_1, v_2, ..., v_n$ be the cycle vertices and $v_{i1}, v_{i2}, ..., v_{im}, v_{21}, v_{22}, ..., v_{2m}, v_{31}, v_{32}, ..., v_{3m}, ...; v_{i1}, v_{i2}, ..., v_{i1}, ..., v_{im}, ..., v_{n1}, v_{n2}, ..., v_{nj}, ..., v_{nm}$ be the vertices of $m$ complete graph $K_1$, where $1 \leq i \leq n$ and $1 \leq j \leq m$. It is noted that $|V(G)| = (m+1)n$ and $|E(G)| = (m+1)n$.

To define $V_4$-cordial labeling $f : V(G) \rightarrow V_4$ following cases are considered:

**Case 1:** $n \equiv 0 \text{(mod 4)}$.

Label the vertices $v_1, v_2, ..., v_n$ of cycle as follows.

- $f(v_i) = 0; \quad i \equiv 0 \text{(mod 4)}$;
- $f(v_i) = a; \quad i \equiv 1 \text{(mod 4)}$;
- $f(v_i) = b; \quad i \equiv 2 \text{(mod 4)}$;
\( f(v_i) = c; \quad i \equiv 3(\text{mod } 4); \quad 1 \leq i \leq n. \)

The labeling pattern of the vertices \( v_{1j}, v_{2j}, \ldots, v_{nj} \), where \( 1 \leq j \leq m \) is divided into following two subcases.

**Subcase 1:** \( m = 1. \)

\[
\begin{align*}
    f(v_{i1}) &= 0; \quad i \equiv 2(\text{mod } 4); \\
    f(v_{i1}) &= a; \quad i \equiv 1(\text{mod } 4); \\
    f(v_{i1}) &= b; \quad i \equiv 0(\text{mod } 4); \\
    f(v_{i1}) &= c; \quad i \equiv 3(\text{mod } 4); \quad 1 \leq i \leq n.
\end{align*}
\]

**Subcase 2:** \( m > 1. \)

\[
\begin{align*}
    f(v_{ij}) &= 0; \quad i \equiv 2(\text{mod } 4); \\
    f(v_{ij}) &= a; \quad i \equiv 1(\text{mod } 4); \\
    f(v_{ij}) &= b; \quad i \equiv 3(\text{mod } 4); \\
    f(v_{ij}) &= c; \quad i \equiv 0(\text{mod } 4); \quad 1 \leq i \leq n \text{ and } 2 \leq j \leq m.
\end{align*}
\]

**Case 2:** \( n \equiv 1(\text{mod } 4). \)

Label the vertices \( v_1, v_2, \ldots, v_n \) of cycle as follows.

\[
\begin{align*}
    f(v_i) &= 0; \quad i \equiv 0(\text{mod } 4); \\
    f(v_i) &= a; \quad i \equiv 1(\text{mod } 4); \\
    f(v_i) &= b; \quad i \equiv 2(\text{mod } 4); \\
    f(v_i) &= c; \quad i \equiv 3(\text{mod } 4); \quad 1 \leq i \leq n - 1, \\
    f(v_n) &= 0.
\end{align*}
\]

The labeling pattern of the vertices \( v_{1j}, v_{2j}, \ldots, v_{nj} \), where \( 1 \leq j \leq m \) is divided into following subcases.

**Subcase 1:** \( m = 1. \)

\[
\begin{align*}
    f(v_{i1}) &= 0; \quad i \equiv 2(\text{mod } 4); \\
    f(v_{i1}) &= a; \quad i \equiv 1(\text{mod } 4); \\
    f(v_{i1}) &= b; \quad i \equiv 0(\text{mod } 4); \\
    f(v_{i1}) &= c; \quad i \equiv 3(\text{mod } 4); \quad 1 \leq i \leq n - 1, \\
    f(v_{n1}) &= a.
\end{align*}
\]

**Subcase 2:** \( m \equiv 0(\text{mod } 4). \)

\[
\begin{align*}
    f(v_{ij}) &= 0; \quad i \equiv 2(\text{mod } 4); \\
    f(v_{ij}) &= a; \quad i \equiv 1(\text{mod } 4); \\
    f(v_{ij}) &= b; \quad i \equiv 3(\text{mod } 4);
\end{align*}
\]
\[ f(v_{ij}) = c; \quad i \equiv 0(\text{mod } 4); \quad 1 \leq i \leq n - 1 \text{ and } 2 \leq j \leq m, \]
\[ f(v_{nj}) = 0. \]

**Subcase 3:** \( m \equiv 1(\text{mod } 4). \)
\[ f(v_{ij}) = 0; \quad i \equiv 2(\text{mod } 4); \]
\[ f(v_{ij}) = a; \quad i \equiv 1(\text{mod } 4); \]
\[ f(v_{ij}) = b; \quad i \equiv 3(\text{mod } 4); \]
\[ f(v_{ij}) = c; \quad i \equiv 0(\text{mod } 4); \quad 1 \leq i \leq n - 1 \text{ and } 2 \leq j \leq m, \]
\[ f(v_{nj}) = a. \]

**Subcase 4:** \( m \equiv 2(\text{mod } 4). \)
\[ f(v_{ij}) = 0; \quad i \equiv 2(\text{mod } 4); \]
\[ f(v_{ij}) = a; \quad i \equiv 1(\text{mod } 4); \]
\[ f(v_{ij}) = b; \quad i \equiv 3(\text{mod } 4); \]
\[ f(v_{ij}) = c; \quad i \equiv 0(\text{mod } 4); \quad 1 \leq i \leq n - 1 \text{ and } 2 \leq j \leq m, \]
\[ f(v_{nj}) = b. \]

**Subcase 5:** \( m \equiv 3(\text{mod } 4). \)
\[ f(v_{ij}) = 0; \quad i \equiv 2(\text{mod } 4); \]
\[ f(v_{ij}) = a; \quad i \equiv 1(\text{mod } 4); \]
\[ f(v_{ij}) = b; \quad i \equiv 3(\text{mod } 4); \]
\[ f(v_{ij}) = c; \quad i \equiv 0(\text{mod } 4); \quad 1 \leq i \leq n - 1 \text{ and } 2 \leq j \leq m, \]
\[ f(v_{nj}) = c. \]

**Case 3:** \( n \equiv 2(\text{mod } 4). \)

Label the vertices \( v_1, v_2, \ldots, v_n \) of cycle as follows.

\[ f(v_i) = 0; \quad i \equiv 0(\text{mod } 4); \]
\[ f(v_i) = a; \quad i \equiv 1(\text{mod } 4); \]
\[ f(v_i) = b; \quad i \equiv 2(\text{mod } 4); \]
\[ f(v_i) = c; \quad i \equiv 3(\text{mod } 4); \quad 1 \leq i \leq n - 2, \]
\[ f(v_{n-1}) = 0; \]
\[ f(v_n) = c. \]

The labeling pattern of the vertices \( v_{1j}, v_{2j}, \ldots, v_{nj} \), where \( 1 \leq j \leq m \) is divided into following subcases.

**Subcase 1:** \( m = 1. \)
\[ f(v_{i1}) = 0; \quad i \equiv 2(\text{mod } 4); \]
\[ f(v_{i1}) = a; \quad i \equiv 1(\text{mod } 4); \]
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\[ f(v_i) = b; \quad i \equiv 0(\text{mod } 4); \]
\[ f(v_i) = c; \quad i \equiv 3(\text{mod } 4); \quad 1 \leq i \leq n - 1, \]
\[ f(v_{n1}) = b. \]

**Subcase 2:** \( m \equiv 0(\text{mod } 2). \)
\[ f(v_{ij}) = 0; \quad i \equiv 2(\text{mod } 4); \]
\[ f(v_{ij}) = a; \quad i \equiv 1(\text{mod } 4); \]
\[ f(v_{ij}) = b; \quad i \equiv 3(\text{mod } 4); \]
\[ f(v_{ij}) = c; \quad i \equiv 0(\text{mod } 4); \quad 1 \leq i \leq n \text{ and } 2 \leq j \leq m. \]

**Subcase 3:** \( m \equiv 1(\text{mod } 2). \)
\[ f(v_{ij}) = 0; \quad i \equiv 2(\text{mod } 4); \]
\[ f(v_{ij}) = a; \quad i \equiv 1(\text{mod } 4); \]
\[ f(v_{ij}) = b; \quad i \equiv 3(\text{mod } 4); \]
\[ f(v_{ij}) = c; \quad i \equiv 0(\text{mod } 4); \quad 1 \leq i \leq n - 2 \text{ and } 2 \leq j \leq m, \]
\[ f(v_{(n-1)j}) = b; \]
\[ f(v_{nj}) = c. \]

**Case 4:** \( n \equiv 3(\text{mod } 4). \)

Label the vertices \( v_1, v_2, \ldots, v_n \) of cycle as follows.
\[ f(v_i) = 0; \quad i \equiv 0(\text{mod } 4); \]
\[ f(v_i) = a; \quad i \equiv 1(\text{mod } 4); \]
\[ f(v_i) = b; \quad i \equiv 2(\text{mod } 4); \]
\[ f(v_i) = c; \quad i \equiv 3(\text{mod } 4); \quad 1 \leq i \leq n. \]

The labeling pattern of the vertices \( v_{1j}, v_{2j}, \ldots, v_{nj} \), where \( 1 \leq j \leq m \) is divided into following subcases.

**Subcase 1:** \( m = 1. \)
\[ f(v_{ij}) = 0; \quad i \equiv 2(\text{mod } 4); \]
\[ f(v_{ij}) = a; \quad i \equiv 1(\text{mod } 4); \]
\[ f(v_{ij}) = b; \quad i \equiv 0(\text{mod } 4); \]
\[ f(v_{ij}) = c; \quad i \equiv 3(\text{mod } 4); \quad 1 \leq i \leq n. \]

**Subcase 2:** \( m \equiv 0(\text{mod } 8). \)
\[ f(v_{ij}) = 0; \quad i \equiv 2(\text{mod } 4); \]
\[ f(v_{ij}) = a; \quad i \equiv 1(\text{mod } 4); \]
\[ f(v_{ij}) = b; \quad i \equiv 3(\text{mod } 4); \]
\[ f(v_{ij}) = c; \quad i \equiv 0(\text{mod } 4); \quad 1 \leq i \leq n - 3 \text{ and } 2 \leq j \leq m, \]
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\[ f(v_{(n-2)j}) = b; \]
\[ f(v_{(n-1)j}) = c; \]
\[ f(v_{nj}) = a. \]

**Subcase 3:** \( m \equiv 1(\text{mod } 8). \)
\[ f(v_{ij}) = 0; \quad i \equiv 2(\text{mod } 4); \]
\[ f(v_{ij}) = a; \quad i \equiv 1(\text{mod } 4); \]
\[ f(v_{ij}) = b; \quad i \equiv 3(\text{mod } 4); \]
\[ f(v_{ij}) = c; \quad i \equiv 0(\text{mod } 4); \quad 1 \leq i \leq n - 1 \text{ and } 2 \leq j \leq m, \]
\[ f(v_{nj}) = c. \]

**Subcase 4:** \( m \equiv 2(\text{mod } 8). \)
\[ f(v_{ij}) = 0; \quad i \equiv 2(\text{mod } 4); \]
\[ f(v_{ij}) = a; \quad i \equiv 1(\text{mod } 4); \]
\[ f(v_{ij}) = b; \quad i \equiv 3(\text{mod } 4); \]
\[ f(v_{ij}) = c; \quad i \equiv 0(\text{mod } 4); \quad 1 \leq i \leq n - 3 \text{ and } 2 \leq j \leq m, \]
\[ f(v_{(n-2)j}) = 0; \]
\[ f(v_{(n-1)j}) = a; \]
\[ f(v_{nj}) = b. \]

**Subcase 5:** \( m \equiv 3(\text{mod } 8). \)
\[ f(v_{ij}) = 0; \quad i \equiv 2(\text{mod } 4); \]
\[ f(v_{ij}) = a; \quad i \equiv 1(\text{mod } 4); \]
\[ f(v_{ij}) = b; \quad i \equiv 3(\text{mod } 4); \]
\[ f(v_{ij}) = c; \quad i \equiv 0(\text{mod } 4); \quad 1 \leq i \leq n - 3 \text{ and } 2 \leq j \leq m, \]
\[ f(v_{(n-2)j}) = b; \]
\[ f(v_{(n-1)j}) = 0; \]
\[ f(v_{nj}) = c. \]

