Chapter 1

Introduction

1.1 About Graph Theory

- Graph theory is an important area of mathematics which is used in structural models of science and technology. It is widely used in biology for statistical data analysis, in electrical engineering for communication network and coding theory, in computer science for algorithms and computation, in operations research for scheduling.

- The graph coloring which is one of the areas of graph theory has theoretical results and practical applications. Colouring theory started with the problem of colouring the countries on a map in such a way that no two countries that have a common border receive the same colour.

- Domination is an extremely wide field of research in graph theory. The work on domination in graph theory started in 1960. Vizing’s conjecture is the biggest open problem in the domination of graphs. It was introduced by Ore[29] and Berge[8],[9]. Domination in graphs started with the problem of placing a minimum number of queens on a $n \times n$ chess board so as to cover or to dominate every square.

- The present work provides a completely new direction by showing the group theory can be used in graph theory. The group theory is used to construct the labelled graphs.
1.2 Basic Terminology and Preliminaries

Definition 1.2.1 A graph $G = (V(G), E(G))$ consists of two sets, $V(G) = \{v_1, v_2, \ldots\}$ called vertex set of $G$ and $E(G) = \{e_1, e_2, \ldots\}$ called edge set of $G$. Denote vertex set of $G$ as $V$ and edge set of $G$ as $E$. Elements of $V(G)$ and $E(G)$ are called vertices and edges respectively.

Definition 1.2.2 The number of vertices in a given graph is called order of the graph. The order of a graph $G$ is denoted by $p$ or $|V(G)|$.

Definition 1.2.3 The number of edges in a given graph is called size of the graph. The size of a graph $G$ is denoted by $q$ or $|E(G)|$.

Definition 1.2.4 An edge of a graph that joins a vertex to itself is called a loop. A loop is an edge $e = v_iv_i$.

Definition 1.2.5 If two vertices of a graph are joined by more than one edge then these edges are called multiple edges or parallel edges.

Definition 1.2.6 A graph which has neither loops nor parallel edges is called a simple graph.

Definition 1.2.7 If two vertices of a graph are joined by an edge then these vertices are called adjacent vertices.

Definition 1.2.8 Two vertices of a graph which are adjacent are said to be neighbours. The set of all neighbours of a vertex $v$ of $G$ is called the neighbourhood set of $v$. It is denoted by $N(v)$ or $N[v]$ and they are respectively known as open and closed neighbourhood set.

$$N(v) = \{u \in V(G)/u \text{ adjacent to } v \text{ and } u \neq v\}.$$  
$$N[v] = N(v) \cup \{v\}.$$  

Definition 1.2.9 If two or more edges of a graph have a common vertex then these edges are called incident edges.

Definition 1.2.10 Degree of a vertex $v$ of any graph $G$ is defined as the number of edges incident on $v$, counting twice the number of loops. It is denoted by $deg(v)$ or $d(v)$.

Definition 1.2.11 A walk is defined as a finite alternating sequence of vertices and edges of the form $v_i e_j v_{i+1} e_{j+1} \ldots e_k v_m$ which begins and ends with vertices such that each edge in the sequence is incident on the vertex preceding and succeeding it in the sequence. A walk from $v_0$ to $v_n$ is denoted as $v_0 - v_n$ walk. A walk $v_0 - v_0$ is called a closed walk.

Definition 1.2.12 A walk is called a trail if no edge is repeated.
Definition 1.2.13 A walk in which no vertex is repeated is called a path. A path with \(n\) vertices is denoted as \(P_n\). A path from \(v_0\) to \(v_n\) is denoted as \(v_0 - v_n\) path.

Definition 1.2.14 A closed path is called a cycle. A cycle with \(n\) vertices is denoted as \(C_n\).

Definition 1.2.15 A graph \(G = (V(G), E(G))\) is said to be connected if there is a path between every pair of vertices of \(G\). A graph which is not connected is called a disconnected graph.

Definition 1.2.16 A closed trail which covers all the edges of given graph is called an Eulerian trail. A graph which has an Eulerian trail is called an Eulerian graph.

