Chapter 5

Perturbed soliton excitations in hydrogen bonded polypeptide chain
5.1 Introduction

In the living systems many biological processes, such as muscle contraction, DNA replication, neuroelectric pulse transfer on the membranes of as well as work of calcium pump and sodium pump and so on, are associated with bioenergy transport. The bio-energies needed for biological processes in the bio-tissues are provided by the energy released during the hydrolysis of adenosine triphosphate (ATP) [268]. Thus, there are always biological processes of energy transport from the generated place to the tissues of needed energy through protein molecules. Hence, one can confirm that the bio-energy is transported along the polypeptide chains in the protein molecules. Therefore the study of the energy transfer along polypeptide chains is a very interesting subject in biology and has an important significance in life science. However, understanding the mechanism of energy transfer in living systems has been a long-standing problem that remains of great interest up to now. Although, various models [131, 198, 269-274] have been developed to explain the energy transfer in biological systems. The bioenergy can be transported through polypeptide chains as a localized wave packet or soliton which can travel over macroscopic distances along the molecular chains and is believed to be the energy carrier, retaining the wave shape, energy, momentum and other properties of the quasiparticle. Thus, the theoretical framework may enable one to gradually unveil the mysterious process of energy transport in life. Polypeptide chains of protein structure are the major workers within the cells and carrying out the functions described by the genetic codes: providing structural support, transmitting signals, catalyzing reactions, and transporting molecules. Thus the fundamental understanding on the energy transport mechanism in long chains of hydrogen bonded polypeptide groups will also have great values to protein folding, conformational changes, and misfolding disease research. Mostly all protein molecules are composed by a linear polymerization of twenty different amino acids and represent very long polypeptide chains with periodic repetition of peptide groups (PG) which contains H, N, C, O atoms. These polypeptide chains can exist different configurations, among which the α-helices are of special in-
terest. When two amino acids join together, they release one water molecule and form a peptide bond. When the polypeptide chain has been formed, it can fold into a variety of complex three-dimensional conformations [275, 276]. In \( \alpha \)-helical proteins, the polypeptide chains are coiled into long helices, stabilized by hydrogen bonds between neighboring carbon atoms within the primary polypeptide chain. In each helical protein molecule, the peptide groups are situated along three chains of hydrogen bonds equidistant from one another, and forming periodic structures. The energy of the hydrogen bond is approximately in the order of magnitude less than that of ordinary chemical bonds. Therefore, the displacements of peptide groups from their equilibrium positions along the chains of hydrogen bonds occur much more readily than their displacements along the main polypeptide chain [277].

As an alternative to electronic mechanisms, the energy released by ATP hydrolysis might stay localized and stored as vibrational energy in the C=O bonds of hydrogen bonded peptide groups [23]. In the polypeptide chain of protein molecule, a likely recipient should be the amide-I vibration, or stretch and contraction of the C=O bond of the peptide groups. The vibration energy can be transported along the polypeptide chains due to the resonant or dipole-dipole interaction between neighboring C=O bonds. However, the C=O bonds are involved in the amino acid residues or peptide groups, then their vibrations will necessarily lead to the deformation of amino acid residues, which reacts, again through phonon-exciton coupling, with the amide-I vibrational quanta and make them become a soliton. Thus a soliton is formed in this process. The latter is just a carrier of the bio-energy absorbed from ATP hydrolysis. Thus the bio-energy can transport along the protein molecules in virtue of the motion of the soliton. Even though many of the theoretical models [202-204] have been proposed to explain the energy transfer process in biological systems, the proton transfer through hydrogen bonded polypeptide chains remains questionable as yet. The purpose of present work is to explore some possible mechanisms by which protons may move successively through hydrogen bonded polypeptide chains with a clear picture of peptide molecular
structure as shown in Fig. (5.1). In this chapter, we study the energy transfer

and propagation of solitons during the migration of protons between neighboring peptide groups of hydrogen bonded polypeptide chains. These solitons can arise due to interaction of amide-I excitation with acoustic phonon and the interaction of peptide groups with its local environment chosen in the form of nonlinear interaction potential [205, 278, 279]. Thus, to describe an efficient solitonic transport of amide-I energy along the chains of hydrogen bonded peptide groups, it is essential to consider the interpeptide proton displacements with higher order molecular interactions and excitations.

In the next section, we construct a developed model Hamiltonian for dynamics of protons hydrogen bonded polypeptide chains with higher order molecular interactions and excitations and deduce the dynamical equations for the proton transfer along the chain.
A modified model for dynamics of protons in polypeptide chains

An inspection of $\alpha$-helical structure reveals three channels of hydrogen bonded peptide groups situated approximately in the longitudinal direction with sequence $\text{O}=\text{C}-\text{N-H} - - - \text{O}=\text{C}-\text{N-H} - - - \text{O}=\text{C}-\text{N-H}$ etc., where the dashed lines represent hydrogen bonds. To illustrate the present work, in comparison with previous theories [22, 271, 280], we consider our model as the highly idealized case of molecule (peptide group) per unit cell as shown in Fig. (5.2).

In order to explain the dynamics of protons, it is necessary to consider a co-operative proton migration through the hydrogen bond between each peptide group. Every proton which affects the hydrogen bond is connected with nitrogen atom by covalent bond on one hand and with an oxygen atom by a hydrogen bond on the other hand. The local contraction of the chain, formed by the nonlinear lattice distortion, may serve as a potential well for proton transfer. This vibration allows a comparatively easy compression of hydrogen bonds. In this case hydrogen bond lengths are variable, i.e., when the proton in the hydrogen bridges goes over the local potential barrier which alters the hydrogen bond length. The protons in the hydrogen bridges can move in the double well potentials created by the pair of nearest neighbor peptide molecules. The flexible proton transfer along the polypeptide chains are subjected to the step-wise mechanism. In each step only one proton jumps along the hydrogen bond which schematically indicated in Fig. (5.3). To elucidate the detailed understanding of proton dynamics in hydrogen bonded polypeptide chains, it is essential to introduce a developed model Hamiltonian which betterly explains
the propagation of proton soliton in the chain along the hydrogen bridges in
the presence of higher order molecular excitations and interactions. Thus, the
new Hamiltonian of the systems can be read as follows,

\[
H = - \sum_n B_n^\dagger (2E_0 + W)B_n + J (B_{n+1} + B_{n-1}) - \frac{\hbar \omega_1^2}{4\omega_0} (a_n^\dagger + a_n)(a_{n+1}^\dagger + a_{n+1})
\]

\[
+ \frac{\hbar}{m\omega_0^2} (a_n + a_n^\dagger)^2 + \frac{\hbar^2}{4m^2\omega_0^2} (a_n + a_n^\dagger)^4
\]

\[
+ \chi B_n^\dagger B_n (u_{n+1} + u_{n-1}) + \chi_1 B_n^\dagger B_{n+1} B_{n+1}^\dagger B_{n-1} (u_{n+1} - u_{n-1})
\]

\[
+ E_1 B_n^\dagger B_{n+1} B_{n+1}^\dagger B_{n-1} + B_n^\dagger B_{n-1} B_{n-1}^\dagger B_{n}.
\]

where, \( W = \frac{1}{2} \int_n \frac{1}{M} v_n^2(t) + k(b_n - b_{n-1})^2 \). \( W \) is the deformation energy of
the chain. Here, \( E_0 \) represents the excitation energy of each peptide groups in
the hydrogen bonding spine including the resonant interaction energy between
neighbouring peptide groups. \( J \) represents the dipole-dipole coupling coeffi-
cient between adjacent peptide group along the spine. \( \chi \) is a nonlinear cou-
ing coefficient representing the change in energy of the amide-I bond caused
by stretching of the chain between two neighbouring peptide groups. \( E_1 \)
represents the higher order excitation energy of the peptide group and \( J_1 \)
corresponds to the higher order coupling coefficients. \( \chi_1 \) is the new nonlinear

Figure 5.3: Stepwise mechanism of proton transfer in polypeptide chain.
coupling coefficient due to higher order molecular excitations. $u_n$ is the displacement operator of lattice oscillator at site $n$, $P_n$ is its conjugate momentum operator, $M$ is the mass of an peptide molecule, $k$ is the elasticity constant of polypeptide chains. $m$ is the mass of the proton. $a_n^\dagger$ and $a_n$ are the creation and annihilation operators of the proton. $B_n$ and $B_n^\dagger$ are the boson creation and annihilation operators for quanta of amide-I vibrations. For the state vector of the exciton-phonon system the wave function for the collective excitations of the molecular chain may be sought in the form [202],

$$\Psi_n(t) = \sum_n a_n(t) \exp \sigma(t) B_n^\dagger |0\rangle,$$

where $|0\rangle$ represents a vacuum state and

$$\sigma(t) = -\frac{i}{\hbar} b_n(t) p_n - v(t) u_n.$$

(5.2)

Here $b_n$ and $v_n$ correspond to the coherent states of the displacements and conjugate momenta of the unit cells in the state given by $\Psi(t)$, where $b_n(t)$ and $v_n(t)$ are given as,

$$b_n(t) = \langle \Psi_n(t) | u_n | \Psi_n(t) \rangle,$$

(5.4)

$$v_n(t) = \langle \Psi_n(t) | p_n | \Psi_n(t) \rangle.$$

(5.5)

The position and momentum operators satisfy the commutation relations,

$$[u_n, p_n] = i\hbar \sigma_n n',$$

(5.6)

and the coherent state representation of the operator $B_n$ and $B_n^\dagger$ are given by

$$a_n(t) = \langle \Psi_n(t) | B_n | \Psi_n(t) \rangle,$$

(5.7)

$$a_n^\dagger(t) = \langle \Psi_n(t) | B_n^\dagger | \Psi_n(t) \rangle.$$

(5.8)

$|a_n(t)|^2$ characterizes the probability of excitation of the $n^{th}$ peptide group and the normalization condition for the wavefunction Eq. (5.2) yields,

$$\sum_n |a_n(t)|^2 = 1.$$
In order to obtain the equations of motion, Eq. (5.2) can be used to form the expectation value of the total energy which is then used as the Hamiltonian for constructing the equation of motion for \( a_n \) and \( b_n \) which are assumed to evolve as classical variables. Now the Hamiltonian can be written as,

\[
H = \langle \psi(t)|H|\psi(t)\rangle, \tag{5.10}
\]

Using Eq. (5.1) and Eq. (5.2) in Eq. (5.10), we can again recast the Hamiltonian as,

\[
H = -\sum_{n} a_n^{+} (2E_0 + W - J) a_{n+1} - a_{n-1} + \chi a_n^{+} a_n b_{n+1} - b_{n-1} + \chi_1 a_n^{+} a_{n-1} a_n b_{n+1} - b_{n-1} + E_1 a_n^{+} a_n a_n - J a_n^{+} a_{n+1} a_n + \hbar \omega_1^2 a_n^{+} a_n + \frac{\hbar^2}{4m_0^2 \omega_0^2} a_n^{+} a_n^2 + \frac{\hbar}{4m_0^2 \omega_0^2} a_n^{+} a_n^4 + U_0 \left( 1 - \frac{\hbar^2}{m_0^2 \omega_0^2 r_0^2} a_n^{+} a_n^2 + \frac{\hbar^2}{4m_0^2 \omega_0^2 r_0^4} a_n^{+} a_n^4 \right). \tag{5.11}
\]

Having developed the Hamiltonian, we can write the equations of motion using the Heisenberg equation for the polypeptide chain,

\[
i\hbar \frac{\partial a_n}{\partial t} = [a_n, H]. \tag{5.12}
\]

By using Eq. (5.12) we can write the equations of motion after the evaluation of commutation relations,

\[
i\hbar \frac{\partial a_n}{\partial t} = (2E_0 + W) a_n^{+} b_{n+1} - b_{n-1} + 2E_1 a_n^{+} a_n^2 a_{n+1} + a_{n-1} - 2J a_n^{+} a_n^2 a_{n+1} + a_{n-1} + 2\chi_1 |a_n|^2 a_n b_{n+1} - b_{n-1} - \frac{\hbar \omega_1^2}{4m_0^2} a_n^{+} a_{n+1} + a_n^{+} a_{n-1} + a_{n+1} - 2J U_0 a_n^{+} a_n + a_{n+1} + U_0 \frac{\hbar^2}{2} a_n^{+} a_n^2 + 4a_n^{+} a_n^3 + 4a_n^3 + 12|a_n|^2 a_n + 12|a_n|^2 a_n^{+}, \tag{5.13}
\]
\[ m \frac{\partial^2 b_n}{\partial t^2} = k (b_{n+1} + b_{n-1} - 2b_n) - \chi |a_{n+1}|^2 - |a_{n-1}|^2 + \chi_1 |a_{n+1}|^4 - |a_{n-1}|^4. \tag{5.14} \]

The equations of motion describe the energy transfer during the process of ATP hydrolysis through large amide-I vibrations along the hydrogen bonding spine of polypeptide chains.

### 5.3 Dynamical equations in the continuum limit

However, the Eqs. (5.13) and (5.14) cannot be understood analytically due to its nonlinear nature. Hence, in the long wavelength and low temperature limit it is reasonable to rewrite the equations under continuum approximation this is also a valid approximation when we assume that the distance between two lattice points is very small. Hence, \( a_n(t) \) and \( b_n(t) \) are replaced by \( a(x,t) \) and \( b(x,t) \) respectively, such that \( x = n \epsilon \) in a HB chain. Thus, to derive the dynamical equations in the continuum limit, one can use the series expansion given below,

\[
a_{n+1}(t) = 1 + \epsilon \frac{\partial}{\partial x} + \frac{\epsilon^2}{2!} \frac{\partial^2}{\partial x^2} \pm \frac{\epsilon^3}{3!} \frac{\partial^3}{\partial x^3} + \frac{\epsilon^4}{4!} \frac{\partial^4}{\partial x^4} \pm \cdots a(x,t), \tag{5.15} \]

\[
b_{n+1}(t) = 1 + \epsilon \frac{\partial}{\partial x} + \frac{\epsilon^2}{2!} \frac{\partial^2}{\partial x^2} \pm \cdots b(x,t). \tag{5.16} \]

In order to see the effect of discreteness we include the higher order terms upto \( O(\epsilon^4) \). Thus, we can obtain the equations of motion for our Hamiltonian in the continuum limit as,
\[
\begin{align*}
\frac{i\hbar}{\partial t} \frac{\partial a}{\partial t} &= -2E_0 + W - 2J - 2E_1 - 4J_1 |a|^2 a - J \epsilon^2 - \frac{\partial^2 a}{\partial x^2} - \frac{\epsilon^2 - \partial^4 a}{12} \\
&- 2J_1 \epsilon^2 2 |a|^2 \frac{\partial^2 a}{\partial x^2} + 2a^* \frac{\partial a}{\partial x} + \frac{\epsilon^2}{6} |a|^2 \frac{\partial^4 a}{\partial x^4} + \frac{\epsilon^2}{2} a^* \frac{\partial^2 a}{\partial x^2} \\
&+ \frac{2}{3} \epsilon^2 a^* \frac{\partial a}{\partial x} \frac{\partial^3 a}{\partial x^3} + 2 \epsilon \chi + 2 \chi |a|^2 a - \frac{\partial b}{\partial x} + U_0 2J^* a + a^* \\
&+ (J^* a^3 + a^3 + 3 |a|^2 a + 3 |a|^2 a^*) - \hbar \omega_i \frac{\partial^2 a^*}{\partial t^2} \frac{a^*}{2} a + a^* + \epsilon^2 \frac{\partial^2 a^*}{\partial x^2} \\
&+ \frac{\partial^2 a}{\partial x^2} + \frac{\epsilon^4 - \partial^4 a}{12} \frac{\partial^4 a}{\partial x^4} + \frac{\partial^4 a}{\partial x^4}, \tag{5.17}
\end{align*}
\]

\[
\begin{align*}
m \frac{\partial^2 b}{\partial t^2} &= k \epsilon^2 \frac{\partial^2 b}{\partial x^2} + 2 \chi \epsilon \frac{\partial}{\partial x} |a|^2 + \frac{\epsilon^2}{6} a \frac{\partial^3 a^*}{\partial x^3} + a^* \frac{\partial^3 a^*}{\partial x^3} + \frac{\epsilon^2}{2} \frac{\partial}{\partial x} \frac{\partial a}{\partial x} \frac{\partial a^*}{\partial x} \\
&+ 2 \chi_1 \epsilon \frac{\partial}{\partial x} |a|^4 + \frac{\epsilon^2}{3} |a|^2 a \frac{\partial^3 a^*}{\partial x^3} + a^* \frac{\partial^3 a^*}{\partial x^3} + \epsilon^2 a^* \frac{\partial a}{\partial x} \frac{\partial a^*}{\partial x} + \epsilon^2 a^{12} \\
&- \frac{\partial a}{\partial x} \frac{\partial^2 a}{\partial x^2} + 2 \epsilon \frac{\partial x}{\partial x} |a| \frac{\partial}{\partial x} \frac{\partial}{\partial x} \frac{\partial}{\partial x}, \tag{5.18}
\end{align*}
\]