**Subcase 6:** \( m \equiv 4(\text{mod } 8). \)
\[ f(v_{ij}) = 0; \quad i \equiv 2(\text{mod } 4); \]
\[ f(v_{ij}) = a; \quad i \equiv 1(\text{mod } 4); \]
\[ f(v_{ij}) = b; \quad i \equiv 3(\text{mod } 4); \]
\[ f(v_{ij}) = c; \quad i \equiv 0(\text{mod } 4); \quad 1 \leq i \leq n - 3 \text{ and } 2 \leq j \leq m, \]
\[ f(v_{(n-2)j}) = a; \]
\[ f(v_{(n-1)j}) = c; \]
\[ f(v_{nj}) = 0. \]

**Subcase 7:** \( m \equiv 5(\text{mod } 8). \)
\[ f(v_{ij}) = 0; \quad i \equiv 2(\text{mod } 4); \]
\[ f(v_{ij}) = a; \quad i \equiv 1(\text{mod } 4); \]
\[ f(v_{ij}) = b; \quad i \equiv 3 (\text{mod} \ 4); \]
\[ f(v_{ij}) = c; \quad i \equiv 0 (\text{mod} \ 4); \quad 1 \leq i \leq n - 3 \text{ and } 2 \leq j \leq m, \]
\[ f(v_{(n-2)j}) = b; \]
\[ f(v_{(n-1)j}) = 0; \]
\[ f(v_{nj}) = c. \]

**Subcase 8:** \( m \equiv 6 (\text{mod} \ 8). \)
\[ f(v_{ij}) = 0; \quad i \equiv 2 (\text{mod} \ 4); \]
\[ f(v_i) = a; \quad i \equiv 1 (\text{mod} \ 4); \]
\[ f(v_{ij}) = b; \quad i \equiv 3 (\text{mod} \ 4); \]
\[ f(v_{ij}) = c; \quad i \equiv 0 (\text{mod} \ 4); \quad 1 \leq i \leq n - 3 \text{ and } 2 \leq j \leq m, \]
\[ f(v_{(n-2)j}) = c; \]
\[ f(v_{(n-1)j}) = a; \]
\[ f(v_{nj}) = b. \]

**Subcase 9:** \( m \equiv 7 (\text{mod} \ 8). \)
\[ f(v_{ij}) = 0; \quad i \equiv 2 (\text{mod} \ 4); \]
\[ f(v_{ij}) = a; \quad i \equiv 1 (\text{mod} \ 4); \]
\[ f(v_{ij}) = b; \quad i \equiv 3 (\text{mod} \ 4); \]
\[ f(v_{ij}) = c; \quad i \equiv 0 (\text{mod} \ 4); \quad 1 \leq i \leq n - 3 \text{ and } 2 \leq j \leq m, \]
\[ f(v_{(n-2)j}) = 0; \]
\[ f(v_{(n-1)j}) = b; \]
\[ f(v_{nj}) = a. \]

The labeling pattern defined above covers all possible arrangements of vertices. In each possibility the graph under consideration satisfies the vertex conditions and edge conditions of \( V_4 \)-cordial labeling as shown in Table 4.3.6. That is \( G \) admits \( V_4 \)-cordial labeling.

Let \( n = 4a + b \) and \( (j = 1, 2, 3, \ldots, m) \), where \( m, a, b \in N \cup \{ 0 \}. \)

<table>
<thead>
<tr>
<th>( b )</th>
<th>( m )</th>
<th>Vertex conditions</th>
<th>Edge conditions</th>
</tr>
</thead>
</table>
| 0    | \( m \equiv 0, 1 (\text{mod} \ 2) \) | \[ v_f(0) = v_f(a) \]
\[ v_f(b) = v_f(c) \] | \[ e_f(b) = e_f(c) \]
|      | \( m \equiv 0 (\text{mod} \ 4) \) | \[ v_f(0) = v_f(a) + 1 \]
\[ v_f(b) + 1 = v_f(c) + 1 \] | \[ e_f(0) = e_f(a) + 1 \]
\[ e_f(b) + 1 = e_f(c) + 1 \] |
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Vertex conditions

<table>
<thead>
<tr>
<th>b</th>
<th>m (mod 2)</th>
<th>Vertex conditions</th>
<th>Edge conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>m ≡ 0 (mod 2)</td>
<td>( v_f(0) = v_f(a) )</td>
<td>( e_f(0) + 1 = e_f(a) )</td>
</tr>
<tr>
<td></td>
<td>m ≡ 1 (mod 2)</td>
<td>( v_f(0) = v_f(a), v_f(b) = v_f(c) )</td>
<td>( e_f(0) = e_f(a), e_f(b) = e_f(c) )</td>
</tr>
<tr>
<td></td>
<td>m ≡ 2 (mod 8)</td>
<td>( v_f(0) + 1 = v_f(a) )</td>
<td>( e_f(0) + 1 = e_f(a) )</td>
</tr>
<tr>
<td></td>
<td>m ≡ 1 (mod 8)</td>
<td>( v_f(0) + 1 = v_f(a), v_f(b) = v_f(c) )</td>
<td>( e_f(0) = e_f(a) + 1, e_f(b) = e_f(c) + 1 )</td>
</tr>
<tr>
<td></td>
<td>m ≡ 2 (mod 8)</td>
<td>( v_f(0) + 1 = v_f(a), v_f(b) + 1 = v_f(c) + 1 )</td>
<td>( e_f(0) + 1 = e_f(a), e_f(b) + 1 = e_f(c) + 1 )</td>
</tr>
<tr>
<td></td>
<td>m ≡ 3, 7 (mod 8)</td>
<td>( v_f(0) = v_f(a), v_f(b) = v_f(c) )</td>
<td>( e_f(0) = e_f(a), e_f(b) = e_f(c) )</td>
</tr>
<tr>
<td></td>
<td>m ≡ 4 (mod 8)</td>
<td>( v_f(0) = v_f(a), v_f(b) + 1 = v_f(c) )</td>
<td>( e_f(0) = e_f(a), e_f(b) + 1 = e_f(c) )</td>
</tr>
<tr>
<td></td>
<td>m ≡ 5 (mod 8)</td>
<td>( v_f(0) = v_f(a) + 1 )</td>
<td>( e_f(0) = e_f(a) + 1 )</td>
</tr>
<tr>
<td></td>
<td>m ≡ 6 (mod 8)</td>
<td>( v_f(0) + 1 = v_f(a) + 1 )</td>
<td>( e_f(0) + 1 = e_f(a) + 1 )</td>
</tr>
</tbody>
</table>

Table 4.3.6: \( V_4 \)-cordial labeling of corona graph \( C_n \odot mK_1 \).

**Illustration 4.3.6** The corona graph \( C_4 \odot 5K_1 \) and its \( V_4 \)-cordial labeling is shown in Figure 4.3.6.

![Figure 4.3.6: \( V_4 \)-cordial labeling of corona graph \( C_4 \odot 5K_1 \).](image)
**Theorem 4.3.7** The theta graph \( \theta(|G|) \) is \( V_4 \)-cordial.

**Proof:** Let \( G = \theta(|G|) \) be the theta graph. Theta graph consists \( k \geq 3 \) pair wise internally disjoint paths of length 8. Let \( v_1^1, v_2^1, v_3^1, v_4^1, v_5^1, v_6^1, v_7^1, v_8^1 \) be the vertices of first path of length 8, \( v_1^2, v_2^2, v_3^2, v_4^2, v_5^2, v_6^2, v_7^2, v_8^2 \) be the vertices of second path of length 8, \( \ldots, v_1^m, v_2^m, v_3^m, v_4^m, v_5^m, v_6^m, v_7^m, v_8^m \) be the vertices of \( m^{th} \) path of length 8. It is noted that \( |V(G)| = 7m + 2 \) and \( |E(G)| = 7m. \)

Define \( V_4 \)-cordial labeling \( f : V(G) \to V_4 \) as follows:

\[
f(u) = 0 \text{ and } f(v) = 0.
\]

**Case 1:** \( m = 1. \)
\[
f(v_1^1) = a, f(v_2^1) = b, f(v_3^1) = b, f(v_4^1) = 0,
f(v_5^1) = c, f(v_6^1) = c, f(v_7^1) = a.
\]

**Case 2:** \( m = 2. \)
\[
f(v_1^2) = a, f(v_2^2) = b, f(v_3^2) = b, f(v_4^2) = 0,
f(v_5^2) = c, f(v_6^2) = c, f(v_7^2) = a.
\]

**Case 3:** \( m \equiv 0 \text{(mod 4)}. \)
\[
f(v_1^3) = a, f(v_2^3) = b, f(v_3^3) = b, f(v_4^3) = 0,
f(v_5^3) = 0, f(v_6^3) = c, f(v_7^3) = a, \quad 3 \leq i \leq m.
\]

**Case 4:** \( m \equiv 1 \text{(mod 4)}. \)
\[
f(v_1^4) = a, f(v_2^4) = b, f(v_3^4) = 0, f(v_4^4) = 0,
f(v_5^4) = c, f(v_6^4) = c, f(v_7^4) = b, \quad 3 \leq i \leq m.
\]

**Case 5:** \( m \equiv 2 \text{(mod 4)}. \)
\[
f(v_1^5) = a, f(v_2^5) = b, f(v_3^5) = c, f(v_4^5) = a,
f(v_5^5) = c, f(v_6^5) = 0, f(v_7^5) = 0, \quad 3 \leq i \leq m.
\]

**Case 6:** \( m \equiv 3 \text{(mod 4)}. \)
\[
f(v_1^6) = a, f(v_2^6) = b, f(v_3^6) = b, f(v_4^6) = c,
f(v_5^6) = a, f(v_6^6) = c, f(v_7^6) = 0, \quad 3 \leq i \leq m.
\]

The labeling pattern defined above covers all possible arrangements of vertices. In each possibility the graph under consideration satisfies the vertex conditions and edge conditions of \( V_4 \)-cordial labeling as shown in table 4.3.7. That is \( G \) admits \( V_4 \)-cordial labeling. Let \( n = 4a + b, a, b \in N \cup \{0\}. \)
<table>
<thead>
<tr>
<th>b</th>
<th>Vertex conditions</th>
<th>Edge conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$v_f(0) + 1 = v_f(a) = v_f(b) = v_f(c) + 1$</td>
<td>$e_f(0) = e_f(a) = e_f(b) = e_f(c)$</td>
</tr>
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<td>1</td>
<td>$v_f(0) + 1 = v_f(a) + 1 = v_f(b) = v_f(c) + 1$</td>
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<td>2</td>
<td>$v_f(0) = v_f(a) = v_f(b) = v_f(c)$</td>
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</tr>
<tr>
<td>3</td>
<td>$v_f(0) + 1 = v_f(a) = v_f(b) = v_f(c)$</td>
<td>$e_f(0) = e_f(a) = e_f(b) = e_f(c)$</td>
</tr>
</tbody>
</table>

Table 4.3.7: $V_4$-cordial labeling of theta graph $\theta(8^{[m]})$.

Illustration 4.3.7 The $V_4$-cordial labeling of theta graph $\theta(8^{[d]})$ is shown in Figure 4.3.7.

Figure 4.3.7: $V_4$-cordial labeling of theta graph $\theta(8^{[d]})$. 
4.4 \hspace{1.5ex} V_4\hspace{.5ex}-\hspace{.5ex}cordial Labeling of Snake Related Graphs

Theorem 4.4.1  \hspace{1.5ex} The alternate triangular snake $A(TS_n)$ is $V_4$-cordial for all $n$.

**Proof:** Let $G = A(TS_n)$ be the alternate triangular snake obtained from the path $P_n$ with vertices $v_1, v_2, ..., v_n$ by joining $v_i$ and $v_{i+1}$ alternatively to the new vertex $v'_i$ for all $1 \leq i \leq n$.

It is noted that

$|V(G)| = \frac{3n}{2}$ and $|E(G)| = 2n - 1$, if $n \equiv 0(mod\ 2)$ and snake starts with a cycle.

$|V(G)| = \frac{3n}{2} - 1$ and $|E(G)| = 2n - 3$, if $n \equiv 0(mod\ 2)$ and snake starts with a pendant edge.