Definition 1.2.17 A graph which does not contain any cycle is known as acyclic graph.

Definition 1.2.18 A connected acyclic graph is called a tree.

Definition 1.2.19 Let \(G\) and \(H\) be two graphs. Then \(H\) is said to be a subgraph of \(G\) if \(V(H) \subseteq V(G)\) and \(E(H) \subseteq E(G)\). Here \(G\) is called super graph of \(H\).

Definition 1.2.20 A simple, connected graph is said to be complete if every pair of vertices of \(G = (V(G), E(G))\) is connected by an edge. A complete graph on \(n\) vertices is denoted by \(K_n\).

Definition 1.2.21 A graph \(G = (V(G), E(G))\) is said to be bipartite if the vertex set can be partitioned into two disjoint subsets \(V_1\) and \(V_2\) such that for every edge \(e_i = v_i v_j \in E(G), v_i \in V_1\) and \(v_j \in V_2\).

Definition 1.2.22 A complete bipartite graph is a simple bipartite graph such that two vertices are adjacent if and only if they are in different partite sets. If partite sets \(V_1\) and \(V_2\) are having \(m\) and \(n\) vertices respectively then the related complete bipartite graph is denoted by \(K_{m,n}\) and \(V_1\) is called \(m\)-vertices part and \(V_2\) is called \(n\)-vertices part of \(K_{m,n}\).

Definition 1.2.23 A complete bipartite graph \(K_{1,n}\) is known as star graph.

Definition 1.2.24 A graph \(G = (V(G), E(G))\) is called \(n\)-partite graph if the vertex set \(V\) can be partitioned into \(n\) nonempty sets \(V_1, V_2, \ldots, V_n\) such that every edge of \(G\) joins the vertices from different subsets. It is often called a multipartite graph.

Definition 1.2.25 Let \(G\) and \(H\) be two graphs such that \(V(G) \cap V(H) = \phi\). Then join of \(G\) and \(H\) is denoted by \(G + H\). It is the graph with \(V(G + H) = V(G) \cup V(H)\), \(E(G + H) = E(G) \cup E(H) \cup J\), where \(J = \{uv/u \in V(G), v \in V(H)\}\).

Definition 1.2.26 The wheel graph \(W_n\) is join of the graphs \(C_n\) and \(K_1\). i.e. \(W_n = C_n + K_1\). Here vertices corresponding to \(C_n\) are called rim vertices and \(C_n\) is called rim of \(W_n\) while the vertex corresponds to \(K_1\) is called apex vertex.
Definition 1.2.27 A fan $f_{n-1}$ is the graph obtained by taking $n - 1$ concurrent chords in a cycle $C_n$. The vertex at which all the chords are concurrent is called the apex. That is, $f_{n-1} = P_{n-1} + K_1$.

Definition 1.2.28 The double fan $DF_n$ consists of two fan graph that have a common path.

Definition 1.2.29 The one-point union of $n$ copies of cycle $C_3$ is known as friendship graph. It is denoted by $F_n$. That is $F_n = C_3^{(n)}$.

Definition 1.2.30 A helm $H_{n}$, $n \geq 3$ is the graph obtained from the wheel $W_n$ by adding a pendant edge at each vertex on the wheel’s rim.

Definition 1.2.31 The flower graph $Fl_n$ is the graph obtained from a helm $H_n$ by joining each pendant vertex to the apex of the helm.

Definition 1.2.32 The gear graph $G_n$ is obtained by subdividing each rim edge of the wheel $W_n$.

Definition 1.2.33 The double wheel graph $DW_n$ of size $n$ can be composed of $2C_n + K_1$ it consists of two cycles of size $n$ where vertices of two cycles are all connected to a central vertex.

Definition 1.2.34 The triangular book with $n$-pages is defined as $n$ copies of cycle $C_3$ sharing a common edge. The common edge is called the spine or base of the book. This graph is denoted by $B(3, n)$ or $B_n$. In other words it is the complete tripartite graph $K_{1,1,n}$.