Now redefining the terms,

\[
\begin{align*}
\lambda &= 2E_0 + W - 2J, \\
k &= \frac{\hbar}{m}, \\
C_0 &= \frac{\partial b(x, t)}{\partial x}, \\
\rho(x, t) &= -\epsilon \frac{\partial b(x, t)}{\partial x},
\end{align*}
\]

and after differentiating Eq. (5.18) with respect to x once and introducing the wave variable \( \xi = x - ct \) upon integrating the Eq. (5.18) twice, we obtain

\[
\begin{align*}
\rho(\xi) &= \frac{1}{k(1 - s^2)} 2 \chi |a|^2 + \frac{\epsilon^2}{6} a \frac{\partial^2 a^*}{\partial x^2} + a^* \frac{\partial^2 a}{\partial x^2} + \frac{\epsilon^2}{3} \frac{\partial a}{\partial x} \frac{\partial a^*}{\partial x} \frac{\partial a^*}{\partial x}
\end{align*}
\]
\[ + \frac{2}{3} \chi_{1} e^{2} |a|^2 \frac{\partial^2 a}{\partial x^2} + a^* \frac{\partial^2 a}{\partial x^2} - 2 \frac{\partial a}{\partial x} \frac{\partial a^*}{\partial x} \]
\[ + a^2 \frac{\partial a^*}{\partial x} - 2 a \frac{\partial a^*}{\partial x} + 6|a|^2 \frac{\partial a}{\partial x} \frac{\partial a^*}{\partial x} + 3|a|^4. \]

(5.19)

Choosing \( \lambda = 0, J = \hbar = 1, \chi_{1} = \frac{\chi}{4}, J_{1} = \frac{2k}{2k(1-s^2)}, E_{1} = \frac{\hbar q}{k(1-s^2)} \) and substituting Eq. (5.19) in Eq. (5.17), after making the transformation, \( a = -\frac{\hbar q}{4x^2} \), \( t = e^\frac{1}{2} \) we obtain,

\[ i \frac{q_t}{q} + 2|q|^2 q + \lambda q_{xxx} + 8|q|^2 q_{xx} + 2q^2 q_{xx} + 4q|q|q + 6q^2 q + 6|q|^4 q \]
\[ + \alpha^* |q|^2 + \frac{1}{2} q_{xxx} + \frac{3}{2} q_{xx} + 4q^2 q_{xx} + 5|q|^2 q + q^4 q_x^2 + \frac{3}{2} q^4 q_{xx}^2 + 2q^4 q_{xxx} \]
\[ + q^2 q_x^2 + \beta |q|^2 |q|^4 q_{xx} + |q|^2 q_{xx} + |q|^4 q_{xx} + |q|^2 q_{xx}^2 + |q|^4 q + 4|q|^2 q_{xx} q_x^2 \]
\[ + b_1 q + q^* - b_2 q^3 + q^2 q^* - 3b_2 q^2 q + |q|^2 q^* - b_3 \frac{\epsilon^2}{2} q_{xx} + q_x^4 \]
\[ - b_3 \frac{\epsilon^4}{24} q_{xxx} + q_x^4 = 0. \]

(5.20)

Where,

\[ \lambda = \frac{\epsilon^2}{12}, \alpha = \frac{k(1-s^2)}{4x^2 \epsilon^2}, \beta = \frac{1}{4} \left( \frac{k(1-s^2)}{4x^2} \right)^2, b_1 = \frac{24U_0 j \alpha}{\epsilon^2} - \frac{12 \hbar}{w_0 \epsilon^2}, \]
\[ b_2 = \frac{12 j^2 \alpha^3}{\epsilon^2}, b_3 = \frac{12 \hbar w_0^2 \alpha}{w_0 \epsilon^2}. \]

In the above equation, suffix denotes the partial derivative with respect to \( x \), where \( \alpha, \beta, \lambda \), are the various parameters. The equations of motion representing the dynamics of proton transfer in hydrogen bonded system leads to the higher order nonlinear Schrödinger (NLS) equation which contain soliton solution. Soon after this discovery, there was lots of interest in studying the nonlinear excitations and molecular interaction. When the effect of discreteness and higher order excitations and interactions are induced, it is expected that the dynamics is found to be governed by perturbed nonlinear Schrödinger equation (NLS). Solving Eq. (5.20) analytically is very difficult due to its mathematical complexity which arises because of high degree of nonlinearity. For
example, when $\lambda = 0$, the equation reduces to the integrable cubic nonlinear Schrödinger equation,

$$i\dot{q} + q_{xx} + 2|q|^2q = 0. \tag{5.21}$$

When $\alpha = \beta = 0$, $\lambda=0$ Eq. (5.20) reduces to the integrable higher order (fourth order) nonlinear Schrödinger equation

$$i\dot{q} + q_{xx} + 2|q|^2q + \lambda \qdot{xxxx} + 8|q|^2 q_{xx} + 2q^2 q_{xx} + 4q|q_x|^2 + 6q^* q_x^2 + 6|q|^4 q = 0. \tag{5.22}$$

So, it implies that the coupled interactions between the proton and peptide group play the main role for determining the properties of protons and it makes the proton dynamics governed by solitons to shift over the barriers in the interbonds by the mechanism of quasi-self trapping whereas the double well potential playing only a minor role. In the next section, we perform a multiple scale analysis of the perturbed higher order nonlinear Schrödinger equation obtained in this section which shows the velocity and amplitude of the soliton varies with time.

### 5.4 Evolution of soliton parameters

The dynamics of protons in hydrogen bonded polypeptide chains when higher order excitations, interactions and the effect of discreteness are introduced is found to be governed by the perturbed higher order nonlinear Schrödinger equation (5.20). In order to understand the nature of proton dynamics in polypeptide chains under the influence of higher order excitations, interactions and the effect of discreteness, we employ multiple scale analysis by treating the terms proportional to $\lambda$ as weak perturbation along the lines of Kodama and Ablowitz [179]. In this section, we drop terms proportional to $\beta$ and consider the perturbed equation (5.20) with $\beta = 0$ as,

$$i\dot{q} + q_{xx} + 2|q|^2q + \lambda \qdot{xxxx} + 8|q|^2 q_{xx} + 2q^2 q_{xx} + 4q|q_x|^2 + 6q^* q_x^2 + 6|q|^4 q$$

$$+ \alpha \left|q\right|^2 \frac{1}{2} q_{xxxx} + \frac{3}{2} |q|^2 q_{xx} + 4q^2 q_{xx} + 5|q_x|^2 q + q^* q_x^2 + \frac{3}{2} q^* q_{xx} + 2q^* q_x q_{xxx}$$
\[ + q^2 q_x^2 + b_1 (q + q^4) + b_2 (q^3 + q^3) + 3b_2 (|q|^2 q + |q|^2 q^4) - b_3 \frac{\epsilon^2}{2} (q_{xx} + q_{x}^4) \]

\[ - b_3 \frac{\epsilon^4}{24} (q_{xxxx} + q_{xxx}) = 0. \] (5.23)

When \( \lambda = 0 \), Eq. (5.23) reduces to the completely integrable cubic NLS equation for which the envelope soliton solution can be written as,

\[ q = \eta \sech \eta (\theta - \theta_0) \exp \left[ i \xi (\theta - \theta_0) + i (\sigma - \sigma_0) \right], \] (5.24)

where, \( \theta_t = -2\xi, \theta_x = 1, \sigma_t = \eta^2 + \xi^2, \sigma_x = 0 \). The parameters \( \eta \) and \( \xi \) are the amplitude and velocity of the proton soliton. Making use of \( q \) in Eq. (5.23) and then collecting the coefficients of different powers of \( \lambda \), we obtain at \( O(\lambda) \)

\[-\eta^2 q_t + q_{168} + 2\eta_0 q_1^2 + 4\eta_0^2 q_1 = F_1 (q_0), \] (5.25)

where,

\[ F_1 (q_0) = - q_{0000} - 6q_{000} q_2^2 + q_{00} q_4^2 + 10q_{00} q_{00} - 20q_{0} q_{00} - 20q_{00} q_{00} + 20q_{10} q_{00} + 10q_{10} q_{00} + 20q_{00} q_{0} + 60q_{0} q_{0} + \frac{1}{2} \alpha q_{00} q_{000} - 3q_{00} q_{00} q_2^2 + \frac{\alpha^2}{2} q_{00} q_{00}^2 + \alpha q_{00} q_{000} \]

\[-15q_{00} q_{000} q_2^2 + \frac{7}{2} \alpha q_{00} q_{000} q_2^2 + 33\alpha q_{00} q_{000} q_2^2 - 40q_{00} q_{00} q_{00} - 22q_{00} q_{000} q_{00} + 21q_{00} q_{000} q_{00} + 6\alpha q_{00} q_{000} q_2^2 + 12q_{00} q_{000} q_2^2 + 3q_{00} q_{00} q_{000} q_2^2 - 2q_{00} q_{000} q_{00} + 2q_{00} q_{000} q_{00} \]

\[-12q_{00} q_{000} q_{000} q_2^2 + 2q_{00} q_{000} q_{000} q_2^2 + 2q_{00} q_{000} q_{000} q_2^2 - 6q_{00} q_{000} q_{000} q_2^2 + 2q_{00} q_{000} q_{000} q_2^2 + 6q_{00} q_{000} q_{000} q_2^2 - q_0 q_2 (\theta - \theta_0) + 2b_1 q_0 + 24b_2 q_3 \]

\[-b_3 \epsilon^2 q_{000} - b_3 \epsilon^2 q_2 q_0 - \frac{b_3 \epsilon^2}{12} q_{0000} + \frac{b_3 \epsilon^2}{2} q_2 q_{000} - \frac{b_3 \epsilon^2}{12} q_2 q_{000} - \frac{12}{\epsilon^2} \omega_0 q_0 + \frac{48}{\epsilon^2} q_{00}^3 \]

\[-i 4q_{0000} - 4q_{0000} - 4q_{000} q_{00} - 2q_{00} q_{00} q_2^2 + 2q_{00} q_{00} q_{00} q_2^2 + 6q_{00} q_{00} q_{00} q_{00} - 8q_{00} q_{00} q_{00} q_{00} \]

\[-22q_{00} q_{00} q_{00} q_{00} - 66q_{00} q_{00} q_{00} - 14q_{00} q_{00} q_{00} q_{00} + 6q_{00} q_{00} q_{00} q_{00} - 8q_{00} q_{00} q_{00} q_{00} + q_{00} \]

\[-2b_3 \epsilon^2 q_{000} + 12q_{00} q_{000} q_{000} + 2q_{00} q_{000} q_{000} + 6q_{00} q_{000} q_{000} - \frac{b_3 \epsilon^2}{3} q_{00} q_{000} \]

\[ + \frac{b_3 \epsilon^2}{3} \xi^3 q_{000} \] (5.26)
Assuming $\hat{q}_1 = \hat{\Phi}_1 + i\hat{\Psi}_1$, where $\hat{\Phi}_1$ and $\hat{\Psi}_1$ are real, Eq. (5.25) can be written as,

\[
L_1 \hat{\Phi}_1 = [-\eta^2 \hat{\Phi}_1 + \hat{\Phi}_{100} + 6\hat{q}_{00}^2 \hat{\Phi}_1] = \text{Re}\hat{F}_1(\hat{q}_0),
\]

\[
L_2 \hat{\Psi}_1 = [-\eta^2 \hat{\Psi}_1 + \hat{\Psi}_{100} + 2\hat{q}_{00}^2 \hat{\Psi}_1] = \text{Im}\hat{F}_1(\hat{q}_0).
\]

Where $L_1$ and $L_2$ are self-adjoint operators and $\text{Re}\hat{F}_1(\hat{q}_0)$ and $\text{Im}\hat{F}_1(\hat{q}_0)$ are the real and imaginary parts of $F_1(\hat{q}_0)$ given by

\[
\text{Re}\ F_1(\hat{q}_0) = -q_{000ee} - 6\hat{q}_{000} \xi^2 + q_0 \xi^4 + 10q_{00} \hat{q}_{00} - 20q_{30} \hat{q}_{00} + 20q_{000}^2 \hat{q}_{00} + 10q_{00}^2 \hat{q}_{00}
+ 20q_{00}^2 \hat{q}_{00} + 60q_{00}^2 + \frac{1}{2} \alpha q_{000}^2 - 3\alpha q_{000}^2 \hat{q}_{00} - \frac{\alpha}{2} q_{00}^2 \hat{q}_{00} - 15\alpha q_{000}^2 \hat{q}_{00}^2
+ \frac{7}{2} \alpha q_{00}^2 \hat{q}_{00} + 33\alpha q_{000}^2 \hat{q}_{00} - 40\alpha q_{000}^2 \hat{q}_{00} + 22\alpha q_{000}^2 \hat{q}_{00} + 21\alpha q_{000}^2 \hat{q}_{00} + 6\alpha q_{000}^2 \hat{q}_{00} + 42\alpha q_{000}^2 \hat{q}_{00}
+ 3\alpha q_{000}^2 \hat{q}_{00} - 6\alpha q_{000}^2 \hat{q}_{00} - 3\alpha q_{000}^2 \hat{q}_{00} - 12\alpha q_{000}^2 \hat{q}_{00} - 3\alpha q_{000}^2 \hat{q}_{00}^2
+ 2\alpha q_{000}^2 \hat{q}_{00} - 2\alpha q_{000}^2 \hat{q}_{00} - 6\alpha q_{000}^2 \hat{q}_{00} - 2\alpha q_{000}^2 \hat{q}_{00} + 6\alpha q_{000}^2 \hat{q}_{00} + 6\alpha q_{000}^2 \hat{q}_{00}
+ 6\alpha q_{000}^2 \hat{q}_{00} - q_0 \xi^2 (\theta - \theta_0) + 2b_1 q_0 + 24b_2 q_0^3 - b_3 e^2 q_{000} - b_3 e^2 \xi^2 q_0
- \frac{b_3 e^4}{12} q_{000} + \frac{b_3 e^4}{2} \xi^2 q_{000} - \frac{b_3 e^4}{12} \xi^4 q_0 - \frac{12}{\epsilon^2} w_0 q_0 + \frac{48}{\epsilon^2} q_0^3,
\]

\[
\text{Im}\ F_1(\hat{q}_0) = -4\xi q_{000} - 4\xi^3 q_{000} - 40\xi^2 q_{000} - 2\alpha q_{000}^2 \hat{q}_{00} + 2\alpha q_{000}^2 \hat{q}_{00}
+ 4\alpha q_{000}^2 \hat{q}_{00} - 22\alpha q_{000}^2 \hat{q}_{00} + 66\alpha q_{000}^2 \hat{q}_{00} - 14\alpha q_{000}^2 \hat{q}_{00} + 6\alpha q_{000}^2 \hat{q}_{00}
- 8\alpha q_{000}^2 \hat{q}_{00} + 12\alpha q_{000}^2 \hat{q}_{00} + 2\alpha q_{000}^2 \hat{q}_{00} + 6\alpha q_{000}^2 \hat{q}_{00} + \alpha_0 + 2b_3 e^2 \xi q_{000}
- \frac{b_3 e^4}{3} \xi q_{000} + \frac{b_3 e^4}{3} \xi^3 q_{000}.
\]