$|V(G)| = \frac{3n-1}{2}$ and $|E(G)| = 2n - 2$, if $n \equiv 1(mod\ 2)$.

To define $V_4$-cordial labeling $f : V(G) \rightarrow V_4$ following cases are considered:

**Case 1:** $n \equiv 0(mod\ 2)$.

**Subcase 1:** The alternate triangular snake $A(TS_n)$ where snake starts with a cycle.

In this subcase the labeling pattern is defined as follows:

- $f(v_i) = 0; \ i \equiv 4, 5(mod\ 16)$;
- $f(v_i) = a; \ i \equiv 0, 1, 6, 7, 10, 11(mod\ 16)$;
- $f(v_i) = b; \ i \equiv 8, 9, 14, 15(mod\ 16)$;
- $f(v_i) = c; \ i \equiv 2, 3, 12, 13(mod\ 16); 1 \leq i \leq n$;
- $f(v'_i) = 0; \ i \equiv 0, 4, 6, 7(mod\ 8)$;
- $f(v'_i) = b; \ i \equiv 1, 2(mod\ 8)$;
- $f(v'_i) = c; \ i \equiv 3, 5(mod\ 8); 1 \leq i \leq \frac{n}{2}$.

Let $n = 16a + b, a, b \in N \cup \{0\}$.

<table>
<thead>
<tr>
<th>b</th>
<th>Vertex conditions</th>
<th>Edge conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.8</td>
<td>$v_f(0) = v_f(a) = v_f(b) = v_f(c)$</td>
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<td>2.10</td>
<td>$v_f(0) + 1 = v_f(a) = v_f(b) = v_f(c)$</td>
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<tr>
<td>4</td>
<td>$v_f(0) + 1 = v_f(a) + 1 = v_f(b) = v_f(c)$</td>
<td>$e_f(0) + 1 = e_f(a) = e_f(b) = e_f(c)$</td>
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<tr>
<td>6.14</td>
<td>$v_f(0) + 1 = v_f(a) + 1 = v_f(b) + 1 = v_f(c)$</td>
<td>$e_f(0) + 1 = e_f(a) = e_f(b) = e_f(c)$</td>
</tr>
<tr>
<td>12</td>
<td>$v_f(0) + 1 = v_f(a) = v_f(b) + 1 = v_f(c)$</td>
<td>$e_f(0) + 1 = e_f(a) = e_f(b) = e_f(c)$</td>
</tr>
</tbody>
</table>

Table 4.4.1: $V_4$-cordial labeling of an alternate triangular snake $A(TS_n)$ where snake starts with a cycle and $n$ is even.

**Subcase 2:** The alternate triangular snake $A(TS_n)$ where snake starts with a pendant edge. In this subcase the labeling pattern is divided into following three sub-subcases.
Sub-subcase 1: $n \equiv 0 \,(mod \, 16)$

$f(v_1) = a;

f(v_i) = 0; \quad i \equiv 5, 6 (mod \, 16);

f(v_i) = a; \quad i \equiv 1, 2, 7, 8, 11, 12 (mod \, 16);

f(v_i) = b; \quad i \equiv 0, 9, 10, 15 (mod \, 16);

f(v_i) = c; \quad i \equiv 3, 4, 13, 14 (mod \, 16); \quad 2 \leq i \leq n - 1,$

$f(v_n) = b;

f(v'_i) = 0; \quad i \equiv 0, 6, 7 (mod \, 8);

f(v'_i) = b; \quad i \equiv 1, 2 (mod \, 8);

f(v'_i) = c; \quad i \equiv 3, 5 (mod \, 8); \quad 1 \leq i \leq \frac{n}{2} - 1.$

Sub-subcase 2: $n \equiv 2, 4, 6, 8, 10, 12 \,(mod \, 16)$

$f(v_1) = a;

f(v_i) = 0; \quad i \equiv 5, 6 (mod \, 16);

f(v_i) = a; \quad i \equiv 1, 2, 7, 8, 11, 12 (mod \, 16);

f(v_i) = b; \quad i \equiv 0, 9, 10, 15 (mod \, 16);

f(v_i) = c; \quad i \equiv 3, 4, 13, 14 (mod \, 16); \quad 2 \leq i \leq n - 1,$

$f(v_n) = 0;

f(v'_i) = 0; \quad i \equiv 0, 4, 6, 7 (mod \, 8);

f(v'_i) = b; \quad i \equiv 1, 2 (mod \, 8);

f(v'_i) = c; \quad i \equiv 3, 5 (mod \, 8); \quad 1 \leq i \leq \frac{n}{2} - 1.$

Sub-subcase 3: $n \equiv 14 \,(mod \, 16)$

$f(v_1) = a;

f(v_i) = 0; \quad i \equiv 5, 6 (mod \, 16);

f(v_i) = a; \quad i \equiv 1, 2, 7, 8, 11, 12 (mod \, 16);

f(v_i) = b; \quad i \equiv 0, 9, 10, 15 (mod \, 16);

f(v_i) = c; \quad i \equiv 3, 4, 13, 14 (mod \, 16); \quad 2 \leq i \leq n - 4,$

$f(v_{n-3}) = c;

f(v_{n-2}) = 0;

f(v_{n-1}) = b;

f(v_n) = a;

f(v'_i) = 0; \quad i \equiv 0, 4, 6, 7 (mod \, 8);

f(v'_i) = b; \quad i \equiv 1, 2 (mod \, 8);

f(v'_i) = c; \quad i \equiv 3, 5 (mod \, 8); \quad 1 \leq i \leq \frac{n}{2} - 1.$

Let $n = 16a + b, \, a, \, b \in N \cup \{0\}.$
Table 4.4.2: $V_i$-cordial labeling of an alternate triangular snake $A(TS_n)$ where snake starts with a pendant edge and $n$ is even.

**Case 2:** $n \equiv 1 \pmod{2}$.

In this case the labeling pattern is divided into following four subcases.

**Subcase 1:** $n \equiv 1, 5, 7, 9, 15 \pmod{16}$

$f(v_i) = 0; \quad i \equiv 4, 5 \pmod{16}$;

$f(v_i) = a; \quad i \equiv 0, 1, 6, 7, 10, 11 \pmod{16}$;

$f(v_i) = b; \quad i \equiv 8, 9, 14, 15 \pmod{16}$;

$f(v_i) = c; \quad i \equiv 2, 3, 12, 13 \pmod{16}; \quad 1 \leq i \leq n$,

$f(v'_i) = 0; \quad i \equiv 0, 4, 6, 7 \pmod{8}$;

$f(v'_i) = b; \quad i \equiv 1, 2 \pmod{8}$;

$f(v'_i) = c; \quad i \equiv 3, 5 \pmod{8}; \quad 1 \leq i \leq \frac{n-1}{2}$.

**Subcase 2:** $n \equiv 3 \pmod{16}$

$f(v_i) = 0; \quad i \equiv 4, 5 \pmod{16}$;

$f(v_i) = a; \quad i \equiv 0, 1, 6, 7, 10, 11 \pmod{16}$;

$f(v_i) = b; \quad i \equiv 8, 9, 14, 15 \pmod{16}$;

$f(v_i) = c; \quad i \equiv 2, 3, 12, 13 \pmod{16}; \quad 1 \leq i \leq n - 4$,

$f(v_{n-3}) = 0$;

$f(v_{n-2}) = a$;

$f(v_{n-1}) = b$;

$f(v_{n}) = c$;

$f(v'_{i}) = 0; \quad i \equiv 0, 4, 6, 7 \pmod{8}$;

$f(v'_{i}) = b; \quad i \equiv 1, 2 \pmod{8}$;

$f(v'_{i}) = c; \quad i \equiv 3, 5 \pmod{8}; \quad 1 \leq i \leq \frac{n-1}{2} - 1$,

$f(v'_{\frac{n-1}{2}}) = a$.

**Subcase 3:** $n \equiv 11 \pmod{16}$

$f(v_i) = 0; \quad i \equiv 4, 5 \pmod{16}$;
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\[ f(v_i) = a; \quad i \equiv 0, 1, 6, 7, 10, 11 (\mod 16); \]
\[ f(v_i) = b; \quad i \equiv 8, 9, 14, 15 (\mod 16); \]
\[ f(v_i) = c; \quad i \equiv 2, 3, 12, 13 (\mod 16); \quad 1 \leq i \leq n - 4, \]
\[ f(v_{n-3}) = 0; \]
\[ f(v_{n-2}) = b; \]
\[ f(v_{n-1}) = a; \]
\[ f(v_n) = c; \]
\[ f(v'_i) = 0; \quad i \equiv 0, 4, 6, 7 (\mod 8); \]
\[ f(v'_i) = b; \quad i \equiv 1, 2 (\mod 8); \]
\[ f(v'_i) = c; \quad i \equiv 3, 5 (\mod 8); \quad 1 \leq i \leq \frac{n-1}{2} - 1, \]
\[ f(v'_{\frac{n-1}{2}}) = b. \]

**Subcase 4:** \( n \equiv 13 (\mod 16) \)

\[ f(v_i) = 0; \quad i \equiv 4, 5 (\mod 16); \]
\[ f(v_i) = a; \quad i \equiv 0, 1, 6, 7, 10, 11 (\mod 16); \]
\[ f(v_i) = b; \quad i \equiv 8, 9, 14, 15 (\mod 16); \]
\[ f(v_i) = c; \quad i \equiv 2, 3, 12, 13 (\mod 16); \quad 1 \leq i \leq n - 4, \]
\[ f(v_{n-3}) = c; \]
\[ f(v_{n-2}) = 0; \]
\[ f(v_{n-1}) = b; \]
\[ f(v_n) = a; \]
\[ f(v'_i) = 0; \quad i \equiv 0, 4, 6, 7 (\mod 8); \]
\[ f(v'_i) = b; \quad i \equiv 1, 2 (\mod 8); \]
\[ f(v'_i) = c; \quad i \equiv 3, 5 (\mod 8); \quad 1 \leq i \leq \frac{n-1}{2} - 1, \]
\[ f(v'_{\frac{n-1}{2}}) = 0. \]

Let \( n = 16a + b, a, b \in N \cup \{0\}. \)

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<tr>
<th>b</th>
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<th>Edges conditions</th>
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<td>3,11</td>
<td>( v_f(0) = v_f(a) = v_f(b) = v_f(c) )</td>
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</tr>
<tr>
<td>5,13</td>
<td>( v_f(0) = v_f(a) + 1 = v_f(b) = v_f(c) )</td>
<td>( e_f(0) = e_f(a) = e_f(b) = e_f(c) )</td>
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<tr>
<td>9</td>
<td>( v_f(0) + 1 = v_f(a) + 1 = v_f(b) = v_f(c) + 1 )</td>
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<td>( v_f(0) + 1 = v_f(a) + 1 = v_f(b) = v_f(c) )</td>
<td>( e_f(0) = e_f(a) = e_f(b) = e_f(c) )</td>
</tr>
</tbody>
</table>

Table 4.4.3: \( V_4 \)-cordial labeling of an alternate triangular snake \( A(TS_n) \) where \( n \) is odd.

The labeling pattern defined above covers all possible arrangements of vertices. In each possibility the graph under consideration satisfies the vertex conditions and edge conditions of \( V_4 \)-cordial labeling. That is \( G \) admits \( V_4 \)-cordial labeling.
Illustration 4.4.1 The alternate triangular snake $A(TS_8)$ where snake starts with a cycle and its $V_4$-cordial labeling is shown in Figure 4.4.1.

![Diagram](image1)

Figure 4.4.1: $V_4$-cordial labeling of an alternate triangular snake $A(TS_8)$ where snake starts with a cycle.

Illustration 4.4.2 The alternate triangular snake $A(TS_8)$ where snake starts with a pendant edge and its $V_4$-cordial labeling is shown in Figure 4.4.2.

![Diagram](image2)

Figure 4.4.2: $V_4$-cordial labeling of an alternate triangular snake $A(TS_8)$ where snake starts with a pendant edge.

Illustration 4.4.3 The alternate triangular snake $A(TS_5)$ where $n$ is odd and its $V_4$-cordial labeling is shown in Figure 4.4.3.

![Diagram](image3)

Figure 4.4.3: $V_4$-cordial labeling of an alternate triangular snake $A(TS_5)$. 
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**Theorem 4.4.2** The alternate double triangular snake $AD(TS_n)$ is $V_4$-cordial for all $n$.