Definition 1.2.35 The triangular book with book mark is a triangular book $B(3, n)$ with a pendant edge attached at any one of the end vertices of the spine. This graph is denoted by $TB_n(u,v)(v,w)$.

Definition 1.2.36 The triangular book with base path $P_m$ is a graph with the vertex set $V(G) = \{u_i, v_j \mid 1 \leq i \leq m, 1 \leq j \leq n\}$ and the edge set $E(G) = \{u_iu_{(i+1)}, u_iv_j, u_mv_j : 1 \leq i \leq (m - 1), 1 \leq j \leq n\}$. This graph is denoted by $B(3, n)(P_m)$. It is also denoted by $P_m(+)K_n$.

Definition 1.2.37 The jewel graph $J_n$ is a graph with vertex set $V(J_n) = \{u, x, v, y, v_i : 1 \leq i \leq n\}$ and the edge set $E(J_n) = \{ux, vx, uy, vy, xy, uv_i, v_iw_i : 1 \leq i \leq n\}$.

Definition 1.2.38 A triangular snake $TS_n$ is the graph obtained from a path $v_1, v_2, \ldots v_n$ by joining $v_i$ and $v_{i+1}$ to a new vertex $w_i$ for $i = 1, 2, \ldots, n - 1$.

Definition 1.2.39 An alternate triangular snake $A(TS_n)$ is obtained from the path $P_n$ with vertices $v_1, v_2, \ldots v_n$ by joining $v_i$ and $v_{i+1}$ alternatively to a new vertex $v'_i$ for all $1 \leq i \leq n$. In other words every alternate edge of path $P_n$ is replaced by a cycle $C_3$. 
Definition 1.2.40 The double triangular snake $D(TS_n)$ consists of two triangular snakes that have a common path.

Definition 1.2.41 The alternate double triangular snake $AD(TS_n)$ consists of two alternate triangular snakes that have a common path.

Definition 1.2.42 The triple triangular snake $T(TS_n)$ consists of three triangular snakes that have a common path.

Definition 1.2.43 The alternate triple triangular snake $AT(TS_n)$ consists of three alternate triangular snakes that have a common path.

Definition 1.2.44 The quadrilateral snake $QS_n$ is obtained from a path $v_1, v_2, v_3, \ldots, v_n$ by joining $v_i, v_{i+1}$ to new vertices $v'_i, v''_i$ respectively and then joining $v'_i$ and $v''_i$ by an edge for all $1 \leq i \leq n$. That is every edge of path $P_n$ is replaced by a cycle $C_4$.

Definition 1.2.45 An alternate quadrilateral snake $A(QS_n)$ is obtained from a path $v_1, v_2, v_3, \ldots, v_n$ by joining $v_i, v_{i+1}$ alternatively to new vertices $v'_i, v''_i$ respectively and then joining $v'_i$ and $v''_i$ by an edge for all $1 \leq i \leq n$. That is every alternate edge of path $P_n$ is replaced by a cycle $C_4$.

Definition 1.2.46 The double quadrilateral snake $D(QS_n)$ consists of two quadrilateral snakes that have a common path.

Definition 1.2.47 The double alternate quadrilateral snake $DA(QS_n)$ consists of two alternate quadrilateral snakes that have a common path.

Definition 1.2.48 The irregular quadrilateral snake $I(QS_n)$ is obtained from the path $P_n$. Let $u_1, u_2, \ldots, u_n$ be the vertices of path $P_n$ and $v_1, v_2, \ldots, v_{n-2}$ & $w_1, w_2, \ldots, w_{n-2}$ are the newly added vertices. To obtain irregular quadrilateral snake join the vertices $u_i w_i, w_i u_{i+2}$ and $v_i w_i$ for all $1 \leq i \leq n - 2$.

Definition 1.2.49 The cartesian product of two paths is known as grid graph which is denoted by $P_m \times P_n$. In particular the graph $L_n = P_n \times P_2$ is known as ladder graph.