Considering the homogeneous parts of Eqs. (5.27) and (5.28) one can check that $\hat{q}_{00}$ and $\hat{q}_0$ are solutions of the homogeneous parts respectively and hence we have the secularity conditions

\[
\lim_{\theta \to +\infty} q_{00} \text{Re}(\hat{F}_1) d\theta = 0, \quad (5.31)
\]

\[
\lim_{\theta \to -\infty} q_{00} \text{Im}(\hat{F}_1) d\theta = 0. \quad (5.32)
\]
Substituting the values of \( \xi_0, \eta_0, \) \( \text{Re}(F_1) \) and \( \text{Im}(F_1) \) into the above integrals and after evolution, we finally obtain
\[
\xi_T = -\frac{b_3 \epsilon^4}{2} + 6 \eta + \frac{b_3 \epsilon^2}{2} - 3 \xi^2 + \frac{b_3 \epsilon^4}{4} \eta^3, \tag{5.33}
\]
and
\[
\eta_T = \frac{b_3 \epsilon^4}{6} - 2 \xi \eta^4. \tag{5.34}
\]
The Eqs. (5.33) and (5.34) indicate that the velocity and amplitude of the proton soliton which traverses in the hydrogen bonded polypeptide channels, are changing with time when the higher order molecular excitations, interactions and the discreteness effects are included. Also to know the soliton parameters, we solve the evolution equations numerically using the fourth order Runge-Kutta method and investigate the proton dynamics in hydrogen bonded polypeptide chain under the influence of higher order molecular excitations, interactions and the effect of discreteness. We have investigated the propagation of proton soliton by choosing the initial values \( \eta(T) = \eta(0) = 0.3 \) and \( \xi(T) = \xi(0) = -1 \) with a step size \( h = 0.1 \). From Figs. (5.4) and (5.5) we could observe the changes in the velocity and amplitude of the soliton during the proton transfer in polypeptide channels. Under perturbation the soliton traverses with greater velocity and lesser amplitude which is clearly depicted in Figs. (5.4) and (5.5). When the lattice parameter is varied from \( \epsilon = 0.1 \) to 1.5, the plots clearly explain the propagation of proton soliton for the fundamental parameters of polypeptide channels in protein molecule. From this, we can describe the proton dynamics as follows. A quasi-free incoming proton moves towards a peptide group of protein molecule which is trapped by the double well potential and then bonds to oxygen atom of neighboring peptide group. This bonding energy would transform smoothly in to the soliton formation energy and can transfer along the peptide chain without energy loss. It is evident from the Fig. (5.4) that the velocity of the soliton increases as the increase in lattice parameter \( \epsilon \). On the other hand physically speaking, if we alter the length of hydrogen bond through by stretching and deformation of the chain, the proton may easily hop to the nearest peptide group with greater velocity by crossing
over the double well potential barrier. Further, the increase in lattice parameter could faster the mobility of protons in hydrogen bonded polypeptide chains. We also have plotted the variation of the amplitude of the soliton with respect to time. Surprisingly, we observe that at a particular instant time, when the velocity of the soliton reaches a maximum value, the amplitude of the soliton is highly decelerating in nature as evident from the plots.

5.5 Perturbed solitons

In order to construct the explicit perturbed solution to Eq. (5.23), we now solve Eqs. (5.27) and (5.28). The homogeneous parts of Eqs. (5.27) and (5.28) admit, respectively, the following particular solution,

\[ \hat{\tau}_{11} = \text{sech} \tau \tanh \tau, \]
\[ \hat{\phi}_{12} = -\frac{1}{\eta} \left[ \text{sech} \tau - \frac{3}{2} \tau \text{sech} \tau \tanh \tau - \frac{1}{2} \tanh \tau \sinh \tau \right]. \]

and

\[ \hat{\psi}_{11} = \text{sech} \tau, \]
\[ \hat{\psi}_{12} = \frac{1}{2\eta} \left[ \tau \text{sech} \tau + \sinh \tau \right]. \]

where, \( \tau = \eta (\theta - \theta_0) \), with two particular solutions known for both Eqs. (5.27) and (5.28), the general solutions of the second order inhomogeneous Eqs. (5.27) and (5.28) respectively can be written as,

\[ \hat{\phi}_1 = \delta_1 \phi_{11} + \delta_2 \phi_{12} - \phi_{11} \int_{-\infty}^{\theta} \text{Re} \hat{F}_1 d\theta' + \phi_{12} \int_{-\infty}^{\theta} \text{Re} \hat{F}_1 d\theta', \]  \hspace{1cm} (5.35)

\[ \hat{\psi}_1 = \delta_3 \psi_{11} + \delta_4 \psi_{12} - \psi_{11} \int_{-\infty}^{\theta} \text{Im} \hat{F}_1 d\theta' + \psi_{12} \int_{-\infty}^{\theta} \psi_{11} \text{Im} \hat{F}_1 d\theta'. \]  \hspace{1cm} (5.36)

where, \( \delta_1, \delta_2, \delta_3 \) and \( \delta_4 \) are integration constants. The construction of explicit form of the general solution of \( \phi_{11}, \phi_{12} \) requires the evaluation of the integrals in the last two terms in Eq. (5.35). In order to find the solutions of \( \hat{\phi}_1 \) we
remove the secular terms i.e., the terms proportional to \( \sinh \eta (\theta - \theta_0) \) using the boundary conditions,

\[
\hat{\psi}_1(0) |_{\theta_0=0} = 0, \\
\hat{\psi}_1(0) |_{\theta_0=0} = 0.
\]  

(5.37)

Using the value of \( \psi_{11}, \psi_{12} \) and after evaluating the integrals and removing the secular terms and using the boundary conditions,

\[
\hat{\psi}_1(0) |_{\theta_0=0} = 0, \\
\hat{\psi}_1(0) |_{\theta_0=0} = 0
\]  

(5.38)