**Proof:** Let $G = AD(TS_n)$ be the alternate double triangular snake obtained from the path $P_n$ with vertices $v_1, v_2, ..., v_n$ by joining $v_i$ and $v_{i+1}$ alternatively to the new vertices $v_i'$ and $v_i''$ by four new edges, for all $1 \leq i \leq n$.

It is noted that

- $|V(G)| = 2n$ and $|E(G)| = 3n - 1$; if $n \equiv 0(\mod 2)$ and snake starts with a cycle.
- $|V(G)| = 2n - 2$ and $|E(G)| = 3n - 5$; if $n \equiv 0(\mod 2)$ and snake starts with a pendant edge.
- $|V(G)| = 2n - 1$ and $|E(G)| = 3n - 3$; if $n \equiv 1(\mod 2)$.

To define $V_4$-cordial labeling $f : V(G) \to V_4$ following cases are considered:

**Case 1:** $n \equiv 0(\mod 2)$.

**Subcase 1:** The alternate double triangular snake $AD(TS_n)$ where snake starts with a cycle.

In this subcase the labeling pattern is divided into following two sub-subcases.

**Sub-subcase 1:** $n \equiv 0, 4, 6(\mod 8)$
- $f(v_i) = 0; \quad i \equiv 4, 5(\mod 8)$;
- $f(v_i) = a; \quad i \equiv 0, 1(\mod 8)$;
- $f(v_i) = b; \quad i \equiv 2, 3, 6, 7(\mod 8); \quad 1 \leq i \leq n$,
- $f(v_i') = 0; \quad i \equiv 2(\mod 4)$;
- $f(v_i') = a; \quad i \equiv 1, 3(\mod 4)$;
- $f(v_i'') = c; \quad i \equiv 0(\mod 4); \quad 1 \leq i \leq \frac{n}{2}$,
- $f(v_i''') = 0; \quad i \equiv 0(\mod 4)$;
- $f(v_i'''') = c; \quad i \equiv 1, 2, 3(\mod 4); \quad 1 \leq i \leq \frac{n}{2}$.

**Sub-subcase 2:** $n \equiv 2(\mod 8)$
- $f(v_i) = 0; \quad i \equiv 4, 5(\mod 8)$;
- $f(v_i) = a; \quad i \equiv 0, 1(\mod 8)$;
- $f(v_i) = b; \quad i \equiv 2, 3, 6, 7(\mod 8); \quad 1 \leq i \leq n - 3$,
- $f(v_{n-2}) = 0$,
- $f(v_{n-1}) = a$,
- $f(v_n) = b$,
- $f(v_i') = 0; \quad i \equiv 2(\mod 4)$;
- $f(v_i') = a; \quad i \equiv 1, 3(\mod 4)$;
- $f(v_i'') = c; \quad i \equiv 0(\mod 4); \quad 1 \leq i \leq \frac{n}{2}$,
- $f(v_i''') = 0; \quad i \equiv 0(\mod 4)$;
\[ f(v_i^n) = c; \quad i \equiv 1, 2, 3(\text{mod } 4); \quad 1 \leq i \leq \frac{n}{2}. \]

Let \( n = 8a + b, a, b \in N \cup \{0\} \).

<table>
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<tr>
<th>b</th>
<th>Vertex conditions</th>
<th>Edge conditions</th>
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<tbody>
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<td>0</td>
<td>( v_f(0) = v_f(a) = v_f(b) = v_f(c) )</td>
<td>( e_f(0) + 1 = e_f(a) = e_f(b) = e_f(c) )</td>
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<tr>
<td>2</td>
<td>( v_f(0) = v_f(a) = v_f(b) = v_f(c) )</td>
<td>( e_f(0) + 1 = e_f(a) + 1 = e_f(b) + 1 = e_f(c) )</td>
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<tr>
<td>4</td>
<td>( v_f(0) = v_f(a) = v_f(b) = v_f(c) )</td>
<td>( e_f(0) = e_f(a) + 1 = e_f(b) = e_f(c) )</td>
</tr>
<tr>
<td>6</td>
<td>( v_f(0) = v_f(a) = v_f(b) = v_f(c) )</td>
<td>( e_f(0) + 1 = e_f(a) + 1 = e_f(b) + 1 = e_f(c) )</td>
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Table 4.4.4: \( V_1 \)-cordial labeling of an alternate double triangular snake \( AD(TS_n) \) where snake starts with a cycle and \( n \) is even.

Subcase 2: The alternate double triangular snake \( AD(TS_n) \) where snake starts with a pendant edge.

In this subcase the labeling pattern is divided into following four Sub-subcases.

Sub-subcase 1: \( n \equiv 0(\text{mod } 8) \)

\[
\begin{align*}
 f(v_1) &= 0; \\
 f(v_i) &= 0; \quad i \equiv 5, 6(\text{mod } 8); \\
 f(v_i) &= a; \quad i \equiv 1, 2(\text{mod } 8); \\
 f(v_i) &= b; \quad i \equiv 0, 3, 4, 7(\text{mod } 8); \quad 2 \leq i \leq n - 1, \\
 f(v_n) &= b; \\
 f(v_i) &= 0; \quad i \equiv 2(\text{mod } 4); \\
 f(v_i) &= a; \quad i \equiv 1, 3(\text{mod } 4); \\
 f(v_i) &= c; \quad i \equiv 0(\text{mod } 4); \quad 1 \leq i \leq \frac{n}{2} - 1, \\
 f(v_i) &= 0; \quad i \equiv 0(\text{mod } 4); \\
 f(v_i) &= 0; \quad i \equiv 1, 2, 3(\text{mod } 4); \quad 1 \leq i \leq \frac{n}{2} - 1.
\end{align*}
\]

Sub-subcase 2: \( n \equiv 2(\text{mod } 8) \)

\[
\begin{align*}
 f(v_1) &= a; \\
 f(v_i) &= 0; \quad i \equiv 5, 6(\text{mod } 8); \\
 f(v_i) &= a; \quad i \equiv 1, 2(\text{mod } 8); \\
 f(v_i) &= b; \quad i \equiv 0, 3, 4, 7(\text{mod } 8); \quad 2 \leq i \leq n - 1, \\
 f(v_n) &= 0; \\
 f(v_i) &= 0; \quad i \equiv 2(\text{mod } 4); \\
 f(v_i) &= a; \quad i \equiv 1, 3(\text{mod } 4); \\
 f(v_i) &= c; \quad i \equiv 0(\text{mod } 4); \quad 1 \leq i \leq \frac{n}{2} - 1, \\
 f(v_i) &= 0; \quad i \equiv 0(\text{mod } 4); \\
 f(v_i) &= 0; \quad i \equiv 1, 2, 3(\text{mod } 4); \quad 1 \leq i \leq \frac{n}{2} - 1.
\end{align*}
\]
Sub-case 3: $n \equiv 4 \pmod{8}$

\[
\begin{align*}
f(v_1) &= 0; \\
f(v_i) &= 0; \quad i \equiv 5, 6 \pmod{8}; \\
f(v_i) &= a; \quad i \equiv 1, 2 \pmod{8}; \\
f(v_i) &= b; \quad i \equiv 0, 3, 4, 7 \pmod{8}; \quad 2 \leq i \leq n - 1, \\
f(v_i) &= 0; \\
f(v_i') &= 0; \quad i \equiv 2 \pmod{4}; \\
f(v_i') &= a; \quad i \equiv 1, 3 \pmod{4}; \\
f(v_i') &= c; \quad i \equiv 0 \pmod{4}; \quad 1 \leq i \leq \frac{n}{2} - 1, \\
f(v_i''') &= 0; \quad i \equiv 0 \pmod{4}; \\
f(v_i''') &= c; \quad i \equiv 1, 2, 3 \pmod{4}; \quad 1 \leq i \leq \frac{n}{2} - 1.
\end{align*}
\]

Sub-case 4: $n \equiv 6 \pmod{8}$

\[
\begin{align*}
f(v_1) &= 0; \\
f(v_i) &= 0; \quad i \equiv 5, 6 \pmod{8}; \\
f(v_i) &= a; \quad i \equiv 1, 2 \pmod{8}; \\
f(v_i) &= b; \quad i \equiv 0, 3, 4, 7 \pmod{8}; \quad 2 \leq i \leq n - 1, \\
f(v_i) &= a; \\
f(v_i') &= 0; \quad i \equiv 2 \pmod{4}; \\
f(v_i') &= a; \quad i \equiv 1, 3 \pmod{4}; \\
f(v_i') &= c; \quad i \equiv 0 \pmod{4}; \quad 1 \leq i \leq \frac{n}{2} - 1, \\
f(v_i''') &= 0; \quad i \equiv 0 \pmod{4}; \\
f(v_i''') &= c; \quad i \equiv 1, 2, 3 \pmod{4}; \quad 1 \leq i \leq \frac{n}{2} - 1.
\end{align*}
\]

Let $n = 8a + b, a, b \in N \cup \{0\}$.

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<thead>
<tr>
<th>$b$</th>
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<th>Edge conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$v_f(0) = v_f(a) + 1 = v_f(b) = v_f(c) + 1$</td>
<td>$e_f(0) = e_f(a) = e_f(b) + 1 = e_f(c)$</td>
</tr>
<tr>
<td>2</td>
<td>$v_f(0) = v_f(a) = v_f(b) + 1 = v_f(c) + 1$</td>
<td>$e_f(0) + 1 = e_f(a) = e_f(b) + 1 = e_f(c) + 1$</td>
</tr>
<tr>
<td>4</td>
<td>$v_f(0) = v_f(a) = v_f(b) + 1 = v_f(c) + 1$</td>
<td>$e_f(0) + 1 = e_f(a) = e_f(b) = e_f(c)$</td>
</tr>
<tr>
<td>6</td>
<td>$v_f(0) = v_f(a) = v_f(b) + 1 = v_f(c) + 1$</td>
<td>$e_f(0) + 1 = e_f(a) = e_f(b) + 1 = e_f(c) + 1$</td>
</tr>
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Table 4.4.5: $V_4$-cordial labeling of an alternate double triangular snake $AD(TS_n)$ where snake starts with a pendant edge and $n$ is even.

Case 2: $n \equiv 1 \pmod{2}$.

In this case the labeling pattern is divided into following four subcases.

Sub-case 1: $n \equiv 1 \pmod{8}$

\[
\begin{align*}
f(v_i) &= 0; \quad i \equiv 4, 5 \pmod{8}; \\
f(v_i) &= a; \quad i \equiv 0, 1 \pmod{8};
\end{align*}
\]
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Subcase 2: \( n \equiv 3(\text{mod } 8) \)

\[
\begin{align*}
    f(v_i) &= 0; \quad i \equiv 4, 5(\text{mod } 8); \\
    f(v_i) &= a; \quad i \equiv 0, 1(\text{mod } 8); \\
    f(v_i) &= b; \quad i \equiv 2, 3, 6, 7(\text{mod } 8); \quad 1 \leq i \leq n - 1, \\
    f(v_n) &= 0; \\
    f(v'_i) &= 0; \quad i \equiv 2(\text{mod } 4); \\
    f(v'_i) &= a; \quad i \equiv 1, 3(\text{mod } 4); \\
    f(v''_i) &= c; \quad i \equiv 0(\text{mod } 4); \quad 1 \leq i \leq \frac{n-1}{2}, \\
    f(v''_i) &= 0; \quad i \equiv 0(\text{mod } 4); \\
    f(v'''_i) &= c; \quad i \equiv 1, 2, 3(\text{mod } 4); \quad 1 \leq i \leq \frac{n-1}{2}.
\end{align*}
\]

Subcase 3: \( n \equiv 5(\text{mod } 8) \)

\[
\begin{align*}
    f(v_i) &= 0; \quad i \equiv 4, 5(\text{mod } 8); \\
    f(v_i) &= a; \quad i \equiv 0, 1(\text{mod } 8); \\
    f(v_i) &= b; \quad i \equiv 2, 3, 6, 7(\text{mod } 8); \quad 1 \leq i \leq n - 1, \\
    f(v_n) &= a; \\
    f(v'_i) &= 0; \quad i \equiv 2(\text{mod } 4); \\
    f(v'_i) &= a; \quad i \equiv 1, 3(\text{mod } 4); \\
    f(v''_i) &= c; \quad i \equiv 0(\text{mod } 4); \quad 1 \leq i \leq \frac{n-1}{2}, \\
    f(v''_i) &= 0; \quad i \equiv 0(\text{mod } 4); \\
    f(v'''_i) &= c; \quad i \equiv 1, 2, 3(\text{mod } 4); \quad 1 \leq i \leq \frac{n-1}{2}.
\end{align*}
\]

Subcase 4: \( n \equiv 7(\text{mod } 8) \)

\[
\begin{align*}
    f(v_i) &= 0; \quad i \equiv 4, 5(\text{mod } 8); \\
    f(v_i) &= a; \quad i \equiv 0, 1(\text{mod } 8); \\
    f(v_i) &= b; \quad i \equiv 2, 3, 6, 7(\text{mod } 8); \quad 1 \leq i \leq n - 1, \\
    f(v_n) &= b; \\
    f(v'_i) &= 0; \quad i \equiv 2(\text{mod } 4); \\
    f(v'_i) &= a; \quad i \equiv 1, 3(\text{mod } 4); \\
    f(v''_i) &= c; \quad i \equiv 0(\text{mod } 4); \quad 1 \leq i \leq \frac{n-1}{2}, \\
    f(v''_i) &= 0; \quad i \equiv 0(\text{mod } 4); \\
    f(v'''_i) &= c; \quad i \equiv 1, 2, 3(\text{mod } 4); \quad 1 \leq i \leq \frac{n-1}{2}.
\end{align*}
\]
Let \( n = 8a + b, a, b \in N \cup \{0\} \).