Definition 1.2.50 The triangular ladder $TL_n$ is obtained from the ladder $L_n = P_n \times P_2$ ($n \geq 2$) by adding the edges $u_i v_{i+1}$ for all $1 \leq i \leq n - 1$, where the consecutive vertices of two copies of paths are $v_1, v_2, \ldots, v_n$ and $u_1, u_2, \ldots, u_n$ and the edges are $u_iv_i$. It is also known as triangular belt and denoted by $TB_n(\downarrow \uparrow \uparrow \ldots)$.

Definition 1.2.51 Let $L_n = P_n \times P_2$ ($n \geq 2$) be the ladder graph with vertex set $u_i$ and $v_i$, $i = 1, 2, \ldots, n$. The alternate triangular belt is obtained from the ladder by adding the edges $u_{2i+1}v_{2i+2}$ for all $i = 0, 1, 2, \ldots, n - 1$ and $v_{2i}u_{2i+1}$ for all $i = 1, 2, \ldots, n - 1$. This graph is denoted by $ATB(n)(\downarrow \uparrow \downarrow \uparrow \ldots)$. 

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Definition 1.2.52 The graph $Z-P_n$ is obtained from the pair of paths $P_n'$ and $P_n''$. Let $v_1, v_2, \ldots, v_n$ be the vertices of path $P_n'$ and $u_1, u_2, \ldots, u_n$ are the vertices of path $P_n''$. To find $Z-P_n$ join $i^{th}$ vertex of path $P_n'$ with $(i+1)^{th}$ vertex of path $P_n''$ for all $1 \leq i \leq n-1$.

Definition 1.2.53 The braid graph $B(n)$ is obtained from the pair of paths $P_n'$ and $P_n''$. Let $v_1, v_2, \ldots, v_n$ be the vertices of path $P_n'$ and $u_1, u_2, \ldots, u_n$ are the vertices of path $P_n''$. To find braid graph join $i^{th}$ vertex of path $P_n'$ with $(i+1)^{th}$ vertex of path $P_n''$ and $i^{th}$ vertex of path $P_n''$ with $(i+2)^{th}$ vertex of path $P_n'$ with the new edges for all $1 \leq i \leq n-2$.

Definition 1.2.54 The middle graph, $M(G)$ of a graph $G$ is the graph whose vertex set is $V(G) \cup E(G)$ in which two vertices are adjacent if and only if either they are adjacent edges of $G$ or one is a vertex of $G$ and the other is an edge incident with it.

Definition 1.2.55 The total graph, $T(G)$ of graph $G$ is the graph whose vertex set is $V(G) \cup E(G)$ in which two vertices are adjacent whenever they are either adjacent or incident in $G$.

Definition 1.2.56 The splitting graph of $G$ is obtained by adding to each vertex $v$ a new vertex $v'$ such that $v'$ is adjacent to every vertex which is adjacent to $v$ in $G$ in other words $N(v) = N(v')$. The Splitting graph is denoted by $S'(G)$.

Definition 1.2.57 Let $G$ be a simple connected graph. The square of graph $G$ denoted by $G^2$ is defined to be the graph with the same vertex set as $G$ and in which two vertices $u$ and $v$ are joined by an edge $\iff$ in $G$ we have $1 \leq d(u, v) \leq 2$.

Definition 1.2.58 A Graph $G$ in which a vertex is distinguished from other vertices is called a rooted graph and the vertex is called the root of $G$. Let $G$ be a rooted graph. The graph $G^n$ obtained by identifying the roots of $n$ copies of $G$ is called a one-point union of the $n$ copies of graph $G$.

Definition 1.2.59 The $n$-pan graph $C_n^{+1}$ is the graph obtained by joining a cycle graph $C_n$ to a complete graph $K_1$ with a bridge. In other words the pan graph is obtained by adding a pendant edge to any vertex of cycle $C_n$.