we get,

\[
\begin{align*}
\phi_{11} & = \text{Re} \int \text{e}^{it} \text{d} \theta = \alpha \eta^5 \left[ \frac{-49}{10} \text{sech} \Delta \tanh^2 \Delta - \frac{14}{5} \cosh(2\Delta) \text{sech}^3 \Delta \right. \\
& \left. + \frac{56}{5} \text{sech}^3 \Delta \tanh^4 \Delta + \frac{21}{2} \Delta \text{sech}^7 \Delta \tanh \Delta - \frac{63}{4} \Delta \text{sech}^5 \Delta \right] \\
& + \frac{21}{2} \Delta \text{sech} \Delta \tanh^6 \Delta + \frac{21}{2} \Delta \text{sech} \Delta \tanh \Delta - \frac{36}{35} \cosh(2\Delta) \text{sech}^3 \Delta \tanh^6 \Delta \\
& + \frac{216}{35} \text{sech}^3 \Delta \tanh^6 \Delta - \frac{87}{35} \text{sech} \Delta \tanh^2 \Delta + \frac{27}{4} \text{sech}^9 \Delta \tanh \Delta - 18 \Delta \text{sech}^7 \Delta \tanh \Delta + \frac{27}{28} \text{sech}^5 \Delta \tanh^2 \Delta + \frac{171}{70} \text{sech}^5 \Delta \tanh^2 \Delta \\
& - \frac{87}{70} \text{sech}^3 \Delta \tanh^2 \Delta - \frac{18}{35} \text{sech} \Delta \tanh \Delta - \frac{18}{7} \text{sech} \Delta \\
& + \frac{18}{7} \text{sech} \Delta \tanh \Delta + \frac{5}{2} \text{sech}^3 \Delta \tanh^2 \Delta + 10 \text{sech} \Delta \tanh \Delta \\
& - \frac{15}{4} \text{sech} \Delta \tanh^2 \Delta - \frac{45}{16} \Delta \text{sech}^5 \Delta \tanh \Delta - \frac{45}{8} \text{sech}^3 \Delta \tanh^2 \Delta - \frac{15}{8} \text{sech} \Delta \tanh \Delta - \frac{5}{4} \text{sech} \Delta \tanh^4 \Delta - \frac{5}{4} \text{sech} \Delta \tanh \Delta + \frac{88}{21} \text{sech} \Delta \tanh^2 \Delta \\
& - \frac{55}{7} \text{sech}^7 \Delta \tanh^2 \Delta - \frac{55}{7} \text{sech}^7 \Delta \tanh^2 \Delta + \frac{11}{7} \text{sech}^5 \Delta \tanh^2 \Delta \\
& + \frac{44}{21} \text{sech}^3 \Delta \tanh^2 \Delta + \frac{88}{21} \text{sech} \Delta \tanh \Delta + \frac{44}{21} \Delta \text{sech} \Delta \tanh^2 \Delta + \frac{55}{48} \text{sech} \Delta \tanh^2 \Delta \right].
\end{align*}
\]
\[
\text{sech}^2 \Delta \tanh \Delta - \frac{275}{126} \text{sech}^7 \tanh \Delta + \frac{11}{56} \text{sech}^5 \tanh \Delta + \frac{11}{21} \\
\text{sech}^3 \tanh \Delta + \frac{11}{14} \Delta \text{sech}^5 \tanh^2 \Delta + \frac{11}{14} \cosh(2\Delta) \text{sech}^3 \tanh^6 \Delta \\
+ \frac{22}{21} \Delta \text{sech}^3 \tanh^2 \Delta + \frac{11}{14} \text{sech}^3 \tanh^6 \Delta + \frac{36}{35} \text{sech}^3 \tanh^6 \Delta \\
+ \frac{6}{35} \cosh(2\Delta) \text{sech}^3 \tanh^6 \Delta - \frac{29}{70} \text{sech} \Delta \tanh^2 \Delta - 3 \Delta \text{sech}^7 \tanh \Delta \\
+ \frac{9}{8} \Delta \text{sech}^9 \tanh \Delta + \frac{9}{4} \Delta \text{sech}^5 \tanh \Delta - \frac{9}{56} \text{sech}^7 \tanh^2 \Delta + \frac{57}{140} \text{sech}^5 \tanh^2 \Delta \\
- \frac{29}{140} \text{sech}^3 \tanh^2 \Delta - \frac{29}{70} \text{sech} \Delta \tanh \Delta - \frac{3}{7} \text{sech} \Delta \\
\tanh^8 \Delta - \frac{3}{7} \text{sech} \Delta \tanh \Delta - \frac{16}{7} \text{sech} \Delta \tanh^2 \Delta + \frac{30}{7} \text{sech}^7 \tanh^2 \Delta \\
- \frac{6}{7} \text{sech}^5 \tanh^2 \Delta - \frac{8}{7} \text{sech}^3 \tanh^2 \Delta - \frac{16}{7} \text{sech} \Delta \tanh \Delta + \frac{10}{7} \text{sech} \Delta \tanh^2 \Delta + \frac{45}{8} \Delta \text{sech}^9 \tanh \Delta \\
- \frac{15}{2} \Delta \text{sech}^7 \tanh \Delta - \frac{45}{56} \text{sech}^7 \tanh \Delta \\
\text{sech}^7 \tanh^2 \Delta - \frac{10}{7} \text{sech} \Delta \tanh \Delta + \frac{45}{8} \text{sech}^9 \tanh \Delta - \frac{15}{2} \text{sech}^7 \tanh \Delta \\
\text{sech}^7 \tanh \Delta - \frac{15}{28} \text{sech}^5 \tanh^2 \Delta + \frac{5}{7} \text{sech}^3 \Delta \\
\text{tanh}^2 \Delta + \frac{184}{35} \text{sech}^3 \tanh^2 \Delta + \frac{92}{35} \text{sech}^3 \tanh^2 \Delta + \frac{184}{35} \text{sech} \tanh \Delta + \frac{92}{35} \text{sech}^3 \tanh^2 \Delta \\
+ \frac{92}{35} \text{sech}^3 \tanh^2 \Delta + \frac{184}{35} \text{sech} \tanh \Delta + \frac{23}{7} \text{sech} \tanh^2 \Delta - \frac{207}{4} \Delta \\
\text{sech}^9 \Delta \tanh \Delta + \frac{69}{4} \Delta \text{sech}^7 \tanh \Delta + \frac{207}{112} \text{sech}^7 \tanh^2 \Delta - \frac{69}{56} \text{sech}^5 \Delta \\
\text{tanh}^2 \Delta - \frac{69}{42} \text{sech}^3 \tanh^2 \Delta - \frac{23}{7} \text{sech} \Delta \tanh \Delta - \frac{69}{70} \cosh(2\Delta) \text{sech}^3 \Delta \\
\text{tanh}^6 \Delta + \frac{9}{70} \cosh(2\Delta) \text{sech}^3 \tanh^6 \Delta - \frac{207}{35} \text{sech}^3 \tanh^6 \Delta + \frac{27}{35} \Delta \text{sech}^9 \tanh \Delta \\
\text{sech} \tanh \Delta + \frac{27}{16} \Delta \text{sech}^5 \tanh \Delta - \frac{27}{224} \text{sech}^7 \tanh^2 \Delta + \frac{1071}{560} \text{sech}^5 \Delta \\
\text{tanh}^2 \Delta - \frac{1071}{1680} \text{sech} \Delta \tanh \Delta - \frac{87}{280} \text{sech} \Delta \tanh \Delta - \frac{9}{28} \text{sech} \tanh^8 \Delta \\
- \frac{9}{28} \text{sech} \tanh \Delta + \frac{b_3 \varepsilon^4}{12} - 1 \eta^3 \frac{28}{3} \text{sech} \tanh^4 \Delta + \frac{28}{3} \text{sech} \tanh \Delta \\
- 14 \text{sech} \tanh^2 \Delta - \frac{21}{2} \Delta \text{sech}^5 \tanh \Delta + 21 \Delta \text{sech}^3 \tanh \Delta + \frac{14}{4} \Delta \text{sech}^5 \tanh \Delta \\
\text{sech}^3 \tanh \Delta - 14 \text{sech} \tanh \Delta - 14 \Delta \text{sech} \tanh \Delta + \frac{56}{3} \text{sech} \tanh^2 \Delta
\]
\[-\frac{14}{3} \cdot \text{sech}^3 \Delta \tanh^2 \Delta + \frac{56}{3} \cdot \text{sech} \Delta \tanh \Delta + 6 \cdot \text{sech} \Delta \tanh^5 \Delta + 6 \cdot \text{sech} \Delta \tanh^7 \Delta + 23 \cdot \text{sech} \Delta \tanh^2 \Delta - 6 \cdot \text{sech}^7 \Delta \tanh \Delta + 18 \cdot \text{sech}^5 \Delta \tanh \Delta - 18 \cdot \text{sech}^3 \Delta \tanh \Delta + \frac{6}{5} \cdot \text{sech}^5 \Delta \tanh \Delta - \frac{33}{5} \cdot \text{sech}^3 \Delta \tanh^2 \Delta + \frac{46}{5} \cdot \text{sech} \Delta \tanh^2 \Delta + 12 \cdot \text{sech} \Delta \tanh \Delta - \frac{52}{5} \cdot \text{sech} \Delta \tanh^2 \Delta - \frac{12}{5} \cdot \text{sech}^5 \Delta \tanh \Delta + \frac{44}{5} \cdot \text{sech}^3 \Delta \tanh^2 \Delta - \frac{92}{5} \cdot \text{sech} \Delta \tanh \Delta - 5 \cdot \text{sech} \Delta \tanh^2 \Delta - 5 \cdot \text{sech} \Delta \tanh \Delta + \frac{15}{4} \cdot \text{sech} \Delta \tanh^2 \Delta - \frac{15}{4} \cdot \text{sech}^3 \Delta \tanh \Delta + \frac{15}{2} \cdot \text{sech} \Delta \tanh \Delta + \frac{5}{2} \cdot \text{sech} \Delta \tanh \Delta - \frac{5}{2} \cdot \text{sech} \Delta \tanh^2 \Delta - \frac{5}{2} \cdot \text{sech} \Delta \tanh \Delta + \eta \cdot 6 \xi^2 + b_3 \epsilon^2 - \frac{b_3 \epsilon^4 \xi^2}{2} - \frac{1}{3} \cdot \text{sech} \Delta \tanh^4 \Delta + \frac{1}{3} \cdot \text{sech} \Delta \tanh \Delta - \frac{1}{2} \cdot \text{sech} \Delta \tanh^2 \Delta - \frac{3}{8} \cdot \text{sech}^5 \Delta \tanh \Delta + \frac{3}{4} \cdot \text{sech}^3 \Delta \tanh \Delta + \frac{1}{8} \cdot \text{sech}^3 \Delta \tanh \Delta - \frac{1}{2} \cdot \text{sech} \Delta \tanh \Delta - \frac{1}{2} \cdot \text{sech} \Delta \tanh^2 \Delta + \frac{2}{3} \cdot \text{sech} \Delta \tanh^2 \Delta - \frac{1}{6} \cdot \text{sech}^3 \Delta \tanh \Delta - \frac{2}{3} \cdot \text{sech} \Delta \tanh \Delta + \eta \cdot 6 \xi^2 + b_3 \epsilon^2 - \frac{b_3 \epsilon^4 \xi}{2} \cdot \frac{1}{\eta} \cdot \frac{1}{2} \cdot \text{sech}^3 \Delta \tanh^4 \Delta + \frac{1}{3} \cdot \text{sech} \Delta \tanh \Delta + \eta \cdot 12 \alpha \xi^2 - 20 \cdot \frac{1}{15} \cdot \cos(2 \Delta) \cdot \text{sech}^3 \Delta \tanh^4 \Delta + \frac{4}{15} \cdot \text{sech}^3 \Delta \tanh^4 \Delta - \frac{7}{60} \cdot \text{sech} \Delta \tanh^2 \Delta - \frac{1}{4} \cdot \text{sech}^7 \Delta \tanh \Delta + \frac{3}{8} \cdot \text{sech}^5 \Delta \tanh \Delta + \frac{1}{60} \cdot \text{sech}^5 \Delta \tanh \Delta - \frac{7}{120} \cdot \text{sech}^3 \Delta \tanh^2 \Delta - \frac{7}{60} \cdot \text{sech} \Delta \tanh \Delta - \frac{1}{10} \cdot \text{sech} \Delta \tanh^6 \Delta - \frac{1}{10} \cdot \text{sech} \Delta \tanh \Delta - \frac{8}{15} \cdot \text{sech} \Delta \tanh \Delta - \frac{1}{5} \cdot \text{sech}^5 \Delta \tanh^2 \Delta - \frac{4}{15} \cdot \text{sech}^3 \Delta \tanh^2 \Delta - \frac{8}{15} \cdot \text{sech} \Delta \tanh \Delta + \frac{6 \alpha \xi^2}{15} - 10 \cdot \frac{4}{15} \cdot \text{sech} \Delta \tanh^2 \Delta - \frac{1}{2} \cdot \text{sech}^7 \Delta \tanh \Delta + \frac{1}{10} \cdot \text{sech}^5 \Delta \tanh^2 \Delta + \frac{2}{15} \cdot \text{sech}^3 \Delta \tanh^2 \Delta + \frac{4}{15} \cdot \text{sech} \Delta \tanh \Delta + \frac{1}{15} \cdot \cos(2 \Delta)\]
\[
\begin{align*}
\text{sech}^3 \Delta \tanh^4 \Delta + & \frac{4}{5} \text{sech}^3 \Delta \tanh^4 \Delta + \alpha \eta^2 \frac{176}{7} \text{sech} \Delta \tanh^2 \Delta + \frac{11}{7} \\
\text{sech}^7 \Delta \tanh^2 \Delta + & \frac{66}{7} \text{sech}^5 \Delta \tanh^2 \Delta + \frac{88}{7} \text{sech}^3 \Delta \tanh^2 \Delta + \frac{176}{7} \text{sech} \Delta \\
\tanh \Delta - & \frac{33}{7} \text{sech} \Delta \tanh \Delta + \frac{165}{16} \Delta \text{sech}^9 \Delta \tanh \Delta - \frac{165}{112} \text{sech}^7 \Delta \\
\tanh^2 \Delta - & \frac{99}{56} \text{sech}^5 \Delta \tanh^2 \Delta - \frac{33}{14} \text{sech}^3 \Delta \tanh^2 \Delta - \frac{33}{7} \text{sech} \Delta \tanh \Delta \\
- & \frac{44}{21} \text{sech} \Delta \tanh^2 \Delta + \frac{55}{14} \text{sech}^7 \Delta \tanh^2 \Delta - \frac{11}{14} \text{sech}^5 \Delta \tanh \Delta - \frac{22}{21} \text{sech}^3 \Delta \tanh^2 \Delta \\
& - \frac{44}{21} \text{sech} \Delta \tanh \Delta + \frac{24}{35} \text{sech} \Delta \tanh^2 \Delta - \frac{9}{7} \text{sech}^7 \Delta \\
\tanh^2 \Delta + & \frac{27}{16} \Delta \text{sech}^9 \Delta \tanh \Delta - \frac{9}{4} \Delta \text{sech}^7 \Delta \tanh \Delta - \frac{27}{112} \text{sech}^7 \Delta \tanh^2 \Delta \\
& + \frac{9}{56} \text{sech}^5 \Delta \tanh^2 \Delta + \frac{3}{14} \text{sech}^3 \Delta \tanh^2 \Delta + \frac{3}{7} \text{sech} \Delta \tanh \Delta + \frac{9}{70} \text{sech} \Delta \\
\cos(2\Delta) & \text{sech}^3 \Delta \tanh^6 \Delta + \frac{27}{35} \text{sech}^3 \Delta \tanh^6 \Delta + \eta^2 \frac{5}{2} \text{sech}^5 \Delta \tanh \Delta \\
& + \frac{5}{2} \text{sech} \Delta \tanh \Delta + 2\Delta \text{sech} \Delta \tanh^2 \Delta - \frac{3}{4} \text{sech}^5 \Delta \tanh \Delta + \frac{1}{2} \text{sech}^3 \Delta \tanh \Delta - 2\log \cosh \Delta \text{sech} \Delta \tanh \Delta - 3\text{sech}^5 \Delta \tanh^2 \Delta - \Delta \text{sech}^3 \Delta \\
\tanh^2 \Delta + & \frac{5}{4} \text{sech} \Delta \tanh^5 \Delta + \frac{5}{4} \text{sech} \Delta \tanh \Delta + \eta^2 \frac{12}{5} \xi^2 - 30 \\
& \frac{1}{15} \cos(2\Delta) \text{sech}^3 \Delta \tanh^4 \Delta + \frac{4}{15} \text{sech}^3 \Delta \tanh^4 \Delta + \frac{7}{60} \text{sech} \Delta \tanh^2 \Delta \\
& + \frac{1}{4} \text{sech}^7 \Delta \tanh \Delta - \frac{3}{8} \Delta \text{sech}^5 \Delta \tanh \Delta - \frac{1}{20} \text{sech}^5 \Delta \tanh^2 \Delta + \frac{7}{120} \text{sech}^3 \Delta \tanh^2 \Delta \\
& + \frac{7}{60} \text{sech} \Delta \tanh \Delta + \frac{1}{10} \text{sech} \Delta \tanh^6 \Delta + \frac{1}{10} \text{sech} \Delta \\
& + \frac{\alpha}{\eta} \frac{72}{35} \text{sech} \Delta \tanh^2 \Delta + \frac{9}{14} \text{sech}^7 \Delta \tanh^2 \Delta + \frac{27}{35} \text{sech}^5 \Delta \\
\tanh^2 \Delta + & \frac{36}{35} \text{sech}^3 \Delta \tanh^2 \Delta + \frac{72}{35} \text{sech}^3 \Delta \tanh \Delta - \frac{27}{70} \text{sech} \Delta \tanh^2 \Delta \\
& + \frac{3}{2} \text{sech}^9 \Delta \tanh \Delta - \frac{27}{224} \text{sech}^7 \Delta \tanh^2 \Delta - \frac{81}{560} \text{sech}^5 \Delta \tanh^2 \Delta \\
& - \frac{27}{140} \text{sech}^3 \Delta \tanh^2 \Delta - \frac{27}{70} \text{sech} \Delta \tanh \Delta - \frac{12}{70} \text{sech} \Delta \tanh^2 \Delta + \frac{9}{28}
\end{align*}
\]
\[
\begin{align*}
\text{sech}^7 \Delta \tanh^2 \Delta &- \frac{9}{140} \text{sech}^5 \Delta \tanh^2 \Delta - \frac{3}{35} \text{sech}^3 \Delta \tanh^2 \Delta - \frac{6}{35} \text{sech} \Delta \\
\tanh \Delta &+ \eta \cdot 20 \xi^2 - \frac{7}{2} a \xi^4 - 24 b_2 + \frac{48}{\epsilon^2} - \frac{1}{3} \text{sech}^3 \Delta \tanh^2 \Delta + \frac{2}{3} \text{sech} \Delta \\
\tanh^2 \Delta &+ \frac{2}{3} \text{sech} \Delta \tanh \Delta + \frac{1}{4} \text{sech} \Delta \tanh^2 \Delta - \frac{3}{8} \text{sech}^5 \Delta \tanh \Delta \\
+8 \text{sech}^3 \Delta \tanh^2 \Delta &+ \frac{1}{4} \text{sech} \Delta \tanh \Delta + \frac{1}{6} \text{sech} \Delta \tanh^4 \Delta + \frac{1}{6} \text{sech} \Delta \\
\tanh \Delta &+ \frac{1}{\eta} - \xi_1 (\theta - \theta_0) + 2b_1 - b_3 \epsilon^2 - \frac{b_3 \epsilon^4 \xi^4}{12} - \frac{12 \omega_0}{\epsilon^2} \text{sech} \Delta \tanh^2 \Delta \\
+ \text{sech} \Delta \tanh \Delta &+ \frac{3}{2} \xi_1 (\theta - \theta_0) - 3b_1 + \frac{3b_3 \epsilon^2 \xi^2}{2} + \frac{b_3 \epsilon^4 \xi^4}{8} + \frac{18 \omega_0}{\epsilon^4} \text{sech} \Delta \tanh^2 \Delta \\
\text{sech} \Delta \tanh \Delta & - \frac{\text{sech} \Delta \tanh \Delta}{2} - \frac{\xi_1 (\theta - \theta_0)}{2} - b_1 + \frac{b_3 \epsilon^2 \xi^2}{2} + \frac{b_3 \epsilon^4 \xi^4}{24} \\
+ \frac{6 \omega_0}{\epsilon^2} \text{sech} \Delta \tanh \Delta - \text{sech} \Delta \tanh^2 \Delta - \text{sech} \Delta \tanh \Delta - \xi^4 \eta^2 \\
- \frac{3}{4} \text{sech} \Delta \tanh \Delta &+ \frac{1}{4} \text{sech}^4 \Delta \tanh^2 \Delta + \frac{3}{8} \text{sech}^2 \Delta \\
\tanh^2 \Delta &+ \frac{3}{4} \text{sech} \Delta \tanh \Delta \tan^{-1} \left( \frac{\Delta}{2} \right) + \frac{3}{10} \text{sech}^3 \Delta \tanh \Delta - \frac{9}{40} \\
\text{sech} \Delta \tanh \Delta & + \frac{1}{2} \text{sech} \Delta \tanh \Delta \tan^{-1} \left( \frac{1}{2} \right) - \frac{3}{40} \text{sech}^3 \Delta \tanh^2 \Delta - \frac{9}{80} \text{sech}^2 \Delta \\
\tanh^2 \Delta &- \frac{9}{40} \text{sech} \Delta \tanh \Delta \tan^{-1} \left( \frac{1}{2} \right) - \frac{1}{8} \text{sech} \Delta \tanh \Delta \tan^{-1} \left( \frac{\Delta}{2} \right) \\
+ \frac{1}{8} \text{sech}^3 \Delta \tanh^2 \Delta - \frac{1}{16} \text{sech}^2 \Delta \tanh^2 \Delta - \frac{1}{8} \text{sech} \Delta \tanh \Delta \tan^{-1} \left( \frac{1}{2} \right) \\
+ \frac{40 \alpha \xi^2 - 60 \eta^3}{15} \text{sech} \Delta \tanh^2 \Delta + \frac{1}{5} \text{sech}^5 \Delta \tanh^2 \Delta + \frac{4}{15} \text{sech}^3 \Delta \\
\tanh^2 \Delta &+ \frac{8}{15} \text{sech} \Delta \tanh \Delta + \frac{2}{15} \text{sech} \Delta \tanh^2 \Delta - \frac{1}{4} \text{sech}^7 \Delta \tanh \Delta \\
+ \frac{1}{20} \text{sech}^5 \Delta \tanh^2 \Delta + \frac{1}{15} \text{sech}^3 \Delta \tanh^2 \Delta + \frac{2}{15} \text{sech} \Delta \tanh \Delta + \frac{2}{15} \\
\text{sech}^3 \Delta \tanh^4 \Delta &+ \frac{1}{30} \cos(2 \Delta) \text{sech}^3 \Delta \tanh^4 \Delta + \frac{1}{4} \text{sech}^2 \Delta \tanh^2 \Delta \\
+ \frac{1}{4} \text{sech}^4 \Delta \tanh \Delta .
\end{align*}
\]
\[ \phi_{12} - \phi_{11} \text{ Re } \mathcal{F}_1 d\theta' = \frac{161}{6} \alpha \tilde{\eta}^5 + \frac{35}{3} \tilde{\eta}^3 + \frac{20}{3} \alpha \xi \tilde{\eta}^3 + 2 \alpha \tilde{\xi}^2 - \frac{10}{3} \text{ sech}^2 \Delta \]

\[ - \frac{139}{5} + \frac{5}{3} + \frac{15}{5} - \frac{3}{2} + b_3 \epsilon^2 \frac{b_3 \epsilon^4 \tilde{\xi}^2}{6} - 2 + 7 \alpha \tilde{\eta} + 4 \eta - 2 \tilde{\xi} + 4 - \frac{1}{8} \eta^2 - 5 \tilde{\xi} - 8 \alpha \tilde{\eta} - 6 b_2 \]

\[ + \frac{12}{\epsilon^2} \eta \text{ sech}^5 \Delta - \frac{103}{2} \alpha \tilde{\eta}^5 + \frac{55}{3} \alpha \eta^2 + \frac{9 \alpha}{16 \eta} \text{ sech}^9 \Delta + \frac{3 \tilde{\xi}^2}{2} + \frac{b_3 \epsilon^2}{4} \frac{b_3 \epsilon^4 \tilde{\xi}^2}{8} \]

\[ \eta \text{ sech} \Delta \text{ tanh}^4 \Delta - \frac{b_3 \epsilon^4}{3} - 4 \eta \text{ sech} \Delta \text{ tanh}^6 \Delta - \frac{5}{2} \frac{5}{2} b_3 \epsilon^4 \eta^2 \text{ sech}^3 \Delta \text{ tanh} \Delta \]

\[ - \frac{2}{3} \eta^2 \cos(2 \Delta) \text{ sech}^3 \Delta \text{ tanh}^3 \Delta - \frac{8}{3} \eta^2 \text{ sech}^3 \Delta \text{ tanh}^3 \Delta - \tilde{\xi}_7 (\alpha^2 - \theta_0) - 2 b_1 + b_3 \epsilon^2 \]

\[ + \frac{b_3 \epsilon^4 \tilde{\xi}^2}{12} + \frac{12 \omega_0}{\epsilon^2} \eta \text{ sech}^2 \Delta - \frac{\xi^4 \eta \text{ sech}^6 \Delta}{4} - \frac{\alpha \epsilon^4 \text{ sech}^4 \Delta}{6} + \frac{7 b_3 \epsilon^4}{12} - \frac{7}{\eta^2} \text{ sech} \Delta \]

\[ + 6 \tilde{\xi}^2 + b_3 \epsilon^2 - \frac{b_3 \epsilon^4 \tilde{\xi}^2}{2} - \frac{\eta^2}{4} \text{ sech} \Delta - \frac{b_3 \epsilon^4 \tilde{\eta}^2}{3} - 4 \eta^2 \text{ sech} \Delta + \frac{75}{4} \alpha \tilde{\eta}^5 - \frac{25}{2} \eta^3 \]