<table>
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<tr>
<th>b</th>
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<tbody>
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<td>1</td>
<td>( v_f(0) + 1 = v_f(a) = v_f(b) + 1 = v_f(c) + 1 )</td>
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<td>( e_f(0) = e_f(a) + 1 = e_f(b) + 1 = e_f(c) )</td>
</tr>
</tbody>
</table>

Table 4.4.6: \( V_4 \)-cordial labeling of an alternate double triangular snake \( AD(TS_n) \) where \( n \) is odd.

The labeling pattern defined above covers all possible arrangements of vertices. In each possibility the graph under consideration satisfies the vertex conditions and edge conditions of \( V_4 \)-cordial labeling. That is \( G \) admits \( V_4 \)-cordial labeling.

**Illustration 4.4.4** The alternate double triangular snake \( AD(TS_8) \) where snake starts with a cycle and its \( V_4 \)-cordial labeling is shown in Figure 4.4.4.

![Figure 4.4.4: V_4-cordial labeling of an alternate double triangular snake AD(TS_8) where snake starts with a cycle.](image-url)
**Illustration 4.4.5** The alternate double triangular snake $AD(TS_8)$ where snake starts with a pendant edge and its $V_4$-cordial labeling is shown in Figure 4.4.5.

![Figure 4.4.5: $V_4$-cordial labeling of an alternate double triangular snake $AD(TS_8)$ where snake starts from with a pendant edge.]

**Illustration 4.4.6** The alternate double triangular snake $AD(TS_5)$ where $n$ is odd and its $V_4$-cordial labeling is shown in Figure 4.4.6.

![Figure 4.4.6: $V_4$-cordial labeling of an alternate double triangular snake $AD(TS_5)$.]

**Theorem 4.4.3** The alternate triple triangular snake $AT(TS_n)$ is $V_4$-cordial for all $n$.

**Proof:** Let $G = AT(TS_n)$ be the alternate triple triangular snake obtained from the path $P_n$ with vertices $u_1, u_2, ..., u_n$ by joining $u_i$ and $u_{i+1}$ alternatively to the new vertices $v_i, v_i'$ and $v_i''$ by six new edges, for all $1 \leq i \leq n$.

It is noted that
\[ |V(G)| = \frac{5n}{2} \quad \text{and} \quad |E(G)| = 4n - 1; \quad \text{if} \ n \equiv 0(\text{mod} \ 2) \ \text{and snake starts with a cycle.} \]
\[ |V(G)| = \frac{5n}{2} - 3 \] and \[ |E(G)| = 4n - 7; \quad \text{if } n \equiv 0 (mod \ 2) \] and snake starts with pendant edge.
\[ |V(G)| = \frac{5n-3}{2} \] and \[ |E(G)| = 4n - 4; \quad \text{if } n \equiv 1 (mod \ 2). \]

To define \( V_4 \)-cordial labeling \( f : V(G) \to V_4 \) following cases are considered:

**Case 1:** \( n \equiv 0 (mod \ 2) \).

**Subcase 1:** The alternate triple triangular snake \( AT(TS_n) \) starts with a cycle.
In this subcase the labeling pattern is defined as follows:

\[
\begin{align*}
f(u_1) &= a; \\
f(u_2) &= c; \\
f(u_3) &= c; \\
f(u_i) &= 0; \quad i \equiv 0, 1 (mod \ 8); \\
f(u_i) &= a; \quad i \equiv 2, 3 (mod \ 8); \\
f(u_i) &= b; \quad i \equiv 4, 5 (mod \ 8); \\
f(u_i) &= c; \quad i \equiv 6, 7 (mod \ 8); \quad 4 \leq i \leq n, \\
f(v_1) &= a; \\
f(v_2) &= a; \\
f(v_i) &= a; \quad i \equiv 0, 1, 3 (mod \ 4); \\
f(v_i) &= c; \quad i \equiv 2 (mod \ 4); \quad 3 \leq i \leq \frac{n}{2}, \\
f(v'_1) &= 0; \\
f(v'_2) &= 0; \\
f(v''_i) &= 0; \quad i \equiv 0, 2, 3 (mod \ 4); \\
f(v''_i) &= c; \quad i \equiv 1 (mod \ 4); \quad 3 \leq i \leq \frac{n}{2}, \\
f(v'_1) &= b; \\
f(v'_2) &= b; \\
f(v''_i) &= b; \quad i \equiv 0, 1, 2 (mod \ 4); \\
f(v''_i) &= c; \quad i \equiv 3 (mod \ 4); \quad 3 \leq i \leq \frac{n}{2}.
\end{align*}
\]

Let \( n = 8a + b, a, b \in N \cup \{0\} \).

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Table 4.4.7: \( V_4 \)-cordial labeling of an alternate triple triangular snake \( AT(TS_n) \) where snake starts with a cycle and \( n \) is even.
Subcase 2: The alternate triple triangular snake $AT(TS_n)$ starts with a pendant edge. In this subcase the labeling pattern is divided into following sub-subcases:

**Sub-subcase 1: $n \equiv 0 \,(mod\ 8)$**

\[
\begin{align*}
    f(u_1) &= 0; \\
    f(u_2) &= a; \\
    f(u_3) &= c; \\
    f(u_4) &= c; \\
    f(u_i) &= 0; & i &\equiv 1,2\,(mod\ 8); \\
    f(u_i) &= a; & i &\equiv 3,4\,(mod\ 8); \\
    f(u_i) &= b; & i &\equiv 5,6\,(mod\ 8); \\
    f(u_i) &= c; & i &\equiv 0,7\,(mod\ 8); & 5 \leq i \leq n - 1, \\
    f(u_n) &= c; \\
    f(v_1) &= a; \\
    f(v_2) &= a; \\
    f(v_i) &= a; & i &\equiv 0,1,3\,(mod\ 4); \\
    f(v_i) &= c; & i &\equiv 2\,(mod\ 4); & 3 \leq i \leq \frac{n}{2}, \\
    f(v'_1) &= 0; \\
    f(v'_2) &= 0; \\
    f(v'_i) &= 0; & i &\equiv 0,2,3\,(mod\ 4); \\
    f(v''_i) &= c; & i &\equiv 1\,(mod\ 4); & 3 \leq i \leq \frac{n}{2}, \\
    f(v''_1) &= b; \\
    f(v''_2) &= b; \\
    f(v''_i) &= b; & i &\equiv 0,1,2\,(mod\ 4); \\
    f(v''_i) &= c; & i &\equiv 3\,(mod\ 4); & 3 \leq i \leq \frac{n}{2}.
\end{align*}
\]

**Sub-subcase 2: $n \equiv 2\,(mod\ 8)$**

\[
\begin{align*}
    f(u_1) &= a; \\
    f(u_2) &= a; \\
    f(u_3) &= c; \\
    f(u_4) &= c; \\
    f(u_i) &= 0; & i &\equiv 1,2\,(mod\ 8); \\
    f(u_i) &= a; & i &\equiv 3,4\,(mod\ 8); \\
    f(u_i) &= b; & i &\equiv 5,6\,(mod\ 8); \\
    f(u_i) &= c; & i &\equiv 0,7\,(mod\ 8); & 5 \leq i \leq n - 1, \\
    f(u_n) &= 0; \\
    f(v_1) &= a; \\
    f(v_2) &= a; \\
    f(v_i) &= a; & i &\equiv 0,1,3\,(mod\ 4); \\
    f(v_i) &= c; & i &\equiv 2\,(mod\ 4); & 3 \leq i \leq \frac{n}{2}, \\
    f(v'_1) &= 0;
\end{align*}
\]
Chapter 4. Results

Sub-subcase 3: \( n \equiv 4(\text{mod} \ 8) \)

\[
\begin{align*}
  f(v'_2) &= 0; \\
  f(v'_i) &= 0; \quad i \equiv 0, 2, 3(\text{mod} \ 4); \\
  f(v'_i) &= c; \quad i \equiv 1(\text{mod} \ 4); \quad 3 \leq i \leq \frac{n}{2}; \\
  f(v''_i) &= b; \\
  f(v''_2) &= b; \\
  f(v''_i) &= b; \quad i \equiv 0, 1, 2(\text{mod} \ 4); \\
  f(v''_i) &= c; \quad i \equiv 3(\text{mod} \ 4); \quad 3 \leq i \leq \frac{n}{2}.
\end{align*}
\]

Sub-subcase 4: \( n \equiv 6(\text{mod} \ 8) \)

\[
\begin{align*}
  f(u_1) &= b; \\
  f(u_2) &= 0; \\
  f(u_3) &= c; \\
  f(u_4) &= c; \\
  f(u_i) &= 0; \quad i \equiv 1, 2(\text{mod} \ 8); \\
  f(u_i) &= a; \quad i \equiv 3, 4(\text{mod} \ 8); \\
  f(u_i) &= b; \quad i \equiv 5, 6(\text{mod} \ 8); \\
  f(u_i) &= c; \quad i \equiv 0, 7(\text{mod} \ 8); \quad 5 \leq i \leq n - 1, \\
  f(u_n) &= a; \\
  f(v_1) &= a; \\
  f(v_2) &= a; \\
  f(v_i) &= a; \quad i \equiv 0, 1, 3(\text{mod} \ 4); \\
  f(v_i) &= c; \quad i \equiv 2(\text{mod} \ 4); \quad 3 \leq i \leq \frac{n}{2}, \\
  f(v'_i) &= 0; \\
  f(v'_2) &= 0; \\
  f(v'_i) &= 0; \quad i \equiv 0, 2, 3(\text{mod} \ 4); \\
  f(v'_i) &= c; \quad i \equiv 1(\text{mod} \ 4); \quad 3 \leq i \leq \frac{n}{2}, \\
  f(v''_i) &= b; \\
  f(v''_2) &= b; \\
  f(v''_i) &= b; \quad i \equiv 0, 1, 2(\text{mod} \ 4); \\
  f(v''_i) &= c; \quad i \equiv 3(\text{mod} \ 4); \quad 3 \leq i \leq \frac{n}{2}.
\end{align*}
\]
Subcase 1
In this case the labeling pattern is divided into following four subcases.

\[ f(v_1) = a; \]
\[ f(v_2) = a; \]
\[ f(v_i) = a; \quad i \equiv 0, 1, 3 \,(\text{mod} \ 4); \]
\[ f(v_i) = c; \quad i \equiv 2 \,(\text{mod} \ 4); \quad 3 \leq i \leq \frac{n}{2}; \]
\[ f(v'_1) = 0; \]
\[ f(v'_2) = 0; \]
\[ f(v'_i) = 0; \quad i \equiv 0, 2, 3 \,(\text{mod} \ 4); \]
\[ f(v'_i) = c; \quad i \equiv 1 \,(\text{mod} \ 4); \quad 3 \leq i \leq \frac{n}{2}; \]
\[ f(v''_1) = 0; \]
\[ f(v''_2) = b; \]
\[ f(v''_i) = b; \quad i \equiv 0, 1, 2 \,(\text{mod} \ 4); \]
\[ f(v''_i) = c; \quad i \equiv 3 \,(\text{mod} \ 4); \quad 3 \leq i \leq \frac{n}{2}. \]

Let \( n = 8a + b, a, b \in N \cup \{0\}. \)

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</tr>
</tbody>
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Table 4.4.8: \( V_4 \)-cordial labeling of an alternate triple triangular snake \( AT(TS_n) \) where snake starts with a pendant edge and \( n \) is even.