Definition 1.2.60 Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. Then cartesian product of $G_1$ and $G_2$ which is denoted by $G_1 \times G_2$, is the graph with vertex set $V = V_1 \times V_2$ consisting of vertices $u = (u_1, u_2), v = (v_1, v_2)$ such that $u$ and $v$ are adjacent in $G_1 \times G_2$ whenever $(u_1 = v_1$ and $u_2$ adjacent to $v_2)$ or $(u_1$ adjacent to $v_1$ and $u_2 = v_2)$.

Definition 1.2.61 The corona $G_1 \odot G_2$ of two graphs $G_1$ and $G_2$ is defined as a graph obtained by taking one copy of $G_1$ (which has $p_1$ vertices) and $p_1$ copies of $G_2$ and attach one copy of $G_2$ at every vertex of $G_1$.

Definition 1.2.62 The crown $C_n \odot K_1$ is obtained by joining a pendant edge to each vertex of cycle $C_n$. 
Definition 1.2.63  An armed crown is a graph in which path $P_m$ is attached at each vertex of cycle $C_n$. This graph is denoted by $C_n \odot P_m$.

Definition 1.2.64  Take $k$ paths of length $l_1, l_2, l_3, ..., l_k$, where $k \geq 3$ and $l_i = 1$ for at most one $i$. Identify their end points to form a new graph. The new graph is called a generalized theta graph and is denoted by $\theta(l_1, l_2, l_3, ..., l_k)$. In other words $\theta(l_1, l_2, l_3, ..., l_k)$ consists $k \geq 3$ pair wise internally disjoint paths of length $l_1, l_2, l_3, ..., l_k$ that share a pair of common end points $u$ and $v$. If each $l_i$ ($i=1,2,3,...,k$) is equal to $l$, then it is denoted by $\theta(l[k])$.

Definition 1.2.65  The graph $kC_n$-snake is a connected graph with $k$ blocks, each of the blocks is isomorphic to the cycle $C_n$. $kC_n$-snake is also known as a cyclic snake. The graph $kC_n$-snake was introduced by Barrientos as generalization of the concept of triangular snake introduced by Rosa.

Definition 1.2.66  The graph $kC_4$-snake be the graph with $4k$ edges. Let $u_1, u_2, ..., u_{k+1}, w_{11}, w_{12}, w_{21}, w_{22}, ..., w_{k1}, w_{k2}$ are the vertices of $kC_4$-snake, such that $w_{ij}$ are put between $u_i$ and $u_{i+1}$, $i = 1, 2, 3, ..., k$, $j = 1, 2$, $w_{ij}$ is above $w_{ij+1}$ where $j = 1, 2$. 
Chapter 1. Introduction

1.3 Review of Literature

The vertices of graph assigned values subject to certain condition then it is known as graph labeling. Any graph labeling will have the following three characteristics:

- a set of numbers from which vertex labels are assigned;
- a rule that assigns a value to each edge;
- a condition that these values must satisfy.

Graph labeling techniques trace their origin to one introduced by Rosa in 1967 or one given by Graham and Sloane in 1980. Labelled graphs give useful models for a broad range of applications like X-ray crystallography, radar, coding theory, astronomy and communication networks etc. The dynamic survey of graph labeling published by J.A.Gallian[17] can be referred for more information.

Following are some standard Graph Labeling techniques:

1. Graceful Labeling
2. Harmonious Labeling
3. Strongly Multiplicative Labeling
4. Mean Labeling

1.3.1 Graceful Labeling

A function \( f \) is called a graceful labeling of a graph \( G \) if \( f : V(G) \rightarrow \{0, 1, 2, ..., q\} \) is injective and the induced function \( f^* : E(G) \rightarrow \{1, 2, 3, ..., q\} \) defined as \( f^*(e = uv) = |f(u) - f(v)| \) is bijective. A graph \( G \) is graceful if it admits graceful labeling.

The concept of graceful labeling was introduced by Rosa[32] in 1967. Initially Rosa named graceful labeling as \( \beta \) valuation. Golomb[19] subsequently called such labeling as graceful labeling.