\[ - 3 \alpha \tilde{\xi}^2 - 10 \alpha \tilde{\xi}^2 \tilde{\eta}^3 + 5 \Delta \text{ sech} \Delta \text{ tanh} \Delta + \frac{15}{4} \eta^3 + \frac{45}{8} \eta^5 + \frac{849}{16} \alpha \tilde{\eta}^5 - \frac{\xi^2}{\eta^2} \]

\[ - \frac{3}{8} b_3 \epsilon^2 - \frac{3}{16} \epsilon^4 \tilde{\xi}^2 - \frac{15 \tilde{\xi}^2}{16} - \frac{21 \alpha \eta \tilde{\xi}^4}{16} - 9 b_2 \eta + \frac{18 \eta}{\epsilon^2} \Delta \text{ sech}^5 \Delta \text{ tanh} \Delta + \frac{165}{16} \alpha \tilde{\eta}^2 \]

\[ + \frac{27 \alpha}{32 \eta} + \frac{309}{2} \alpha \tilde{\eta}^6 \Delta \text{ sech}^9 \Delta \text{ tanh} \Delta - \eta \frac{9}{4} \tilde{\xi}^2 + \frac{3 b_3 \epsilon^2}{8} - \frac{3 b_3 \epsilon^4 \tilde{\xi}^2}{16} \Delta \text{ sech} \Delta \text{ tanh}^6 \Delta \]

\[ + \frac{b_3 \epsilon^4}{2} - 6 \eta \text{ sech} \Delta \text{ tanh}^7 \Delta + \frac{15 \tilde{\xi}^3}{4} - \frac{5 \tilde{\eta}^3 b_3 \epsilon^4}{16} \Delta \text{ sech} \Delta \text{ tanh} \Delta + 4 \eta^2 \]

\[ \text{ sech}^3 \Delta \text{ tanh}^4 \Delta + \eta^2 \Delta \cos(2 \Delta) \text{ sech}^3 \Delta \text{ tanh}^4 \Delta - \frac{3}{2 \eta} \tilde{\xi}_7 (\alpha^2 - \theta_0) - 2 b_1 \]

\[ + b_3 \epsilon^2 \tilde{\xi}^2 + \frac{b_3 \epsilon^4 \tilde{\xi}^2}{12} + \frac{12 \omega_0}{\epsilon^2} \Delta \text{ sech}^2 \Delta \text{ tanh} \Delta + \frac{3}{10 \tilde{\xi}^4 \eta^2} \Delta \text{ sech} \Delta \text{ tanh} \Delta + \frac{1}{4} \alpha \epsilon^4 \]

\[ \Delta \text{ sech}^4 \Delta \text{ tanh} \Delta + \frac{3}{2 \eta} - \frac{7}{12} b_3 \epsilon^4 - \frac{3 \eta}{8} \tilde{\xi}^2 + b_3 \epsilon^2 - \frac{b_3 \epsilon^4 \tilde{\xi}^2}{2} + \frac{3}{2} b_3 \epsilon^4 \eta^3 \]

\[ - 4 \eta^3 \Delta \text{ sech} \Delta \text{ tanh} \Delta + \frac{125}{12} \alpha \tilde{\eta}^5 - \frac{35}{6} \eta^3 - \alpha \tilde{\xi}^2 + \frac{5}{3} - \frac{5}{3} \alpha \tilde{\xi}^2 \eta^3 \text{ sech}^6 \Delta \]
\[ \tanh \Delta \sinh \Delta + \frac{5}{4} \eta^3 + \frac{41}{2} \alpha \eta^5 - \frac{3}{4} \xi^2 + \frac{3b_3 \epsilon^2}{8} - \frac{b_3 \epsilon^4 \xi^2}{16} - \frac{1}{\eta^2} - \frac{5}{2} \xi^2 - \frac{7}{16} \alpha \xi^4 \]

\[ -3b_2 + \frac{6}{\epsilon^2} \eta \tanh \Delta \sinh \Delta + \frac{103}{2} \alpha \eta^5 + \frac{55}{16} \alpha \eta^2 + \frac{9}{32} \eta \tanh^8 \Delta \]

\[ \tanh \Delta \sinh \Delta - \eta \frac{3 \xi^2}{4} + \frac{b_3 \epsilon^2}{8} - \frac{b_3 \epsilon^4 \xi^2}{16} \tanh^5 \Delta \sinh \Delta + \frac{b_3 \epsilon^4}{6} - 2 \eta^3 \]

\[ \tanh^7 \Delta \sinh \Delta + \frac{5}{4} - \frac{5b_3 \epsilon^4}{48} \eta^3 \tanh^2 \Delta \sinh \Delta + \frac{4}{3} \eta^2 \tanh^2 \Delta \tanh^4 \Delta \]

\[ \sinh \Delta + \frac{\eta^2}{3} \cos(2 \Delta) \sinh^2 \Delta \tanh \Delta - \frac{\xi_1(\theta - \theta_0)}{2} - 2b_1 + b_3 \epsilon^2 \xi^2 \]

\[ + \frac{b_3 \epsilon^4 \xi^4}{12} + \frac{12 \omega_0}{\epsilon^2} \frac{1}{2 \eta} \tanh \Delta \sinh \Delta + \frac{\xi^2}{10} \tanh^5 \Delta \sinh \Delta + \frac{\alpha \eta^4}{2} \]

\[ \tanh^3 \Delta \tanh \Delta \sinh \Delta + \frac{\eta}{12} - \frac{b_3 \epsilon^2}{12} \frac{\eta}{2 \eta^2} + \frac{b_3 \epsilon \eta}{6} - 2 \eta^3 - \frac{\eta^5}{8} \frac{4}{6} \eta^2 + b_3 \epsilon^2 \frac{b_3 \epsilon^4 \xi^2}{2} \tanh \Delta \sinh \Delta. \quad (5.40) \]

\[ \hat{\phi}_1 = \delta_1 \tanh \Delta \sinh \Delta + \frac{\delta_2}{2 \eta} \tanh \Delta \tanh \Delta + \tanh \Delta \sinh \Delta - \frac{\xi^2}{\eta} \]

\[ - \alpha \eta^5 - \frac{49}{10} \tanh^2 \Delta - \frac{14}{5} \cos(2 \Delta) \tanh^4 \Delta \sinh \Delta - \frac{56}{5} \]

\[ \tanh^3 \Delta \tanh^4 \Delta + \frac{21}{2} \tanh^7 \Delta \tanh \Delta - \frac{63}{4} \tanh^5 \Delta \tanh \Delta \]

\[ - \frac{21}{10} \tanh^5 \Delta \tanh \Delta + \frac{49}{20} \tanh^3 \Delta \tanh^2 \Delta + \frac{49}{10} \tanh \Delta \tanh \Delta \]

\[ + \frac{21}{5} \tanh \Delta \tanh^6 \Delta + \frac{21}{5} \tanh \Delta \tanh \Delta \Delta - \frac{36}{35} \cos(2 \Delta) \tanh^3 \Delta \tanh^6 \Delta \]

\[ + \frac{2166}{35} \tanh^3 \Delta \tanh^6 \Delta - \frac{87}{35} \tanh^2 \Delta \Delta + \frac{27}{4} \tanh^9 \Delta \tanh \Delta \]

\[ - 18 \tanh^7 \Delta \tanh \Delta + \frac{27}{2} \tanh^5 \Delta \tanh \Delta - \frac{27}{28} \tanh \tanh \tanh \Delta \]

\[ + \frac{171}{70} \tanh^5 \Delta \tanh^2 \Delta - \frac{87}{70} \tanh^3 \tanh^2 \Delta - \frac{87}{35} \tanh \tanh \Delta \]
- \frac{18}{7} \sech \Delta \tanh \Delta - \frac{18}{7} \sech \Delta \tanh \Delta + \frac{5}{2} \sech^3 \Delta \tanh^2 \Delta

+ 10 \sech \Delta \tanh \Delta - \frac{15}{8} \sech \Delta \tanh^2 \Delta + \frac{45}{16} \Delta \sech^5 \Delta \tanh \Delta

- \frac{45}{8} \sech^3 \Delta \tanh^2 \Delta - \frac{15}{8} \sech \Delta \tanh \Delta - \frac{5}{4} \sech \Delta \tanh^4 \Delta

- \frac{5}{4} \sech \Delta \tanh \Delta + \frac{88}{21} \sech \Delta \tanh^2 \Delta - \frac{55}{7} \sech^7 \Delta \tanh^2 \Delta

- \frac{55}{7} \sech^7 \Delta \tanh^2 \Delta + \frac{11}{7} \sech^5 \Delta \tanh^2 \Delta + \frac{44}{21} \sech^3 \Delta \tanh^2 \Delta

+ \frac{88}{21} \sech \Delta \tanh \Delta + \frac{44}{21} \Delta \sech \Delta \tanh^2 \Delta + \frac{55}{48} \sech^9 \Delta \tanh \Delta

- \frac{275}{126} \sech^7 \Delta \tanh \Delta + \frac{11}{56} \sech^5 \Delta \tanh \Delta + \frac{11}{21} \sech^3 \Delta \tanh \Delta

+ \frac{11}{14} \Delta \sech^5 \Delta \tanh \Delta + \frac{11}{14} \cosh(2\Delta) \sech^3 \Delta \tanh^6 \Delta

+ \frac{22}{21} \Delta \sech^3 \Delta \tanh^2 \Delta + \frac{11}{14} \sech^3 \Delta \tanh^6 \Delta + \frac{36}{35} \sech^3 \Delta \tanh^6 \Delta

+ \frac{6}{35} \cosh(2\Delta) \sech^3 \Delta \tanh^6 \Delta - \frac{29}{70} \sech \Delta \tanh^2 \Delta - 3 \Delta \sech^7 \Delta \tanh \Delta

- \frac{9}{8} \Delta \sech^9 \Delta \tanh \Delta + \frac{9}{4} \Delta \sech^5 \Delta \tanh \Delta - \frac{9}{56} \sech^7 \Delta \tanh^2 \Delta

+ \frac{57}{140} \sech^6 \Delta \tanh^2 \Delta - \frac{29}{70} \sech^3 \Delta \tanh^2 \Delta - \frac{29}{70} \sech \Delta \tanh \Delta

- \frac{3}{7} \sech \Delta \tanh^8 \Delta - \frac{3}{7} \sech \Delta \tanh \Delta - \frac{16}{7} \sech \Delta \tanh^2 \Delta

+ \frac{30}{7} \sech^7 \Delta \tanh^2 \Delta - \frac{6}{7} \sech^5 \Delta \tanh^2 \Delta - \frac{8}{7} \sech^3 \Delta \tanh^2 \Delta

- \frac{16}{7} \sech \Delta \tanh \Delta + \frac{10}{7} \sech \Delta \tanh^2 \Delta + \frac{45}{8} \Delta \sech^9 \Delta \tanh \Delta

- \frac{15}{2} \Delta \sech^7 \Delta \tanh \Delta - \frac{45}{56} \sech^7 \Delta \tanh^2 \Delta - \frac{10}{7} \sech \Delta \tanh^2 \Delta

+ \frac{45}{8} \Delta \sech^9 \Delta \tanh \Delta - \frac{15}{2} \Delta \sech^7 \Delta \tanh \Delta - \frac{45}{56} \sech^7 \Delta \tanh^2 \Delta

+ \frac{15}{28} \sech^5 \Delta \tanh^2 \Delta + \frac{5}{7} \sech^3 \Delta \tanh^2 \Delta + \frac{10}{7} \sech \Delta \tanh \Delta

+ \frac{3}{7} \cosh(2\Delta) \sech^3 \Delta \tanh^6 \Delta + \frac{18}{7} \sech^3 \Delta \tanh^6 \Delta + \frac{184}{35} \Delta \tanh^2 \Delta

- \frac{69}{7} \sech^7 \Delta \tanh^2 \Delta + \frac{69}{35} \sech^5 \Delta \tanh^2 \Delta + \frac{92}{35} \sech^3 \Delta \tanh^2 \Delta

+ \frac{184}{105} \sech \Delta \tanh \Delta - \frac{23}{7} \sech \Delta \tanh^2 \Delta - \frac{207}{4} \Delta \sech^6 \Delta \tanh \Delta