**Case 2: \( n \equiv 1 \,(\text{mod} \ 2). \)**

In this case the labeling pattern is divided into following four subcases.

**Subcase 1: \( n \equiv 1 \,(\text{mod} \ 8) \)**
\[ f(u_1) = a; \]
\[ f(u_2) = c; \]
\[ f(u_3) = c; \]
\[ f(u_i) = 0; \quad i \equiv 0, 1 \,(\text{mod} \ 8); \]
\[ f(u_i) = a; \quad i \equiv 2, 3 \,(\text{mod} \ 8); \]
\[ f(u_i) = b; \quad i \equiv 4, 5 \,(\text{mod} \ 8); \]
\[ f(u_i) = c; \quad i \equiv 6, 7 \,(\text{mod} \ 8); \quad 4 \leq i \leq n - 1, \]
\[ f(u_n) = 0; \]
\[ f(v_1) = a; \]
\[ f(v_2) = a; \]
\[ f(v_i) = a; \quad i \equiv 0, 1, 3 \,(\text{mod} \ 4); \]
\[ f(v_i) = c; \quad i \equiv 2 \,(\text{mod} \ 4); \quad 3 \leq i \leq \frac{n-1}{2}, \]
\[ f(v'_1) = 0; \]
Subcase 2: \( n \equiv 3(\text{mod } 8) \)
\[
\begin{align*}
f(u_1) &= b; \\
f(u_2) &= c; \\
f(u_3) &= c; \\
f(u_i) &= 0; \quad i \equiv 0, 1(\text{mod } 8); \\
f(u_i) &= a; \quad i \equiv 2, 3(\text{mod } 8); \\
f(u_i) &= b; \quad i \equiv 4, 5(\text{mod } 8); \\
f(u_i) &= c; \quad i \equiv 6, 7(\text{mod } 8); \\
f(u_{n-1}) &= a; \\
f(v_1) &= a; \\
f(v_2) &= a; \\
f(v_i) &= a; \quad i \equiv 0, 1, 3(\text{mod } 4); \\
f(v_i) &= c; \quad i \equiv 2(\text{mod } 4); \\
f(v_1) &= 0; \\
f(v_2) &= 0; \\
f(v_i) &= 0; \quad i \equiv 0, 2, 3(\text{mod } 4); \\
f(v_i) &= c; \quad i \equiv 1(\text{mod } 4); \\
f(v_1') &= b; \\
f(v_2') &= b; \\
f(v_i') &= b; \quad i \equiv 0, 1, 2(\text{mod } 4); \\
f(v_i') &= c; \quad i \equiv 3(\text{mod } 4); \\
f(v_{n-1}) &= b; \\
f(v_1) &= a; \\
f(v_2) &= a;
\end{align*}
\]
\[ f(v_i) = a; \quad i \equiv 0, 1, 3 \pmod{4}; \]
\[ f(v_i) = c; \quad i \equiv 2 \pmod{4}; \quad 3 \leq i \leq \frac{n-1}{2}, \]
\[ f(v'_1) = 0; \]
\[ f(v'_2) = 0; \]
\[ f(v'_3) = 0; \quad i \equiv 0, 2, 3 \pmod{4}; \]
\[ f(v'_i) = c; \quad i \equiv 1 \pmod{4}; \quad 3 \leq i \leq \frac{n-1}{2}; \]
\[ f(v''_1) = 0; \]
\[ f(v''_2) = b; \]
\[ f(v''_i) = b; \quad i \equiv 0, 1, 2 \pmod{4}; \]
\[ f(v''_i) = c; \quad i \equiv 3 \pmod{4}; \quad 3 \leq i \leq \frac{n-1}{2}. \]

**Subcase 4: \( n \equiv 7 \pmod{8} \)**

\[ f(u_1) = a; \]
\[ f(u_2) = c; \]
\[ f(u_3) = c; \]
\[ f(u_i) = 0; \quad i \equiv 0, 1 \pmod{8}; \]
\[ f(u_i) = a; \quad i \equiv 2, 3 \pmod{8}; \]
\[ f(u_i) = b; \quad i \equiv 4, 5 \pmod{8}; \]
\[ f(u_i) = c; \quad i \equiv 6, 7 \pmod{8}; \quad 4 \leq i \leq n - 1, \]
\[ f(u_n) = c; \]
\[ f(v_1) = a; \]
\[ f(v_2) = a; \]
\[ f(v_i) = a; \quad i \equiv 0, 1, 3 \pmod{4}; \]
\[ f(v_i) = c; \quad i \equiv 2 \pmod{4}; \quad 3 \leq i \leq \frac{n-1}{2}, \]
\[ f(v'_1) = 0; \]
\[ f(v'_2) = 0; \]
\[ f(v'_3) = 0; \quad i \equiv 0, 2, 3 \pmod{4}; \]
\[ f(v'_i) = c; \quad i \equiv 1 \pmod{4}; \quad 3 \leq i \leq \frac{n-1}{2}; \]
\[ f(v''_1) = 0; \]
\[ f(v''_2) = b; \]
\[ f(v''_i) = b; \quad i \equiv 0, 1, 2 \pmod{4}; \]
\[ f(v''_i) = c; \quad i \equiv 3 \pmod{4}; \quad 3 \leq i \leq \frac{n-1}{2} - 1, \]
\[ f(v''_{n-1}) = b. \]

Let \( n = 8a + b, a, b \in N \cup \{0\}. \)
Table 4.4.9: $V_4$-cordial labeling of an alternate triple triangular snake $AT(TS_n)$ where $n$ is odd.

The labeling pattern defined above covers all possible arrangements of vertices. In each possibility the graph under consideration satisfies the vertex conditions and edge conditions of $V_4$-cordial labeling. That is $G$ admits $V_4$-cordial labeling.

**Illustration 4.4.7** The alternate triple triangular snake $AT(TS_8)$ where snake starts with a cycle and its $V_4$-cordial labeling is shown in Figure 4.4.7.

Figure 4.4.7: $V_4$-cordial labeling of an alternate triple triangular snake $AT(TS_8)$ where snake starts with a cycle.
Illustration 4.4.8 The alternate triple triangular snake $AT(TS_8)$ where snake starts with a pendant edge and its $V_4$-cordial labeling is shown in Figure 4.4.8.

Figure 4.4.8: $V_4$-cordial labeling of an alternate triple triangular snake $AT(TS_8)$ where snake starts with a pendant edge.

Illustration 4.4.9 The alternate triple triangular snake $AT(TS_5)$ and its $V_4$-cordial labeling is shown in Figure 4.4.9.

Figure 4.4.9: $V_4$-cordial labeling of an alternate triple triangular snake $AT(TS_5)$. 
Theorem 4.4.4 The quadrilateral snake $QS_n$ is $V_4$-cordial for all $n$.

Proof: Let $G = QS_n$ be the quadrilateral snake obtained from path $v_1, v_2, ..., v_n$ by joining $v_i, v_{i+1}$ to new vertices $v'_i, v''_i$ respectively and then joining $v'_i$ and $v''_i$ by an edge for all $1 \leq i \leq n$. It is noted that $|V(G)| = 3n - 2$ and $|E(G)| = 4n - 4$.

Define $V_4$-cordial labeling $f : V(G) \to V_4$ as follows:

\[
\begin{align*}
  f(v_1) &= a; \\
  f(v_2) &= c; \\
  f(v_3) &= b; \\
  f(v_i) &= 0; & i & \equiv 2 (\text{mod } 4); \\
  f(v_i) &= a; & i & \equiv 1 (\text{mod } 4); \\
  f(v_i) &= b; & i & \equiv 3 (\text{mod } 4); \\
  f(v_i) &= c; & i & \equiv 0 (\text{mod } 4); & 4 \leq i \leq n, \\
  f(v'_1) &= b; \\
  f(v'_2) &= 0; \\
  f(v'_i) &= 0; & i & \equiv 2 (\text{mod } 4); \\
  f(v'_i) &= a; & i & \equiv 1 (\text{mod } 4); \\
  f(v'_i) &= b; & i & \equiv 3 (\text{mod } 4); \\
  f(v'_i) &= c; & i & \equiv 0 (\text{mod } 4); & 3 \leq i \leq n - 1, \\
  f(v''_1) &= c; \\
  f(v''_2) &= 0; \\
  f(v''_i) &= 0; & i & \equiv 0 (\text{mod } 4); \\
  f(v''_i) &= a; & i & \equiv 3 (\text{mod } 4); \\
  f(v''_i) &= b; & i & \equiv 1 (\text{mod } 4); \\
  f(v''_i) &= c; & i & \equiv 2 (\text{mod } 4); & 3 \leq i \leq n - 1.
\end{align*}
\]

Let $n = 4a + b, a, b \in N \cup \{0\}$.

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</table>

Table 4.4.10: $V_4$-cordial labeling of quadrilateral snake $QS_n$.

The labeling pattern defined above covers all possible arrangements of vertices. In each possibility the graph under consideration satisfies the vertex conditions and edge conditions of $V_4$-cordial labeling. That is $G$ admits $V_4$-cordial labeling.
Illustration 4.4.10 The quadrilateral snake $QS_6$ and its $V_4$-cordial labeling is shown in Figure 4.4.10.

Figure 4.4.10: $V_4$-cordial labeling of quadrilateral snake $QS_6$.

Theorem 4.4.5 The alternate quadrilateral snake $A(QS_n)$ is $V_4$-cordial for all $n$.

Proof: Let $G = A(QS_n)$ be the alternate quadrilateral snake obtained from path $v_1, v_2, ..., v_n$ by joining $v_i$ and $v_{i+1}$ (alternatively) to a new vertices $v_i'$ and $v_i''$ by an edge and then joining $v_i'$ and $v_i''$ for all $1 \leq i \leq n$.

It is noted that $|V(G)| = 2n$ and $|E(G)| = \frac{5n-2}{2}$; if $n$ is even and snake starts from $v_1$.

$|V(G)| = 2n - 2$ and $|E(G)| = \frac{5n-8}{2}$; if $n$ is even and snake starts from $v_2$.

$|V(G)| = 2n - 1$ and $|E(G)| = \frac{5n-5}{2}$; if $n$ is odd.

To define $V_4$-cordial labeling $f: V(G) \to V_4$ following cases are considered:

Case 1: If $n$ is even. Then consider the following subcases.

Subcase 1: Alternate quadrilateral snake $A(QS_n)$ where snake starts from $v_1$, then the labeling pattern is divided into following Sub-subcases.

Sub-subcase 1: $n \equiv 0, 4 (mod 8)$

$f(v_i) = 0; \quad i \equiv 0, 3 (mod 8)$;

$f(v_i) = a; \quad i \equiv 1, 4, 5, 6 (mod 8)$;

$f(v_i) = c; \quad i \equiv 2, 7 (mod 8); \quad 1 \leq i \leq n$,

$f(v_i') = 0; \quad i \equiv 2, 3 (mod 4)$;

$f(v_i') = b; \quad i \equiv 1 (mod 4)$;

$f(v_i'') = c; \quad i \equiv 0 (mod 4); \quad 1 \leq i \leq \frac{n}{2}$,

$f(v_i'') = b; \quad i \equiv 0, 2, 3 (mod 4)$;

$f(v_i'') = c; \quad i \equiv 1 (mod 4); \quad 1 \leq i \leq \frac{n}{2}$.

Sub-subcase 2: $n \equiv 2 (mod 8)$

$f(v_i) = 0; \quad i \equiv 0, 3 (mod 8)$;

$f(v_i) = a; \quad i \equiv 1, 4, 5, 6 (mod 8)$;

$f(v_i) = c; \quad i \equiv 2, 7 (mod 8); \quad 1 \leq i \leq n - 5$,

$f(v_{n-4}) = c$,

$f(v_{n-3}) = c$. 