1.3.2 Some Existing Results of Graceful Labeling

- Sub graph of a graceful graph need not be graceful.
- Super graph of a graceful graph need not be graceful.
- All graphs with \( p \leq 5 \) vertices are graceful except \( C_5, K_5 \) and Bowtie.
- The conjecture by Ringel-Kotzing states that "All trees are graceful".
- If \( G \) is a graceful eulerian graph of size \( q \), then \( q \equiv 0(\text{mod } 4) \) or \( q \equiv 3(\text{mod } 4) \).
• Rosa[32] proved that cycle $C_n$ is graceful $\iff n \equiv 0, 3 (mod 4)$.

• Frucht[15],[16], Hoede and Kuiper[25] proved that all wheels $W_n$ are graceful graphs.

• Golomb[19] proved that the complete graph $K_n$ is graceful $\iff n \leq 4$.

• Rosa[32] and Golomb[19] proved that the complete bipartite graphs are graceful.

• Acharya and Gill[1] proved that grid graph $P_m \times P_n$ is graceful.

• Fan $f_n$ and friendship $F_n$ graphs are graceful.

1.3.3 Harmonious Labeling

A function $f$ is called a *harmonious labeling* of a graph $G$ if $f : V(G) \to \{0, 1, 2, ..., q-1\}$ is injective and the induced function $f^* : E(G) \to \{0, 1, 2, ..., q-1\}$ defined as $f^*(e = uv) = (f(u) + f(v))(mod q)$ is bijective. A graph $G$ is *harmonious graph* if it admits harmonious labeling.

1.3.4 Some Existing Results of Harmonious Labeling

The concept of harmonious labeling was introduced by Graham and Slone[20] in 1980. Graham and Slone[20] proved that

• All cycles $C_n$ are harmonious $\iff n \equiv 1, 3 (mod 4)$.

• All wheels $W_n$ are harmonious.

• Fans $f_n$ are harmonious for all $n$.

• Complete graphs $K_n$ are harmonious $\iff n \leq 4$.

• Complete bipartite graphs $K_{m,n}$ are harmonious $\iff m or n = 1$.

Gallian[18] proved that

• $C_m \times P_n$ is harmonious if $n$ is odd.

Jungreis and Reid[27] proved that

• $P_m \times P_n$ is harmonious $\iff (m, n) \neq (2, 2)$.

• In the same paper they proved that $C_m \times P_n$ is harmonious if $m = n$ and $n \geq 3$. 
1.3.5 Strongly Multiplicative Labeling

A graph $G = (V(G), E(G))$ with $p$ vertices is said to be multiplicative, if the vertices of $G$ can be labelled with distinct positive integers such that label induced on the edges by the product of labels of end vertices are all distinct.

Multiplicative labeling was introduced by Beineke and Hegde[7]. In the same paper they proved that every graph $G$ admits multiplicative labeling and defined strongly multiplicative labeling as follows.

A graph $G$ with $p$ vertices is said to be strongly multiplicative if the vertices of $G$ can be labelled with $p$ distinct integers $1, 2, 3, ..., p$ such that label induced on the edges by the product of labels of the end vertices are all distinct.

1.3.6 Some Existing Results of Strongly Multiplicative Labeling

The concept of strongly multiplicative labeling was introduced by Beineke and Hegde[7] and they proved

- Every cycle $C_n$ is strongly multiplicative.
- Every wheel $W_n$ is strongly multiplicative.
- Complete graph $K_n$ is strongly multiplicative $\iff n \leq 5$.
- Complete bipartite graph $K_{n,n}$ is strongly multiplicative $\iff n \leq 4$.

Vaidya and Kanani[41] proved that

- Every cycle with one chord is strongly multiplicative.
- Every cycle with twin chord is strongly multiplicative.
- Every cycle with triangle is strongly multiplicative.