+ \frac{69}{4} \Delta \sech^7 \Delta \tanh \Delta + \frac{207}{112} \sech^7 \Delta \tanh^2 \Delta - \frac{69}{56} \sech^5 \Delta \tanh^2 \Delta
\[
\begin{align*}
- \frac{69}{42} \text{sech}^3 \Delta \tanh^2 \Delta - & \frac{23}{7} \text{sech} \Delta \tanh \Delta - \frac{69}{70} \cosh(2\Delta) \text{sech}^3 \Delta \\
\tanh^6 \Delta + & \frac{9}{70} \cosh(2\Delta) \text{sech}^3 \Delta \tanh^6 \Delta - \frac{207}{35} \text{sech}^3 \Delta \tanh^6 \Delta \\
+ & \frac{27}{35} \text{sech}^3 \Delta \tanh^6 \Delta - \frac{87}{280} \text{sech} \Delta \tanh^2 \Delta + \frac{27}{32} \Delta \text{sech}^9 \Delta \tanh \Delta \\
- & \frac{9}{4} \Delta \text{sech}^7 \Delta \tanh \Delta + \frac{27}{16} \Delta \text{sech}^5 \Delta \tanh \Delta - \frac{27}{224} \text{sech}^7 \Delta \tanh^2 \Delta \\
+ & \frac{1071}{560} \text{sech}^5 \Delta \tanh^2 \Delta - \frac{1071}{1680} \text{sech} \Delta \tanh \Delta - \frac{87}{280} \text{sech} \Delta \tanh \Delta \\
- & \frac{9}{28} \text{sech} \Delta \tanh^8 \Delta - \frac{9}{28} \text{sech} \Delta \tanh \Delta - \frac{b_3 \epsilon^4}{12} - \eta^3 \\
- & \frac{28}{3} \text{sech} \Delta \tanh^4 \Delta + \frac{28}{3} \text{sech} \Delta \tanh \Delta - 14 \text{sech} \Delta \tanh^2 \Delta \\
- & \frac{21}{2} \Delta \text{sech}^5 \Delta \tanh \Delta + 21 \Delta \text{sech}^3 \Delta \tanh \Delta + \frac{14}{4} \text{sech}^3 \Delta \tanh \Delta \\
- & 14 \text{sech} \Delta \tanh \Delta - 14 \text{sech} \Delta \tanh \Delta + \frac{56}{3} \text{sech} \Delta \tanh^2 \Delta \\
- & \frac{14}{3} \text{sech}^3 \Delta \tanh^2 \Delta + \frac{56}{3} \text{sech} \Delta \tanh \Delta + 6 \text{sech} \Delta \tanh^5 \Delta \\
+ & 6 \text{sech} \Delta \tanh \Delta + 23 \text{sech} \Delta \tanh \Delta - 6 \text{sech}^7 \Delta \tanh \Delta \\
+ & 18 \text{sech}^5 \Delta \tanh \Delta - 18 \text{sech}^3 \Delta \tanh \Delta + \frac{6}{5} \text{sech}^5 \Delta \tanh \Delta \\
- & \frac{33}{5} \text{sech}^3 \tanh^2 \Delta + \frac{46}{5} \text{sech} \Delta \tanh \Delta + 12 \text{sech} \Delta \tanh \Delta \\
- & \frac{52}{5} \text{sech} \Delta \tanh^2 \Delta - \frac{12}{5} \text{sech}^5 \Delta \tanh \Delta + \frac{44}{5} \text{sech}^3 \Delta \tanh^2 \Delta \\
- & \frac{92}{5} \text{sech} \Delta \tanh \Delta - 5 \text{sech} \Delta \tanh^2 \Delta - 5 \text{sech} \Delta \tanh \Delta \\
+ & \frac{15}{4} \text{sech} \Delta \tanh^2 \Delta - \frac{15}{4} \Delta \text{sech}^3 \Delta \tanh \Delta + \frac{15}{2} \text{sech} \Delta \tanh \Delta \\
+ & \frac{5}{2} \Delta \text{sech} \Delta \tanh \Delta - \frac{5}{2} \text{sech} \Delta \tanh^2 \Delta - \frac{5}{2} \text{sech} \Delta \tanh \Delta \\
- & \eta \left(6 \xi^2 + b_3 \epsilon^2 - \frac{b_3 \epsilon^4 \xi^2}{2} \right) - \frac{1}{3} \text{sech} \Delta \tanh^4 \Delta + \frac{1}{3} \text{sech} \Delta \tanh \Delta \\
- & \frac{1}{2} \text{sech} \Delta \tanh^2 \Delta - \frac{3}{8} \Delta \text{sech}^5 \Delta \tanh \Delta + \frac{3}{4} \text{sech}^3 \Delta \tanh \Delta \\
+ & \frac{1}{8} \text{sech}^3 \Delta \tanh \Delta - \frac{1}{2} \text{sech} \Delta \tanh \Delta - \frac{1}{2} \Delta \text{sech} \Delta \tanh \Delta \\
+ & \frac{2}{3} \text{sech} \Delta \tanh^2 \Delta - \frac{1}{6} \text{sech}^3 \Delta \tanh^2 \Delta - \frac{2}{3} \text{sech} \Delta \tanh \Delta
\end{align*}
\]
\[
-6\xi^2 + b_3\xi^2 - \frac{b_3\xi^2}{2} - \frac{1}{\eta} - \frac{1}{2} \sech^2 \Delta \tanh^2 \Delta - \frac{2}{3} \sech \Delta \tanh^2 \Delta \\
- \frac{2}{3} \sech \Delta \tanh \Delta + \frac{1}{4} \sech \Delta \tanh^2 \Delta - \frac{3}{8} \sech^5 \Delta \tanh \Delta \\
+ \frac{1}{8} \sech^3 \Delta \tanh^2 \Delta + \frac{1}{4} \sech \Delta \tanh \Delta + \frac{1}{3} \sech \Delta \tanh^4 \Delta \\
+ \frac{1}{3} \sech \Delta \tanh \Delta - \eta \frac{12\xi^2}{3} - 20 - \frac{1}{15} \cos(2\Delta) \sech^3 \Delta \tanh^4 \Delta \\
+ \frac{4}{15} \sech^3 \Delta \tanh^4 \Delta - \frac{7}{60} \sech \Delta \tanh^2 \Delta - \frac{1}{4} \sech^7 \Delta \tanh \Delta \\
+ \frac{3}{8} \sech^5 \Delta \tanh \Delta + \frac{1}{60} \sech^5 \Delta \tanh \Delta - \frac{7}{120} \sech^3 \Delta \tanh^2 \Delta \\
- \frac{7}{60} \sech \Delta \tanh \Delta - \frac{1}{10} \sech \Delta \tanh^6 \Delta - \frac{1}{10} \sech \Delta \tanh \Delta \\
- \frac{8}{15} \sech \Delta \tanh \Delta - \frac{1}{5} \sech^5 \Delta \tanh^2 \Delta - \frac{4}{15} \sech^3 \Delta \tanh^2 \Delta \\
- \frac{8}{15} \sech \Delta \tanh \Delta - 6\alpha \xi^2 - 10 - \frac{4}{15} \sech \Delta \tanh \Delta \\
- \frac{1}{2} \Delta \sech^7 \Delta \tanh \Delta + \frac{1}{10} \sech^5 \Delta \tanh^2 \Delta + \frac{2}{15} \sech^3 \Delta \tanh^2 \Delta \\
+ \frac{4}{15} \sech \Delta \tanh \Delta + \frac{1}{15} \cos(2\Delta) \sech^3 \Delta \tanh^4 \Delta + \frac{4}{5} \sech^3 \Delta \tanh^4 \Delta \\
- \alpha \eta^2 - \frac{176}{7} \sech \Delta \tanh^2 \Delta + \frac{11}{7} \sech^7 \Delta \tanh^2 \Delta + \frac{66}{7} \sech^5 \Delta \tanh^2 \Delta \\
+ \frac{88}{7} \sech^3 \Delta \tanh^2 \Delta + \frac{176}{7} \sech \Delta \tanh \Delta - \frac{33}{7} \sech \Delta \tanh^2 \Delta \\
+ \frac{165}{16} \Delta \sech^9 \Delta \tanh \Delta - \frac{165}{112} \sech^7 \Delta \tanh^2 \Delta - \frac{99}{56} \sech^5 \Delta \tanh^2 \Delta \\
- \frac{33}{14} \sech^3 \Delta \tanh^2 \Delta - \frac{33}{7} \sech \Delta \tanh \Delta - \frac{44}{21} \sech \Delta \tanh^2 \Delta \\
+ \frac{55}{14} \sech^7 \Delta \tanh \Delta - \frac{11}{14} \sech^5 \Delta \tanh \Delta - \frac{22}{21} \sech^3 \Delta \tanh^2 \Delta \\
- \frac{44}{21} \sech \Delta \tanh \Delta + \frac{24}{35} \sech \Delta \tanh^2 \Delta - \frac{9}{7} \sech^7 \Delta \tanh^2 \Delta \\
+ \frac{9}{35} \sech^5 \Delta \tanh^2 \Delta + \frac{24}{35} \sech \Delta \tanh \Delta + \frac{3}{7} \sech \Delta \tanh^2 \Delta \\
+ \frac{27}{16} \Delta \sech^9 \Delta \tanh \Delta - \frac{9}{4} \Delta \sech^7 \Delta \tanh \Delta - \frac{27}{112} \sech^7 \Delta \tanh^2 \Delta \\
+ \frac{9}{56} \sech^5 \Delta \tanh^2 \Delta + \frac{3}{14} \sech^3 \Delta \tanh^2 \Delta + \frac{3}{7} \sech \Delta \tanh \Delta \\
+ \frac{9}{70} \cos(2\Delta) \sech^3 \Delta \tanh^6 \Delta + \frac{27}{35} \sech^3 \Delta \tanh^6 \Delta
\]
\[-\eta^2 \frac{5}{2} \operatorname{sech}^5 \Delta \tanh \Delta + \frac{5}{2} \operatorname{sech} \Delta \tanh \Delta + 2 \Delta \operatorname{sech} \Delta \tanh^2 \Delta \]
\[-\frac{3}{4} \operatorname{sech}^5 \Delta \tanh \Delta + \frac{1}{2} \operatorname{sech}^3 \Delta \tanh \Delta - 2 \log \cosh \Delta \operatorname{sech} \Delta \tanh \Delta \]
\[-3 \Delta \operatorname{sech}^5 \Delta \tanh^2 \Delta - \Delta \operatorname{sech}^3 \tanh \Delta + \frac{5}{4} \operatorname{sech} \Delta \tanh^5 \Delta \]
\[+ \frac{5}{4} \operatorname{sech} \Delta \tanh \Delta \quad - \eta^3 12 \alpha \xi^2 - 30 \quad \frac{1}{15} \cos(2 \Delta) \operatorname{sech}^3 \Delta \tanh^4 \Delta \]
\[+ \frac{4}{15} \operatorname{sech}^3 \Delta \tanh^4 \Delta + \frac{7}{60} \operatorname{sech} \Delta \tanh^2 \Delta + \frac{1}{4} \operatorname{sech}^7 \Delta \tanh \Delta \]
\[-\frac{3}{8} \Delta \operatorname{sech}^5 \Delta \tanh \Delta - \frac{1}{20} \operatorname{sech}^5 \Delta \tanh^2 \Delta + \frac{7}{120} \operatorname{sech}^3 \Delta \tanh^2 \Delta \]
\[+ \frac{7}{60} \operatorname{sech} \Delta \tanh \Delta + \frac{1}{10} \operatorname{sech} \Delta \tanh^6 \Delta + \frac{1}{10} \operatorname{sech} \Delta \tanh \Delta \]
\[-\frac{\alpha}{\eta} \frac{72}{35} \operatorname{sech} \Delta \tanh^2 \Delta + \frac{9}{14} \operatorname{sech}^7 \Delta \tanh^2 \Delta + \frac{27}{35} \operatorname{sech}^5 \Delta \tanh^2 \Delta \]
\[+ \frac{36}{35} \operatorname{sech}^3 \Delta \tanh^2 \Delta + \frac{72}{35} \operatorname{sech} \Delta \tanh \Delta - \frac{27}{70} \operatorname{sech} \Delta \tanh^2 \Delta \]
\[+ \frac{27}{32} \Delta \operatorname{sech}^9 \Delta \tanh \Delta - \frac{27}{224} \operatorname{sech}^7 \Delta \tanh^2 \Delta - \frac{81}{560} \operatorname{sech}^5 \Delta \tanh^2 \Delta \]
\[-\frac{27}{140} \operatorname{sech}^3 \Delta \tanh^2 \Delta - \frac{7}{70} \operatorname{sech} \Delta \tanh \Delta - \frac{12}{70} \operatorname{sech} \Delta \tanh^2 \Delta \]
\[+ \frac{9}{28} \operatorname{sech}^7 \Delta \tanh^2 \Delta - \frac{9}{140} \operatorname{sech}^5 \Delta \tanh^2 \Delta - \frac{3}{35} \operatorname{sech}^3 \Delta \tanh^2 \Delta \]
\[-\frac{6}{35} \operatorname{sech} \Delta \tanh \Delta \quad - \eta^3 20 \xi^2 - \frac{7}{2} \alpha \xi^4 - 24 b_2 + \frac{48}{e^2} \quad \frac{1}{3} \operatorname{sech}^3 \Delta \tanh^2 \Delta \]
\[-\frac{2}{3} \operatorname{sech} \Delta \tanh^2 \Delta + \frac{2}{3} \operatorname{sech} \Delta \tanh \Delta + \frac{1}{4} \operatorname{sech} \Delta \tanh^2 \Delta - \frac{3}{8} \Delta \operatorname{sech}^5 \Delta \tanh \Delta + 8 \operatorname{sech}^3 \Delta \tanh^2 \Delta + \frac{1}{4} \operatorname{sech} \Delta \tanh \Delta + \frac{1}{6} \operatorname{sech} \Delta \tanh^4 \Delta \]
\[-\frac{1}{6} \eta^3 1 - \frac{\xi}{\tau} (\theta - \theta_0) + 2 b - b \epsilon^2 - b_3 \epsilon \xi^4 \quad 12 \omega_0 \quad - \frac{1}{12} - \frac{\epsilon^2}{e^2} \]
\[\quad \operatorname{sech} \Delta \tanh^2 \Delta + \operatorname{sech} \Delta \tanh \Delta + \frac{3}{2} \xi \tau (\theta - \theta_0) - 3 b_1 - \frac{3 b_3 \epsilon \xi^2}{2} + \frac{b_3 \epsilon^4 \xi^4}{8} \]
\[+ \frac{18 \omega_0}{e^4} \quad \frac{2}{2} + \frac{2}{3} \operatorname{sech} \Delta \tanh \Delta - \Delta \operatorname{sech} \Delta \tanh \Delta - \frac{3}{2} \xi \tau (\theta - \theta_0) + \frac{b_1}{2} + \frac{b_3 \epsilon \xi^2}{2} + \frac{b_3 \epsilon^2 \xi^4}{24} + \frac{6 \omega_0}{e^2} \quad \Delta \operatorname{sech} \Delta \tanh \Delta - \operatorname{sech} \Delta \tanh^2 \Delta \]
\[-\text{sech}^2 \Delta \text{tanh} \Delta \cdot -\xi^4 \eta^2 \cdot \frac{3}{4} \text{sech}^2 \Delta \text{tanh} \Delta \tan^{-1} \text{tanh} \left( \frac{\Delta}{2} \right) + \frac{1}{4} \]

\[\text{sech}^2 \Delta \tan^2 \Delta + \frac{3}{8} \text{sech}^2 \Delta \tan^2 \Delta + \frac{3}{4} \text{sech} \Delta \text{tanh} \Delta \tan^{-1} \left( \frac{\Delta}{2} \right) \]

\[+ \frac{3}{10} \Delta \text{sech}^6 \Delta \tan^2 \Delta + \frac{9}{40} \text{sech} \Delta \text{tanh} \Delta \tan^{-1} \left( \frac{\Delta}{2} \right) \]

\[- \frac{3}{40} \text{sech}^4 \Delta \text{tanh}^2 \Delta - \frac{9}{80} \text{sech}^2 \Delta \text{tanh}^2 \Delta - \frac{9}{40} \text{sech} \Delta \text{tanh} \Delta \tan^{-1} \left( \frac{\Delta}{2} \right) \]

\[- \frac{1}{8} \text{sech} \Delta \text{tanh} \Delta \tan^{-1} \left( \frac{\Delta}{2} \right) - \frac{1}{8} \text{sech} \Delta \text{tanh} \Delta \tan^{-1} \left( \frac{\Delta}{2} \right) - 40 \alpha \xi^2 - 60 \eta^3 \]

\[- \frac{8}{15} \text{sech} \Delta \text{tanh}^2 \Delta + \frac{1}{5} \text{sech}^5 \Delta \text{tanh}^2 \Delta + \frac{4}{15} \text{sech}^3 \Delta \text{tanh}^2 \Delta \]

\[+ \frac{8}{15} \text{sech} \Delta \text{tanh}^2 \Delta + \frac{2}{15} \text{sech} \Delta \text{tanh}^2 \Delta - \frac{1}{4} \Delta \text{sech} \Delta \text{tanh} \Delta \]

\[+ \frac{1}{20} \text{sech}^5 \Delta \text{tanh}^2 \Delta + \frac{1}{15} \text{sech}^3 \Delta \text{tanh}^2 \Delta + \frac{2}{15} \text{sech} \Delta \text{tanh} \Delta \]

\[+ \frac{2}{15} \text{sech}^3 \Delta \text{tanh}^4 \Delta + \frac{1}{30} \cos(2\Delta) \text{sech}^3 \Delta \text{tanh}^4 \Delta - \alpha \xi^4 - \frac{1}{4} \text{sech}^2 \Delta \]

\[\tan^2 \Delta + \frac{1}{4} \Delta \text{sech}^4 \Delta \tan^2 \Delta + \frac{161}{6} \alpha \eta^5 + \frac{35}{3} \eta^3 + \frac{20}{3} \alpha \xi^2 \eta^3 + 2 \alpha \xi^2 - \frac{10}{3} \]

\[\text{sech}^7 \Delta = \frac{139}{4} \alpha \eta^5 + \frac{5}{2} \eta^3 + \frac{15}{4} \eta^5 - \frac{3}{2} \xi^2 + \frac{b_3 e^2}{4} - \frac{b_3 e^4 \xi^2}{8} - \frac{1}{\eta^2} - \frac{5}{2} \eta^2 - \frac{7}{8} \alpha \eta^4 \]

\[-6 b_2 + \frac{12}{\eta^2} \eta \text{sech}^5 \Delta - \frac{103}{2} \alpha \eta^5 + \frac{55}{8} \alpha \eta^2 + \frac{9 \alpha}{16 \eta} \text{sech}^3 \Delta + \frac{3 \xi^2}{2} + \frac{b_3 e^2}{4} \]

\[-\frac{b_3 e^4 \xi^2}{8} \eta \text{sech}^4 \Delta - \frac{b_3 e^4}{3} - 4 \eta^3 \text{sech} \Delta \text{tanh}^6 \Delta - \frac{5}{2} - \frac{5}{24} b_3 e^4 \eta^3 \]

\[\text{sech}^3 \Delta \text{tanh} \Delta - \frac{2}{3} \eta^2 \cos(2\Delta) \text{sech}^3 \Delta \text{tanh}^3 \Delta - \frac{8}{3} \eta^2 \text{sech}^3 \Delta \text{tanh}^3 \Delta \]

\[- \xi \left( \theta - \theta_0 \right) - 2 b_1 + b_3 e^2 \xi^2 + \frac{b_3 e^4 \xi^4}{12} + \frac{12 \omega_0}{e^2} \eta \text{sech}^2 \Delta - \frac{\xi^4 \eta^2 \text{sech}^6 \Delta}{4} \]