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(sn) = 0,
\quad f(v'_{i}) = c; \quad i \equiv 0(\text{mod } 4); \quad 1 \leq i \leq \frac{n}{2} - 2,$
\quad f(v''_{2-1}) = a,
\quad f(v''_{2}) = c,
\quad f(v''_{i}) = b; \quad i \equiv 0, 2, 3(\text{mod } 4);
\quad f(v''_{i}) = c; \quad i \equiv 1(\text{mod } 4); \quad 1 \leq i \leq \frac{n}{2} - 2,$
\quad f(v''_{2-1}) = b,
\quad f(v''_{2}) = 0.

Sub-subcase 3: \( n \equiv 6(\text{mod } 8) \)
\quad f(v_{i}) = 0; \quad i \equiv 0, 3(\text{mod } 8);
\quad f(v_{i}) = a; \quad i \equiv 1, 4, 5, 6(\text{mod } 8);
\quad f(v_{i}) = c; \quad i \equiv 2, 7(\text{mod } 8); \quad 1 \leq i \leq n - 1,$
\quad f(v_{n}) = c,
\quad f(v'_{i}) = 0; \quad i \equiv 2, 3(\text{mod } 4);
\quad f(v'_{i}) = b; \quad i \equiv 1(\text{mod } 4);
\quad f(v'_{i}) = c; \quad i \equiv 0(\text{mod } 4); \quad 1 \leq i \leq \frac{n}{2},$
\quad f(v''_{i}) = b; \quad i \equiv 0, 2, 3(\text{mod } 4);
\quad f(v''_{i}) = c; \quad i \equiv 1(\text{mod } 4); \quad 1 \leq i \leq \frac{n}{2}.

Let \( n = 8a + b, a, b \in N \cup \{0\} \).

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<tr>
<th>( b )</th>
<th>Vertex conditions</th>
<th>Edge conditions</th>
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<td>0</td>
<td>( v_{f}(0) = v_{f}(a) = v_{f}(b) = v_{f}(c) )</td>
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<td>( e_{f}(0) + 1 = e_{f}(a) = e_{f}(b) = e_{f}(c) + 1 )</td>
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Table 4.4.11: \( V_{4} \)-cordial labeling of an alternate quadrilateral snake \( A(QS_{n}) \) where snake starts from \( v_{1} \) and \( n \) is even.

Subcase 2: Alternate quadrilateral snake \( A(QS_{n}) \) where snake starts from \( v_{2} \), then the labeling pattern is divided into following Sub-subcases.

Sub-subcase 1: \( n \equiv 0(\text{mod } 8) \)
\quad f(v_{1}) = c,
\quad f(v_{i}) = 0; \quad i \equiv 1, 4(\text{mod } 8);
\quad f(v_{i}) = a; \quad i \equiv 2, 5, 6, 7(\text{mod } 8);
\quad f(v_{i}) = c; \quad i \equiv 0, 3(\text{mod } 8); \quad 2 \leq i \leq n - 1,
Sub-subcase 2: \( n \equiv 2, 4, 6 \pmod{8} \)

\[
\begin{align*}
f(v_1) &= 0, \\
f(v_i) &= 0; \quad i \equiv 1, 4 \pmod{8}; \\
f(v_i) &= a; \quad i \equiv 2, 5, 6, 7 \pmod{8}; \\
f(v_i) &= c; \quad i \equiv 0 \pmod{8}; \quad 2 \leq i \leq n, \\
f(v_i) &= 0; \quad i \equiv 2, 3 \pmod{4}; \\
f(v_i) &= b; \quad i \equiv 1 \pmod{4}; \\
f(v_i) &= c; \quad i \equiv 0 \pmod{4}; \quad 1 \leq i \leq \frac{n}{2} - 1, \\
f(v_i) &= b; \quad i \equiv 0, 2, 3 \pmod{4}; \\
f(v_i) &= c; \quad i \equiv 1 \pmod{4}; \quad 1 \leq i \leq \frac{n}{2} - 1.
\end{align*}
\]

Let \( n = 8a + b, a, b \in N \cup \{0\} \).

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Table 4.4.12: \( V_4 \)-cordial labeling of an alternate quadrilateral snake \( A(QS_n) \) where snake starts from \( v_2 \) and \( n \) is even.

**Case 2:** If \( n \) is odd, then labeling pattern is divided into following subcases.

**Subcase 1:** \( n \equiv 1, 3, 5 \pmod{8} \)

\[
\begin{align*}
f(v_i) &= 0; \quad i \equiv 0, 3 \pmod{8}; \\
f(v_i) &= a; \quad i \equiv 1, 4, 5, 6 \pmod{8}; \\
f(v_i) &= c; \quad i \equiv 2, 7 \pmod{8}; \quad 1 \leq i \leq n, \\
f(v_i) &= 0; \quad i \equiv 2, 3 \pmod{4}; \\
f(v_i) &= b; \quad i \equiv 1 \pmod{4}; \\
f(v_i) &= c; \quad i \equiv 0 \pmod{4}; \quad 1 \leq i \leq \frac{n-1}{2}, \\
f(v_i) &= b; \quad i \equiv 0, 2, 3 \pmod{4}; \\
f(v_i) &= c; \quad i \equiv 1 \pmod{4}; \quad 1 \leq i \leq \frac{n-1}{2}.
\end{align*}
\]

**Subcase 2:** \( n \equiv 7 \pmod{8} \)

\[
\begin{align*}
f(v_i) &= 0; \quad i \equiv 0, 3 \pmod{8}; \\
f(v_i) &= a; \quad i \equiv 1, 4, 5, 6 \pmod{8}; \\
\end{align*}
\]
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\[ f(v_i) = c; \quad i \equiv 2, 7 (\text{mod } 8); \quad 1 \leq i \leq n - 2, \]
\[ f(v_{n-1}) = c, \]
\[ f(v_n) = 0, \]
\[ f(v'_i) = 0; \quad i \equiv 2, 3 (\text{mod } 4); \]
\[ f(v'_i) = b; \quad i \equiv 1 (\text{mod } 4); \]
\[ f(v''_i) = c; \quad i \equiv 1 (\text{mod } 4); \]
\[ 1 \leq i \leq n - 2. \]

Let \( n = 8a + b, a, b \in N \cup \{0\}. \)

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Table 4.4.13: \( V_4 \)-cordial labeling of an alternate quadrilateral snake \( A(QS_n) \) where \( n \) is odd.

The labeling pattern defined above covers all possible arrangements of vertices. In each possibility the graph under consideration satisfies the vertex conditions and edge conditions of \( V_4 \)-cordial labeling. That is \( G \) admits \( V_4 \)-cordial labeling.

**Illustration 4.4.11** The alternate quadrilateral snake \( A(QS_8) \) and its \( V_4 \)-cordial labeling is shown in Figure 4.4.11.

![Figure 4.4.11: \( V_4 \)-cordial labeling of an alternate quadrilateral snake \( A(QS_8) \) where snake starts from \( v_1 \).](image-url)
Illustration 4.4.12 The alternate quadrilateral snake $A(QS_8)$ where snake starts from $v_2$ and its $V_4$-cordial labeling is shown in Figure 4.4.12.

Figure 4.4.12: $V_4$-cordial labeling of an alternate quadrilateral snake $A(QS_8)$ where snake starts from $v_2$.

Illustration 4.4.13 The alternate quadrilateral snake $A(QS_5)$ and its $V_4$-cordial labeling is shown in Figure 4.4.13.

Figure 4.4.13: $V_4$-cordial labeling of an alternate quadrilateral snake $A(QS_5)$.

Theorem 4.4.6 The double quadrilateral snake $D(QS_n)$ is $V_4$-cordial for all $n$.

Proof: Let $G = D(QS_n)$ be the Double quadrilateral snake obtained from path $u_1, u_2, ..., u_n$ by joining $u_i$ and $u_{i+1}$ to a four new vertices $v_i, w_i$ and $v'_i, w'_i$ by the edges $u_iv_i$, $u_{i+1}w_i$, $v_iw_i$, $u_iv'_i$, $u_{i+1}w'_i$ and $v'_iw'_i$ for all $1 \leq i \leq n$. It is noted that $|V(G)| = 5n - 4$ and $|E(G)| = 7(n - 1)$.

Define $V_4$-cordial labeling $f : V(G) \rightarrow V_4$ as follows:

$$f(u_i) = 0; \quad i \equiv 3 (\text{mod } 4);$$
$$f(u_i) = a; \quad i \equiv 1 (\text{mod } 4);$$
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\( f(u_i) = c; \quad i \equiv 0, 2(\text{mod } 4); \quad 1 \leq i \leq n, \)
\( f(v_i) = b; \quad i \equiv 0, 1, 2, 3(\text{mod } 4); \quad 1 \leq i \leq n - 1, \)

\( f(w_i) = 0; \quad i \equiv 2(\text{mod } 4); \)
\( f(w_i) = a; \quad i \equiv 0(\text{mod } 4); \)
\( f(w_i) = b; \quad i \equiv 3(\text{mod } 4); \)
\( f(w_i) = c; \quad i \equiv 1(\text{mod } 4); \quad 1 \leq i \leq n - 1, \)

\( f(v_i') = 0; \quad i \equiv 3(\text{mod } 4); \)
\( f(v_i') = a; \quad i \equiv 1, 2(\text{mod } 4); \)
\( f(v_i') = c; \quad i \equiv 0(\text{mod } 4); \quad 1 \leq i \leq n - 1, \)

\( f(w_i') = 0; \quad i \equiv 0, 1(\text{mod } 4); \)
\( f(w_i') = a; \quad i \equiv 3(\text{mod } 4); \)
\( f(w_i') = c; \quad i \equiv 2(\text{mod } 4); \quad 1 \leq i \leq n - 1. \)

Let \( n = 4a + b, \quad a, b \in N \cup \{0\}. \)

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Table 4.4.14: \( V_4 \)-cordial labeling of double quadrilateral snake \( D(QS_n) \).

The labeling pattern defined above covers all possible arrangements of vertices. In each possibility the graph under consideration satisfies the vertex conditions and edge conditions of \( V_4 \)-cordial labeling. That is \( G \) admits \( V_4 \)-cordial labeling.
Illustration 4.4.14 The double quadrilateral snake $D(QS_6)$ and its $V_4$-cordial labeling is shown in Figure 4.4.14.

![Double Quadrilateral Snake Diagram](image)

Figure 4.4.14: $V_4$-cordial labeling of double quadrilateral snake $D(QS_6)$.

Theorem 4.4.7 The alternate double quadrilateral snake $AD(QS_n)$ is $V_4$-cordial for all $n$.

Proof: Let $G = AD(QS_n)$ be the alternate double quadrilateral snake obtained from path $u_1, u_2, ..., u_n$ by joining $u_i$ and $u_{i+1}$ (alternatively) to a four new vertices $v_i, w_i, v'_i, w'_i$ by the edges $u_iv_i, u_{i+1}w_i, v_iw_i, u_iv'_i, u_{i+1}w'_i, v'_iw'_i$ for all $1 \leq i \leq n$.

It is noted that $|V(G)| = 3n$ and $|E(G)| = 4n - 1$; if $n$ is even and snake starts from $u_1$.

$|V(G)| = 3n - 4$ and $|E(G)| = 4n - 7$; if $n$ is even and snake starts from $u_2$.

$|V(G)| = 3n - 2$ and $|E(G)| = 4n - 4$; if $n$ is odd.

To define $V_4$-cordial labeling $f : V(G) \to V_4$ following cases are considered:

Case 1: If $n$ is even. Then consider the following two subcases.

Subcase 1: Alternate double quadrilateral snake $AD(QS_n)$ where snake starts from $u_1$, then the labeling pattern is define as follows:

- $f(u_i) = 0; \quad i \equiv 0(\text{mod } 4)$
- $f(u_i) = a; \quad i \equiv 1(\text{mod } 4)$
- $f(u_i) = c; \quad i \equiv 2, 3(\text{mod } 4); \quad 1 \leq i \leq n$.
- $f(v_i) = b; \quad i \equiv 0, 1, 2, 3(\text{mod } 4); \quad 1 \leq i \leq n - 2$.
- $f(v'_i) = a; \quad i \equiv 0, 1, 2, 3(\text{mod } 4); \quad 1 \leq i \leq n - 2$.
- $f(w_i) = 0; \quad i \equiv 0, 2(\text{mod } 4)$.
- $f(w_i) = c; \quad i \equiv 1, 3(\text{mod } 4); \quad 1 \leq i \leq n - 2$.
- $f(w'_i) = 0; \quad i \equiv 1, 3(\text{mod } 4)$.
- $f(w'_i) = b; \quad i \equiv 0, 2(\text{mod } 4); \quad 1 \leq i \leq n - 2$. 