Vaidya and Kanani[42] proved that

- Arbitrary super subdivision of any path $P_k$ is strongly multiplicative.
- Arbitrary super subdivision of star $K_{1,n}$ is strongly multiplicative.
- Arbitrary super subdivision of cycle $C_n$ is strongly multiplicative.
- Arbitrary super subdivision of tadpole $T_{n,l}$ is strongly multiplicative.

Vaidya et al.[43] proved that

- Arbitrary super subdivision of tree $T$ is strongly multiplicative.
• Arbitrary super subdivision of complete bipartite graph $K_{m,n}$ is strongly multiplicative.

• Arbitrary super subdivision of grid graph $P_m \times P_n$ is strongly multiplicative.

• Arbitrary super subdivision of armed crown $C_n \odot P_m$ is strongly multiplicative.

• Arbitrary super subdivision of $C_n^m$ is strongly multiplicative.

### 1.3.7 Mean Labeling

A function $f$ is called a *mean labeling* of graph $G$ if $f : V(G) \rightarrow \{0, 1, 2, ..., q\}$ is injective and the induced function $f^* : E(G) \rightarrow \{1, 2, 3, ..., q\}$ defined as

$$f^*(e = uv) = \begin{cases} \frac{f(u)+f(v)}{2}; & \text{if } (f(u) + f(v)) \text{ is even}, \\ \frac{f(u)+f(v)+1}{2}; & \text{if } (f(u) + f(v)) \text{ is odd}, \end{cases}$$

is bijective. The graph $G$ is *mean graph* if it admits mean labeling.

The concept of mean labeling of graph was introduced by Somasunderam and Ponraj[37].

### 1.3.8 Some Existing Results of Mean Labeling

Somasunderam and Ponraj[37] proved that

• $P_n$ is a mean graph for any $n \in N$.

• $W_n$ is not a mean graph for $n > 3$.

• $C_n$ is a mean graph for any $n \in N$.

• Complete graph $K_n$ is a mean graph $\iff n < 3$.

Vaidya and Kanani[45] proved that

• The graph obtained by path union of $k$ copies of cycle $C_n$ is a mean graph.

• The graph obtained by joining two copies of cycle $C_n$ by a path $P_k$ is a mean graph.

• The graph obtained by arbitrary super subdivision of path $P_n$ is a mean graph.

Vaidya and Vyas[50] proved that

• The shadow graph $D_2(P_n)$ of path $P_n$ is a mean graph.

• The middle graph of cycle $C_n$ is a mean graph.

• Double fan $DF_n$ is a mean graph for even $n$. 
Present research work concentrates on $A$-cordial labeling and it is divided into five chapters. The first chapter contains an introduction, basic definitions and literature review, the second chapter contains objectives, the third chapter contains materials and methods. The fourth chapter discusses $4$-cordial labeling of graphs, $k$-cordiality of graphs, $V_4$-cordial labeling of new graph families and snake related $V_4$-cordial graphs. The last chapter contains conclusion.

This research work contains following results:

- The wheel $W_n$ is 4-cordial.
- The fan $f_n$ is 4-cordial.
- The friendship graph $F_n$ is 4-cordial.
- The helm $H_n$ is 4-cordial.
- The flower graph $Fl_n$ is 4-cordial.
- The gear graph $G_n$ is 4-cordial.
- The double fan $Df_n$ is 4-cordial.
- The double wheel graph $DW_n$ is 4-cordial.
- The triangular book graph $B(3, n)$ is 4-cordial.
- The triangular snake $TS_n$ is 4-cordial.
- The irregular quadrilateral snake $IQ(S_n)$ is 4-cordial.
- The triangular belt $TB_n$ is 4-cordial.
- The graph $Z-P_n$ is 4-cordial.
- The braid graph $B(n)$ is 4-cordial.
- The middle graph $M(P_n)$ of the path $P_n$ is 4-cordial.
- The total graph $T(P_n)$ of the path $P_n$ is 4-cordial.
- The splitting graph $S'(P_n)$ of the path $P_n$ is 4-cordial.
- The square graph $P_n^2$ of the path $P_n$ is 4-cordial.
- The middle graph $M(C_n)$ of cycle $C_n$ is 4-cordial.
- The splitting graph $S'(C_n)$ of cycle $C_n$ is 4-cordial.
- The splitting graph $S'(K_{1,n})$ of star $K_{1,n}$ is 4-cordial.
- The one point union $f_3(n)$ of $n$ copies of a fan $f_3$ is 4-cordial for all $n$. 
• The pan graph $C_n^{+1}$ is $k$-cordial for all even $k$ and $n = k + j$, $0 \leq j \leq k - 1$.