\[-\frac{\alpha e^4 \text{sech}^4 \Delta}{6} + \frac{7 b_3 e^4}{12} - \frac{1}{\eta^2} \text{sech} \Delta + 6 \xi^2 + b_3 e^2 - \frac{b_3 e^4 \xi^2}{2} \eta^2 \text{sech} \Delta \]

\[-\frac{b_3 e^4 \eta^5}{3} - 4 \eta^5 \text{sech} \Delta + \frac{75}{4} \alpha \eta^5 - \frac{25}{2} \eta^3 - 3 \alpha \xi^2 - 10 \alpha \xi^2 \eta^3 + 5 \Delta \text{sech}^7 \Delta \]
\[
\begin{align*}
tanh \Delta + & \left( \frac{15}{4} \eta^3 + \frac{45}{8} \eta^5 - \frac{849}{16} \alpha \eta^5 - \frac{\xi^2}{\eta^2} - \frac{3}{8} b_3 \epsilon^2 - \frac{3}{16} \epsilon^4 \xi^2 - \frac{15 \xi^2}{2 \eta} - \frac{21 \alpha \eta \xi^4}{16} \right) \\
- 9b_2 \eta + & \left( \frac{18 \eta}{\epsilon^2} \Delta \text{sech}^5 \Delta \tanh \Delta + \frac{165}{16} \alpha \eta^2 + \frac{27 \alpha}{32 \eta} + \frac{309 \alpha \eta^5}{2} \Delta \text{sech}^9 \Delta \tanh \Delta \right) \\
- \eta & \left( \frac{9 \xi^2}{4} + \frac{3b_3 \epsilon^2}{8} - \frac{3b_3 \epsilon^4 \xi^2}{16} \right) \Delta \text{sech} \Delta \tanh^5 \Delta + \frac{b_3 \epsilon^4}{2} - 6 \eta^3 \Delta \text{sech} \Delta \tanh^7 \Delta \\
+ & \frac{15 \eta^3}{4} - \frac{5 \eta^3 b_3 \epsilon^4}{16} \Delta \text{sech}^3 \Delta \tanh \Delta + 4 \eta^2 \Delta \text{sech}^3 \Delta \tanh^4 \Delta + \eta^2 \Delta \cos(2 \Delta) \\
\Delta \text{sech}^3 \Delta \tanh^4 \Delta - & \frac{3}{2 \eta} \xi_1 (\theta - \theta_0) - 2b_1 + \frac{b_3 \epsilon^2 \xi^2}{12} + \frac{12 \omega_0}{\epsilon^2} \\
\Delta \text{sech}^2 \Delta \tanh \Delta + & \frac{3}{10} \xi^4 \eta^2 \Delta \text{sech}^6 \Delta \tanh \Delta + \frac{1}{4} \alpha \epsilon^4 \Delta \text{sech}^4 \Delta \tanh \Delta \\
+ & \frac{3}{2 \eta} - \frac{7}{12} b_3 \epsilon^4 - \frac{3 \eta}{8} \Delta \tanh^5 \Delta + \frac{5 \alpha \epsilon^2 \eta^3}{6} + 4 \eta^3 \\
\Delta \text{sech} \Delta \tanh \Delta + & - \frac{125 \alpha \eta^5}{12} - \frac{35 \eta^3}{6} - \frac{\alpha \xi^2}{3} + \frac{5 \alpha \xi^2 \eta^3}{3} \\
\Delta \text{sech} \Delta \tanh \Delta & - \frac{7}{16} \alpha \xi^4 - \frac{3b_3 \epsilon^4}{12} + \frac{6 \eta}{\epsilon^2} \eta \Delta \text{sech}^4 \Delta \tanh \Delta \sinh \Delta + \frac{1}{2} \frac{\alpha \eta^5}{16} + \frac{55}{\alpha \eta^2} \Delta + \frac{9 \alpha \eta^2}{32 \eta} \\
\Delta \text{sech}^6 \Delta \tanh \Delta \sinh \Delta & - \frac{3 \xi^2}{4} + \frac{b_3 \epsilon^2}{8} - \frac{b_3 \epsilon^4 \xi^2}{16} \tanh^5 \Delta \sinh \Delta + \frac{b_3 \epsilon^4}{6} + 2 \\
\eta^3 \tanh^7 \Delta \sinh \Delta & - \frac{5}{4} - \frac{b_3 \epsilon^4}{48} \eta^3 \Delta \sinh \Delta + \frac{4 \eta^2 \Delta \sinh \Delta}{3} \\
\tanh^4 \Delta \sinh \Delta & + \frac{\eta^2}{3} \cos(2 \Delta) \Delta \text{sech}^2 \Delta \tanh^4 \Delta \sinh \Delta - \frac{\xi_1 (\theta - \theta_0)}{2} - 2b_1 \\
+ b_3 \epsilon^2 \xi^2 + & \frac{b_3 \epsilon^4 \xi^4}{12} + \frac{12 \omega_0}{\epsilon^2} \frac{1}{2 \eta} \Delta \text{sech} \Delta \tanh \Delta \sinh \Delta + \frac{\xi_4 \eta^2}{10} \Delta \text{sech}^5 \Delta \tanh \Delta \sinh \Delta \\
+ & \frac{\alpha \epsilon^4 \Delta \text{sech}^3 \Delta \tanh \Delta \sinh \Delta + \frac{7}{12} \frac{b_3 \epsilon^4}{2 \eta^2} + \frac{b_3 \epsilon^4 \eta^3}{6} - 2n^3 - \frac{n \Delta \xi^2}{8} + \frac{b_3 \epsilon^4}{2} \
- \frac{b_3 \epsilon^4 \xi^2}{2} & \tanh \Delta \sinh \Delta.
\end{align*}
\]
\[ \delta_1 = 2.15 \alpha \eta^5 - 15.4 \frac{b_3 \varepsilon^4}{12} - 1 \eta^3 + 0.45 \frac{12 \alpha \xi^2}{2} - 20 \eta^5 - 0.1 \frac{40 \alpha \xi^2}{5} \]
\[ -60 \eta^5 - 5 \frac{8}{6 \xi^2} + 3 \varepsilon^2 - \frac{b_3 \varepsilon^2}{2} \eta + 7 \frac{20 \xi^2}{2} - 24 b_2 + 48 \frac{\eta}{\xi^2} \]
\[ -14.65 \alpha \eta^2 - 4 \eta^2 - 0.12 \tan^{-1} \left( \frac{1}{2} \right) \xi^4 \eta^2 + \frac{16}{15} 12 \alpha \xi^2 - 20 - \frac{4}{15} 6 \alpha \xi^2 \]
\[ -10 + 3.15 \alpha + \frac{1}{4} \tan^{-1} \left( \frac{1}{2} \right) \alpha \xi^4 + \frac{3}{4} \xi \tan (\theta - \theta_0) - 2 b_1 + b_3 \varepsilon^2 \xi^2 \]
\[ + \frac{b_3 \varepsilon^4 \xi^4}{12} + 12 \omega_0 \eta + \frac{5}{2} - \frac{5}{24} b_3 \varepsilon^4 \eta^3 \]
\[ (5.42) \]

\[ \delta_2 = \frac{-353}{8} \alpha - \frac{2 b_3 \varepsilon^4}{3} + \frac{17}{4} \eta^6 + \frac{40}{3} \alpha \xi^2 + \frac{85}{3} \eta^4 + \frac{-b_3 \varepsilon^2}{2} \frac{55}{4} \alpha - \frac{9 \xi^4}{20} \]
\[ + 3 \xi^2 - \frac{b_3 \varepsilon^4 \xi^2}{8} \eta^3 + \frac{10 \xi^2 - 12 b_2 - 2 \xi \tan (\theta - \theta_0) + 4 b_1 + 2 b_3 \varepsilon^2 \xi^2}{2} \]
\[ - \frac{b_3 \varepsilon^4 \xi^4}{6} - \frac{24 \omega_0}{2} \frac{b_3 \varepsilon^2 \xi^2}{4} \eta^2 + \frac{4 \alpha \xi^2 - \alpha \xi^4}{3} - \frac{20}{3} \eta + \frac{3 \xi^2}{2} + \frac{b_3 \varepsilon^2}{4} \]
\[ - \frac{b_3 \varepsilon^4 \xi^2}{8} + \frac{7 b_3 \varepsilon^4}{12} - \frac{7}{1} - \frac{9 \alpha}{16} \]
\[ (5.43) \]

\[ \psi_{11} \quad \psi_{12} \quad \text{Im} \quad \hat{F}_1 d \theta' = \frac{4 \xi + 4 \alpha \xi - \frac{b_3 \varepsilon^4 \xi}{2} \eta^2 - \text{sech}^3 \Delta}{2} \]
\[ + \frac{7}{2} \text{sech} \Delta \tanh \Delta - \text{sech}^5 \Delta - \frac{1}{6} \text{sech}^3 \Delta \tanh \Delta + 4 \text{sech} \Delta \]
\[ - \frac{b_3 \varepsilon^4 \xi}{3} - 4 \xi \eta^2 \text{sech} \Delta + \frac{1}{3} \text{sech}^3 \Delta \tanh \Delta - 5 \frac{\text{sech} \Delta \tanh^3 \Delta}{3} \]
\[ \text{sech} \Delta \frac{4}{5} \text{sech} \Delta \tanh^5 \Delta + \frac{4}{3} \cos(2 \Delta) \text{sech}^3 \Delta \tanh^3 \Delta \]
\[ + \frac{289}{9} \text{sech} \Delta + \frac{16}{3} \text{sech}^3 \Delta \tanh^3 \Delta + \frac{118}{45} \text{sech} \Delta \tanh \Delta \]
\[ - \frac{26}{3} \text{sech}^7 \Delta + \frac{26}{15} \text{sech}^5 \Delta \tanh \Delta + \frac{59}{45} \text{sech}^3 \Delta \tanh \Delta + 3 \text{sech}^5 \Delta \]
\[ - 3 \text{sech} \Delta \tanh^4 \Delta + 40 \xi - 16 \alpha \xi^3 \eta^2 \frac{1}{6} \text{sech} \Delta \tanh \Delta - \frac{1}{4} \text{sech}^3 \Delta \]
\[ + \frac{1}{12} \text{sech}^3 \Delta \tanh \Delta + \frac{1}{3} \text{sech} \Delta \tanh^3 \Delta + \frac{1}{3} \text{sech} \Delta - 4 \xi^3 - \frac{1}{2} \text{sech} \Delta \tanh \Delta \]
\[
\begin{align*}
\frac{1}{2} \text{sech}\Delta - \frac{1}{2} \Delta \text{sech}^3\Delta + \Delta \text{sech}\Delta &= -2b_3\epsilon\xi^2 - \frac{1}{2} \text{sech}\Delta \tanh\Delta \\
-\frac{1}{2} \Delta \text{sech}^3\Delta + \Delta \text{sech}\Delta - \frac{1}{2} \text{sech}\Delta + \frac{b_3\epsilon^4\xi^3}{3} &= \text{sech}\Delta - \frac{\eta_\tau}{\eta^2} - \frac{\Delta^2 \text{sech}^3\Delta}{2} \\
-\frac{1}{2} \text{sech}\Delta \tanh\Delta - \frac{1}{2} \Delta \text{sech}^3\Delta + \Delta \text{sech}\Delta &= \frac{\eta_\tau}{\eta^2} - \frac{\Delta^2 \text{sech}^3\Delta}{2} \\
-\Delta \text{sech}\Delta \tanh\Delta + \frac{\Delta^2 \text{sech}\Delta}{2} &= \theta - \theta_0 - \frac{\text{sech}\Delta \tanh\Delta}{2} \\
-\frac{\Delta \text{sech}^3\Delta}{2} - \frac{\text{sech}\Delta}{2} + \Delta \text{sech}\Delta &= , \tag{5.44}
\end{align*}
\]

\[
\begin{align*}
\psi_{12} &= \psi_{11} \text{Im} \int_1 \text{d}\theta' = \frac{\xi}{2} - \frac{b_3\epsilon^4\xi}{24} \eta^2 \Delta \text{sech}\Delta \tanh^6\Delta + \Delta \text{sech}\Delta \\
+5\Delta \text{sech}\Delta + \tanh^6\Delta \sinh\Delta + \sinh\Delta + 5\text{sech}^4\Delta \sinh\Delta \\
+\alpha\xi^4 - \frac{4}{3} \Delta \text{sech}^7\Delta + \Delta \text{sech}^5\Delta - \Delta \text{sech}^4\Delta + \frac{1}{3} \text{sech}^6\Delta \sinh^6\Delta \\
-\frac{7}{3} \text{sech}^6\Delta \sinh\Delta + \text{sech}^4\Delta \sinh\Delta &= -\frac{b_3\epsilon^4\xi^3}{12} - \frac{\xi^3 - b_3\epsilon^2\xi}{2} \Delta \text{sech}^3\Delta \\
+\text{sech}^2\Delta \sinh\Delta &= -\eta^2 - 5\xi - 2\alpha\xi^3 \Delta \text{sech}^5\Delta + \text{sech}^4\Delta \sinh\Delta . \tag{5.45}
\end{align*}
\]

\[
\begin{align*}
\hat{\psi}_1 &= \frac{34}{3} \xi^2 - \frac{14}{3} \alpha\xi^2 - \frac{118}{45} \alpha\xi^4 - 2\xi^3 + 3\epsilon^4\xi^3 + \frac{1}{3} b_3\epsilon^4\xi\eta^3 \\
+ \frac{5}{18} b_3\epsilon^4\xi^3 + \frac{8}{3} \alpha\xi^3\eta^2 - 2b_3\epsilon^2\xi + \frac{\theta - \theta_0}{2} \text{sech}\Delta \tanh\Delta \\
+ \delta_3 - 18\xi\eta^2 - 14\alpha\xi^2 + \frac{3}{2} b_3\epsilon^4\xi^3 + \frac{227}{15} \alpha\xi^4 + 8\alpha\xi^3\eta^2 \\
+ \frac{2}{3} b_3\epsilon^4\xi^3 - 2\xi^3 - 2b_3\epsilon^2\xi + \frac{(\theta - \theta_0)\eta}{2} \text{sech}\Delta + \frac{\delta_4}{2\eta} + \frac{1}{3} b_3\epsilon^4\xi
\end{align*}
\]
\[ -\frac{7}{2} \xi \eta^2 + 4 \xi^3 + 2 b_3 \epsilon^2 \xi - \frac{1}{3} b_3 \epsilon^4 \xi^3 - \frac{1}{24} b_3 \epsilon^4 \xi^2 - (\theta - \theta_0) \tau \]

\[ \Delta \sech \Delta + \frac{23}{3} \xi \eta^2 - 3 b_3 \epsilon^4 \xi \eta^2 + \frac{14}{3} \alpha \xi \eta^2 - 2 \alpha \xi^3 \eta^2 - 2 \alpha \xi^4 \]

\[ - \frac{5}{24} b_3 \epsilon^4 \eta^2 \sech^5 \Delta + 2 \xi \eta^2 + 2 \alpha \xi \eta^2 - \frac{1}{6} b_3 \epsilon^4 \xi - \xi^3 - b_3 \epsilon^2 \]

\[ - \frac{1}{2} b_3 \epsilon^4 \xi^3 + \frac{1}{2} b_3 \epsilon^2 \xi - \frac{(\theta - \theta_0) \tau}{3} \Delta \sech \Delta + \frac{13}{3} \xi \eta^2 \]

\[ + \frac{1}{3} \alpha \xi \eta^2 - \frac{5}{36} \epsilon^4 \xi \eta^2 + \frac{4}{3} \alpha \xi^3 \eta^2 - \frac{59}{45} \alpha \xi^4 \sech^3 \Delta \tanh \Delta \]

\[ + \frac{20}{3} \xi \eta^2 - \frac{5}{9} b_3 \epsilon^4 \xi \eta^2 + \frac{16}{3} \alpha \xi^3 \eta^2 \sech \Delta \tanh \Delta - \frac{112}{15} \alpha \xi^4 \]

\[ \sech^3 \Delta \tanh^3 \Delta \Delta \sech \Delta + \frac{\eta_\tau}{\eta^2} \sech \Delta \tanh \Delta - \frac{4}{3} \alpha \xi^4 \]