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Let \( n = 4a + b, a, b \in N \cup \{0\} \).

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Table 4.4.15: \( V_4 \)-cordial labeling of an alternate double quadrilateral snake \( AD(QS_n) \) where snake starts from \( u_1 \) and \( n \) is even.

**Subcase 2:** Alternate double quadrilateral snake \( AD(QS_n) \) where snake starts from \( u_2 \), then the labeling pattern is divided into following Sub-subcases.

**Sub-subcase 1:** \( n \equiv 0(\text{mod } 4) \)

\[
\begin{align*}
&f(u_1) = c, \\
&f(u_i) = 0; \quad i \equiv 1(\text{mod } 4); \\
&f(u_i) = a; \quad i \equiv 2(\text{mod } 4); \\
&f(u_i) = c; \quad i \equiv 0, 3(\text{mod } 4); \quad 2 \leq i \leq n - 2, \\
&f(u_{n-1}) = 0, \\
&f(u_n) = b, \\
&f(v_i) = b; \quad i \equiv 0, 1, 2, 3(\text{mod } 4); \quad 1 \leq i \leq \frac{n}{2} - 1, \\
&f(v'_i) = a; \quad i \equiv 0, 1, 2, 3(\text{mod } 4); \quad 1 \leq i \leq \frac{n}{2} - 1, \\
&f(w_i) = 0; \quad i \equiv 0, 2(\text{mod } 4); \\
&f(w'_i) = c; \quad i \equiv 1, 3(\text{mod } 4); \quad 1 \leq i \leq \frac{n}{2} - 1, \\
&f(w'_i) = 0; \quad i \equiv 1, 3(\text{mod } 4); \\
&f(w'_i) = b; \quad i \equiv 0, 2(\text{mod } 4); \quad 1 \leq i \leq \frac{n}{2} - 1.
\end{align*}
\]

**Sub-subcase 2:** \( n \equiv 2(\text{mod } 4) \)

\[
\begin{align*}
&f(u_1) = 0, \\
&f(u_i) = 0; \quad i \equiv 1(\text{mod } 4); \\
&f(u_i) = a; \quad i \equiv 2(\text{mod } 4); \\
&f(u_i) = c; \quad i \equiv 0, 3(\text{mod } 4); \quad 2 \leq i \leq n, \\
&f(v_i) = b; \quad i \equiv 0, 1, 2, 3(\text{mod } 4); \quad 1 \leq i \leq \frac{n}{2} - 1, \\
&f(v'_i) = a; \quad i \equiv 0, 1, 2, 3(\text{mod } 4); \quad 1 \leq i \leq \frac{n}{2} - 1, \\
&f(w_i) = 0; \quad i \equiv 0, 2(\text{mod } 4); \\
&f(w'_i) = c; \quad i \equiv 1, 3(\text{mod } 4); \quad 1 \leq i \leq \frac{n}{2} - 1, \\
&f(w'_i) = 0; \quad i \equiv 1, 3(\text{mod } 4); \\
&f(w'_i) = b; \quad i \equiv 0, 2(\text{mod } 4); \quad 1 \leq i \leq \frac{n}{2} - 1.
\end{align*}
\]

Let \( n = 4a + b, a, b \in N \cup \{0\} \).
Case 2: If \( n \) is odd, then labeling pattern is divided into following two subcases.

**Subcase 1:** \( n \equiv 1, 5 \) (mod 8)
\[
f(u_i) = 0; \quad i \equiv 0 \) (mod 4); 
f(u_i) = a; \quad i \equiv 1 \) (mod 4); 
f(u_i) = c; \quad i \equiv 2, 3 \) (mod 4); \quad 1 \leq i \leq n,
\]
\[
f(v_i) = b; \quad i \equiv 0, 1, 2, 3 \) (mod 4); \quad 1 \leq i \leq \frac{n-1}{2},
\]
\[
f(v'_i) = a; \quad i \equiv 0, 1, 2, 3 \) (mod 4); \quad 1 \leq i \leq \frac{n-1}{2},
\]
\[
f(w_i) = 0; \quad i \equiv 0, 2 \) (mod 4); 
f(w_i) = c; \quad i \equiv 1, 3 \) (mod 4); \quad 1 \leq i \leq \frac{n-1}{2},
\]
\[
f(w'_i) = 0; \quad i \equiv 1, 3 \) (mod 4); 
f(w'_i) = b; \quad i \equiv 0, 2 \) (mod 4); \quad 1 \leq i \leq \frac{n-1}{2}.
\]

**Subcase 2:** \( n \equiv 3, 7 \) (mod 8)
\[
f(u_i) = 0; \quad i \equiv 0 \) (mod 4); 
f(u_i) = a; \quad i \equiv 1 \) (mod 4); 
f(u_i) = c; \quad i \equiv 2, 3 \) (mod 4); \quad 1 \leq i \leq n - 1,
\]
\[
f(u_n) = a,
\]
\[
f(v_i) = b; \quad i \equiv 0, 1, 2, 3 \) (mod 4); \quad 1 \leq i \leq \frac{n-1}{2},
\]
\[
f(v'_i) = a; \quad i \equiv 0, 1, 2, 3 \) (mod 4); \quad 1 \leq i \leq \frac{n-1}{2} - 1,
\]
\[
f(v'_{n-1}) = 0,
\]
\[
f(w_i) = 0; \quad i \equiv 0, 2 \) (mod 4); 
f(w_i) = c; \quad i \equiv 1, 3 \) (mod 4); \quad 1 \leq i \leq \frac{n-1}{2},
\]
\[
f(w'_i) = 0; \quad i \equiv 1, 3 \) (mod 4); 
f(w'_i) = b; \quad i \equiv 0, 2 \) (mod 4); \quad 1 \leq i \leq \frac{n-1}{2}.
\]

Let \( n = 8a + b, a, b \in N \cup \{0\} \).

<table>
<thead>
<tr>
<th>b</th>
<th>Vertex conditions</th>
<th>Edge conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>1, 5</td>
<td>( v_f(0) + 1 = v_f(a) = v_f(b) + 1 = v_f(c) + 1 )</td>
<td>( e_f(0) + 1 = e_f(a) = e_f(b) + 1 = e_f(c) + 1 )</td>
</tr>
<tr>
<td>3, 7</td>
<td>( v_f(0) = v_f(a) = v_f(b) + 1 = v_f(c) )</td>
<td>( e_f(0) = e_f(a) = e_f(b) = e_f(c) )</td>
</tr>
</tbody>
</table>

Table 4.4.17: \( V_t \)-cordial labeling of an alternate double quadrilateral snake \( AD(QS_n) \) where \( n \) is odd.
The labeling pattern defined above covers all possible arrangements of vertices. In each possibility the graph under consideration satisfies the vertex conditions and edge conditions of $V_4$-cordial labeling. That is $G$ admits $V_4$-cordial labeling.

**Illustration 4.4.15** The alternate double quadrilateral snake $AD(QS_8)$ where snake starts from $u_1$ and its $V_4$-cordial labeling is shown in Figure 4.4.15.

![Figure 4.4.15: $V_4$-cordial labeling of an alternate double quadrilateral snake $AD(QS_8)$ where snake starts from $u_1$.](image)

**Illustration 4.4.16** The alternate double quadrilateral snake $AD(QS_8)$ where snake starts from $u_2$ and its $V_4$-cordial labeling is shown in Figure 4.4.16.

![Figure 4.4.16: $V_4$-cordial labeling of an alternate double quadrilateral snake $AD(QS_8)$ where snake starts from $u_2$.](image)
**Illustratoin 4.4.17** The alternate double quadrilateral snake $AD(QS_5)$ and its $V_4$-cordial labeling is shown in Figure 4.4.17.

![Diagram of the alternate double quadrilateral snake](image)

Figure 4.4.17: $V_4$-cordial labeling of an alternate double quadrilateral snake $AD(QS_5)$.

**Theorem 4.4.8** The graph $kC_4$-snake is $V_4$-cordial for all $k$.

**Proof:** Let $G = kC_4$-snake be the cyclic graph. Let $u_1, u_2, \ldots, u_{k+1}, w_{11}, w_{12}, w_{21}, w_{22}, \ldots, w_{k1}, w_{k2}$ are the vertices of $kC_4$-snake, such that $w_{ij}$ are put between $u_i$ and $u_{i+1}$, $i = 1, 2, 3, \ldots, k$, $j = 1, 2$, $w_{ij}$ is above $w_{ij+1}$ where $j = 1, 2$. It is noted that $|V(G)| = 3k + 1$ and $|E(G)| = 4k$.

Define $V_4$-cordial labeling $f : V(G) \rightarrow V_4$ as follows:

- $f(u_1) = a$,
- $f(u_2) = c$,
- $f(u_3) = 0$,
- $f(u_i) = b$; \hspace{1em} $i \equiv 0(mod\ 4)$,
- $f(u_i) = c$; \hspace{1em} $i \equiv 1(mod\ 4)$,
- $f(u_i) = 0$; \hspace{1em} $i \equiv 2(mod\ 4)$,
- $f(u_i) = a$; \hspace{1em} $i \equiv 3(mod\ 4)$, \hspace{1em} $4 \leq i \leq k + 1$,

- $f(w_{11}) = a$,
- $f(w_{21}) = c$,
- $f(w_{31}) = 0$,
- $f(w_{11}) = b$; \hspace{1em} $i \equiv 0(mod\ 4)$,
- $f(w_{11}) = c$; \hspace{1em} $i \equiv 1(mod\ 4)$,
- $f(w_{11}) = 0$; \hspace{1em} $i \equiv 2(mod\ 4)$,
- $f(w_{11}) = a$; \hspace{1em} $i \equiv 3(mod\ 4)$, \hspace{1em} $4 \leq i \leq k$, 

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\[ f(w_{12}) = b, \]
\[ f(w_{22}) = b, \]
\[ f(w_{32}) = a, \]
\[ f(w_i) = a; \quad i \equiv 0 (\text{mod} \ 4), \]
\[ f(w_i) = a; \quad i \equiv 1 (\text{mod} \ 4), \]
\[ f(w_i) = b; \quad i \equiv 2 (\text{mod} \ 4), \]
\[ f(w_i) = c; \quad i \equiv 3 (\text{mod} \ 4), \quad 4 \leq i \leq k. \]

Let \( k = 4a + b, a, b \in N \cup \{0\}. \)

<table>
<thead>
<tr>
<th>b</th>
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<th>Edge conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( v_f(0) + 1 = v_f(a) = v_f(b) = v_f(c) + 1 )</td>
<td>( e_f(0) = e_f(a) = e_f(b) = e_f(c) )</td>
</tr>
<tr>
<td>1</td>
<td>( v_f(0) + 1 = v_f(a) + 1 = v_f(b) = v_f(c) + 1 )</td>
<td>( e_f(0) = e_f(a) = e_f(b) = e_f(c) )</td>
</tr>
<tr>
<td>2</td>
<td>( v_f(0) = v_f(a) = v_f(b) = v_f(c) )</td>
<td>( e_f(0) = e_f(a) = e_f(b) = e_f(c) )</td>
</tr>
<tr>
<td>3</td>
<td>( v_f(0) = v_f(a) = v_f(b) = v_f(c) + 1 )</td>
<td>( e_f(0) = e_f(a) = e_f(b) = e_f(c) )</td>
</tr>
</tbody>
</table>

Table 4.4.18: \( V_4 \)-cordial labeling of \( kC_4 \)-snake.

The labeling pattern defined above covers all possible arrangements of vertices. In each possibility the graph under consideration satisfies the vertex conditions and edge conditions of \( V_4 \)-cordial labeling. That is \( G \) admits \( V_4 \)-cordial labeling.

**Illustration 4.4.18** The \( 7C_4 \)-snake and its \( V_4 \)-cordial labeling is shown in Figure 4.4.18.

![Figure 4.4.18: \( V_4 \)-cordial labeling of \( 7C_4 \)-snake.](image-url)

Figure 4.4.18: \( V_4 \)-cordial labeling of \( 7C_4 \)-snake.