• The pan graph $C_n^{+1}$ is $k$-cordial for all even $k$ and $n = 2tk + j$, where $t \in N \cup \{0\}$ and $0 \leq j \leq k - 1$.

• The pan graph $C_n^{+1}$ is $k$-cordial for all even $k$ and $n = 2tk + k + j$, where $t \in N$ and $0 \leq j \leq k - 1$.

• The triangular book $B(3, n)$ with $n$-pages is $k$-cordial.

• The triangular book with book mark $B(3, n)(u, v)(v, w)$ is $k$-cordial for all $n$.

• The graph $B(3, n)(P_{2m})$ is $k$-cordial for all $n$ and $2m = 4$ and $k$ is even.

• The graph $B(3, n)(P_{2m})$ is $k$-cordial for all $n$, $2m = lk$, where $l \in N$ and $k$ is even.

• The graph $B(3, n)(P_{2m})$ is $k$-cordial for all $n$ and $4 < 2m < lk$, where $k$ is even and $l$ is the smallest positive odd integer such that $2m < lk$.

• The graph $B(3, n)(P_{2m})$ is $k$-cordial for all $n$ and $4 < 2m < lk$, where $k$ is even and $l$ is the smallest positive even integer such that $2m < lk$.

• The graph $B(3, n)(P_{2m})$ is $k$-cordial for all $n$, $4 \leq 2m$ and odd $k$.

• The jewel graph $J_n$ is $k$-cordial for all $n$.

• The triangular belt $TB(n)(\downarrow n)$ is $k$-cordial for all $n$.

• The alternate triangular belt $ATB(n)(\downarrow \uparrow \downarrow \uparrow \ldots)$ is $k$-cordial for all $n$.

• The graph $Z-P_n$ is $k$-cordial for all odd $k$ and for all $n$.

• The braid graph $B(n)$ is $k$-cordial for all $n$.

• The square graph $P_n^2$ of path $P_n$ is $k$-cordial.

• The pan graph $C_n^{+1}$ is $V_4$-cordial for all $n$.

• The triangular book $B(3, n)$ with $n$-pages is $V_4$-cordial except $n \equiv 2(mod 4)$.

• The triangular book with book mark $B(3, n)(u, v)(v, w)$ is $V_4$-cordial except $n \equiv 1(mod 4)$.

• The crown $C_n \odot K_1$ is $V_4$-cordial for all $n$.

• The armed crown $AC_n$ is $V_4$-cordial for all $n$.

• The corona $C_n \odot mK_1$ of cycle $C_n$ and $m$ complete graph $K_1$ is $V_4$-cordial for all $n$ and $m$.

• The theta graph $\theta(8[m])$ is $V_4$-cordial for all $m$. 
• The alternate triangular snake $ATS_n$ is $V_4$-cordial for all $n$.
• The alternate double triangular snake $AD(TS_n)$ is $V_4$-cordial for all $n$.
• The alternate triple triangular snake $AT(TS_n)$ is $V_4$-cordial for all $n$.
• The quadrilateral snake $QS_n$ is $V_4$-cordial for all $n$.
• The alternate quadrilateral snake $A(QS_n)$ is $V_4$-cordial for all $n$.
• The double quadrilateral snake $D(QS_n)$ is $V_4$-cordial for all $n$.
• The alternate double quadrilateral snake $AD(QS_n)$ is $V_4$-cordial for all $n$.
• The graph $kC_4$-snake is $V_4$-cordial for all $k$. 