\[ \cos(2 \Delta) \sech \Delta \tanh^3 \Delta - \frac{4}{5} \alpha \xi \eta^4 \sech \Delta \tanh \Delta + \frac{\delta_4}{2 \eta} - \frac{1}{2} b_3 \epsilon^4 \xi \eta^2 \]

\[ + \frac{\xi \eta^2}{2} \sinh \Delta - \frac{26}{15} \alpha \xi \eta^4 \sech \Delta \tanh \Delta + 2 \alpha \xi^4 \sech \Delta \]

\[ + 3 \alpha \xi \eta^4 \sech \Delta \tanh^4 \Delta + \frac{\eta_\tau}{2n^2} \Delta^2 \sech \Delta + \frac{\xi \eta^2}{2} - \frac{1}{24} b_3 \epsilon^4 \xi \eta^2 \]

\[ \Delta \sech \Delta \tanh^6 \Delta - \alpha \xi \eta^4 \sech^4 \Delta + \frac{5}{2} \xi \eta^2 - \frac{5}{8} b_3 \epsilon^4 \eta^2 + \alpha \xi^4 \]

\[ + 2 \alpha \xi^3 \eta^2 \sech^4 \Delta \sinh \Delta + \frac{1}{24} b_3 \epsilon^4 \xi \eta^2 + \frac{1}{2} \xi \eta^2 \tanh^6 \Delta \sinh \Delta \]

\[ - \frac{7}{3} \alpha \xi \eta^4 \sech \Delta \sinh \Delta + \frac{\xi^3}{2} b_3 \epsilon^2 \xi - \frac{1}{12} \sech^2 \Delta \sinh \Delta . \] (5.46)

\[ \delta_3 = \frac{b_3 \epsilon^4 \xi}{3} - 2 \alpha + 2 \alpha \xi \eta^2 - \xi \lambda + \frac{5 b_3 \epsilon^4 \xi \eta^2}{9} - \frac{80 \xi \eta^2}{3} + \frac{53}{45} \alpha \xi \eta^4 - 2 \xi^3 \]

\[ + 8 \alpha \xi^3 \eta^2 + b_3 \epsilon^2 + b_3 \epsilon^2 \xi - \frac{2 b_3 \epsilon^4 \xi^3}{3}, \] (5.47)
\[
\delta_4 = -\frac{32\xi - \frac{5}{2}b_3\epsilon^4\xi + 6\alpha\xi - 8\alpha\xi^3\eta^3 + \frac{1728}{135}\alpha\xi\eta^5 + \frac{3}{8}b_3\epsilon^4\xi^3 - 4\xi^3 - 2b_3\epsilon^2\xi\eta}{2\eta}.
\]

(5.48)

\[
\dot{\xi}_1 = \delta_1\text{sech}\Delta\tanh\Delta + \frac{\delta_2}{2\eta} - 3\Delta\text{sech}\Delta\tanh\Delta + \tanh\Delta\sinh\Delta - \frac{\delta_2}{\eta}\text{sech}\Delta
\]

\[
-\alpha\eta^5 - \frac{49}{10}\text{sech}\Delta\tanh^2\Delta - \frac{14}{5}\cosh(2\Delta)\text{sech}^3\Delta\tanh^4\Delta - \frac{56}{5}
\]

\[
\text{sech}^3\Delta\tanh^4\Delta + \frac{21}{2}\Delta\text{sech}^7\Delta\tanh\Delta - \frac{63}{4}\Delta\text{sech}^5\Delta\tanh\Delta
\]

\[
- \frac{21}{10}\text{sech}\Delta\tanh\Delta + \frac{49}{20}\text{sech}^3\Delta\tanh^2\Delta + \frac{49}{10}\text{sech}\Delta\tanh\Delta
\]

\[
+ \frac{21}{5}\text{sech}\Delta\tanh^6\Delta + \frac{21}{5}\text{sech}\Delta\tanh^6\Delta - \frac{36}{35}\cosh(2\Delta)\text{sech}^3\Delta\tanh^6\Delta
\]

\[
+ \frac{216}{35}\text{sech}^3\Delta\tanh^6\Delta - \frac{87}{35}\text{sech}\Delta\tanh^2\Delta + \frac{27}{4}\Delta\text{sech}^9\Delta\tanh\Delta
\]

\[
- 18\Delta\text{sech}^7\Delta\tanh\Delta + \frac{27}{2}\Delta\text{sech}^5\Delta\tanh\Delta - \frac{27}{28}\text{sech}^7\Delta\tanh^2\Delta
\]

\[
+ \frac{171}{70}\text{sech}^5\Delta\tanh^2\Delta - \frac{87}{70}\text{sech}^3\Delta\tanh^2\Delta - \frac{87}{35}\text{sech}\Delta\tanh\Delta
\]

\[
- \frac{18}{7}\text{sech}\Delta\tanh^8\Delta - \frac{18}{7}\text{sech}\Delta\tanh\Delta + \frac{5}{2}\text{sech}^3\Delta\tanh^2\Delta
\]

\[
+ 10\text{sech}\Delta\tanh\Delta - \frac{15}{8}\text{sech}\Delta\tanh^2\Delta + \frac{45}{16}\Delta\text{sech}^5\Delta\tanh\Delta
\]

\[
- \frac{45}{8}\text{sech}^3\Delta\tanh^2\Delta - \frac{15}{8}\text{sech}\Delta\tanh\Delta - \frac{5}{4}\text{sech}\Delta\tanh^4\Delta
\]

\[
- \frac{5}{4}\text{sech}\Delta\tanh\Delta + \frac{88}{21}\text{sech}\Delta\tanh^2\Delta - \frac{55}{7}\text{sech}^7\Delta\tanh^2\Delta
\]

\[
- \frac{55}{7}\text{sech}\Delta\tanh^2\Delta + \frac{11}{7}\text{sech}^5\Delta\tanh^2\Delta + \frac{44}{21}\text{sech}^3\Delta\tanh^2\Delta
\]

\[
+ \frac{88}{21}\text{sech}\Delta\tanh\Delta + \frac{44}{21}\Delta\text{sech}\Delta\tanh^2\Delta + \frac{55}{48}\text{sech}^6\Delta\tanh\Delta
\]

\[
- \frac{275}{126}\text{sech}^7\Delta\tanh\Delta + \frac{11}{56}\text{sech}\Delta\tanh^2\Delta + \frac{11}{21}\text{sech}^3\Delta\tanh^2\Delta
\]

\[
+ \frac{11}{14}\Delta\text{sech}^5\Delta\tanh^2\Delta + \frac{11}{14}\cosh(2\Delta)\text{sech}^3\Delta\tanh^6\Delta
\]

\[
- \frac{22}{21}\Delta\text{sech}^3\Delta\tanh^2\Delta + \frac{11}{14}\text{sech}^3\Delta\tanh^6\Delta + \frac{36}{35}\text{sech}^3\Delta\tanh^6\Delta
\]

\[
+ \frac{6}{35}\cosh(2\Delta)\text{sech}^3\Delta\tanh^6\Delta - \frac{29}{70}\text{sech}\Delta\tanh^2\Delta - 3\Delta\text{sech}^7\Delta\tanh\Delta
\]
\[\begin{align*}
+ \frac{9}{8} \Delta \text{sech}^3 \Delta \tanh \Delta &+ \frac{9}{4} \Delta \text{sech}^5 \Delta \tanh \Delta - \frac{9}{56} \Delta \text{sech}^7 \Delta \tanh^2 \Delta \\
+ \frac{57}{140} \Delta \text{sech}^5 \Delta \tanh^2 \Delta &- \frac{29}{140} \Delta \text{sech}^3 \Delta \tanh^2 \Delta - \frac{29}{70} \Delta \text{sech} \Delta \tanh \Delta \\
- \frac{3}{7} \Delta \text{sech} \Delta \tanh^8 \Delta &- \frac{3}{7} \Delta \text{sech} \Delta \tanh \Delta - \frac{16}{7} \Delta \text{sech} \Delta \tanh^2 \Delta \\
- \frac{30}{7} \Delta \text{sech}^7 \Delta \tanh^2 \Delta &- \frac{6}{7} \Delta \text{sech}^5 \Delta \tanh^2 \Delta - \frac{8}{7} \Delta \text{sech}^3 \Delta \tanh^2 \Delta \\
- \frac{16}{7} \Delta \text{sech} \Delta \tanh \Delta + \frac{10}{7} \Delta \text{sech} \Delta \tanh^2 \Delta + \frac{45}{8} \Delta \text{sech}^9 \Delta \tanh \Delta \\
- \frac{15}{2} \Delta \text{sech}^7 \Delta \tanh \Delta &- \frac{45}{56} \Delta \text{sech}^7 \Delta \tanh^2 \Delta - \frac{10}{7} \Delta \text{sech} \Delta \tanh^2 \Delta \\
+ \frac{8}{2} \Delta \text{sech}^9 \Delta \tanh \Delta &- \frac{15}{2} \Delta \text{sech}^7 \Delta \tanh \Delta - \frac{45}{56} \Delta \text{sech}^7 \Delta \tanh^2 \Delta \\
+ \frac{15}{28} \Delta \text{sech}^5 \Delta \tanh^2 \Delta &+ \frac{5}{7} \Delta \text{sech}^3 \Delta \tanh^2 \Delta + \frac{10}{7} \Delta \text{sech} \Delta \tanh \Delta \\
- \frac{3}{7} \Delta \text{cosh}(2 \Delta) \Delta \text{sech}^3 \Delta \tanh^6 \Delta &+ \frac{18}{7} \Delta \text{sech}^3 \Delta \tanh^6 \Delta + \frac{184}{35} \Delta \tanh^2 \Delta \\
- \frac{69}{7} \Delta \text{sech}^7 \Delta \tanh^2 \Delta &+ \frac{69}{35} \Delta \text{sech}^5 \Delta \tanh^2 \Delta + \frac{92}{35} \Delta \text{sech}^3 \Delta \tanh^2 \Delta \\
+ \frac{184}{105} \Delta \text{sech} \Delta \tanh \Delta &- \frac{23}{7} \Delta \text{sech} \Delta \tanh^2 \Delta - \frac{207}{4} \Delta \text{sech}^9 \Delta \tanh \Delta \\
+ \frac{69}{4} \Delta \text{sech}^7 \Delta \tanh \Delta &+ \frac{207}{112} \Delta \text{sech}^7 \Delta \tanh^2 \Delta - \frac{61}{56} \Delta \text{sech}^5 \Delta \tanh^2 \Delta \\
- \frac{69}{7} \Delta \text{sech}^3 \Delta \tanh^2 \Delta &- \frac{23}{7} \Delta \text{sech} \Delta \tanh \Delta - \frac{69}{70} \Delta \text{cosh}(2 \Delta) \text{sech}^3 \Delta \tanh^6 \Delta \\
+ \frac{9}{70} \Delta \text{cosh}(2 \Delta) \text{sech}^3 \Delta \tanh^6 \Delta &- \frac{207}{35} \Delta \text{sech}^3 \Delta \tanh^6 \Delta \\
+ \frac{27}{35} \Delta \text{sech}^5 \Delta \tanh^6 \Delta &- \frac{87}{280} \Delta \text{sech} \Delta \tanh^2 \Delta + \frac{27}{32} \Delta \text{sech}^9 \Delta \tanh \Delta \\
- \frac{9}{4} \Delta \text{sech}^7 \Delta \tanh \Delta &+ \frac{27}{16} \Delta \text{sech}^5 \Delta \tanh \Delta - \frac{27}{224} \Delta \text{sech}^7 \Delta \tanh^2 \Delta \\
+ \frac{1071}{560} \Delta \text{sech}^5 \Delta \tanh^2 \Delta &- \frac{1071}{1680} \Delta \text{sech} \Delta \tanh \Delta - \frac{87}{280} \Delta \text{sech} \Delta \tanh \Delta \\
- \frac{9}{28} \Delta \text{sech} \Delta \tanh^8 \Delta &- \frac{9}{28} \Delta \text{sech} \Delta \tanh \Delta - \frac{b_3 \epsilon^4}{12} - \eta^3 \\
- \frac{28}{3} \Delta \text{sech} \Delta \tanh^4 \Delta &+ \frac{28}{3} \Delta \text{sech} \Delta \tanh \Delta - 14 \Delta \text{sech} \Delta \tanh^2 \Delta \\
- \frac{21}{2} \Delta \text{sech}^5 \Delta \tanh \Delta &+ 21 \Delta \text{sech}^3 \Delta \tanh \Delta + \frac{14}{4} \Delta \text{sech}^3 \Delta \tanh \Delta \\
- 14 \Delta \text{sech} \Delta \tanh \Delta - 14 \Delta \text{sech} \Delta \tanh \Delta + \frac{56}{3} \Delta \text{sech} \Delta \tanh^2 \Delta
\end{align*}\]
$-\frac{14}{3} \sech^3 \tanh^2 \Delta + \frac{56}{3} \sech \tanh \Delta + 6 \sech \tanh^5 \Delta$

$-6 \sech \tanh \Delta + 23 \sech \tanh^2 \Delta - 6 \sech^7 \tanh \Delta$

$+18 \Delta \sech^5 \tanh \Delta - 18 \Delta \sech^3 \tanh \Delta + \frac{6}{5} \sech^5 \tanh \Delta$

$-\frac{33}{5} \sech^3 \tanh^2 \Delta + \frac{46}{5} \sech \tanh \Delta + 12 \sech \tanh \Delta$

$-\frac{52}{5} \sech \tanh^2 \Delta - \frac{12}{5} \sech^5 \tanh \Delta + \frac{44}{5} \sech^3 \tanh^2 \Delta$

$-\frac{15}{4} \sech \tanh^2 \Delta - \frac{15}{4} \Delta \sech^3 \tanh \Delta + \frac{15}{2} \sech \tanh \Delta$

$+\frac{5}{2} \Delta \sech \tanh \Delta - \frac{5}{2} \sech \tanh^2 \Delta - \frac{5}{2} \sech \tanh \Delta$

$-\eta \left( 6 \xi^2 + b_3 \epsilon^2 - \frac{b_3 \epsilon^4 \xi^2}{2} - \frac{1}{3} \sech^4 \tanh \Delta + \frac{1}{3} \sech \tanh \Delta \right)$

$-\frac{1}{2} \sech \tanh^2 \Delta - \frac{3}{8} \Delta \sech^5 \tanh \Delta + \frac{3}{4} \Delta \sech^3 \tanh \Delta$

$+\frac{1}{8} \sech^3 \tanh \Delta - \frac{1}{2} \sech \tanh \Delta - \frac{1}{2} \Delta \sech \tanh \Delta$

$+\frac{2}{3} \sech \tanh^2 \Delta - \frac{1}{6} \sech^3 \tanh^2 \Delta - \frac{2}{3} \sech \tanh \Delta$

$-6 \xi^2 + b_3 \epsilon^2 - \frac{b_3 \epsilon^4 \xi^2}{2} - \frac{1}{\eta} \left( -\frac{1}{2} \sech^3 \tanh^2 \Delta - \frac{2}{3} \sech \tanh^2 \Delta \right)$

$-\frac{2}{3} \sech \tanh \Delta + \frac{1}{4} \sech \tanh^2 \Delta - \frac{3}{8} \sech^5 \tanh \Delta$

$+\frac{1}{8} \sech^3 \tanh^2 \Delta + \frac{1}{4} \sech \tanh \Delta + \frac{1}{3} \sech \tanh^4 \Delta$

$+\frac{1}{3} \sech \tanh \Delta - \eta \left( 12 \alpha \xi^2 - 20 \frac{1}{15} \cos(2\Delta) \sech^3 \tanh^4 \Delta \right)$

$+\frac{4}{15} \sech^3 \tanh^4 \Delta - \frac{7}{60} \sech \tanh^2 \Delta - \frac{1}{4} \Delta \sech^7 \tanh \Delta$

$+\frac{3}{8} \Delta \sech^5 \tanh \Delta + \frac{1}{60} \sech^5 \tanh \Delta - \frac{7}{120} \sech^3 \tanh^2 \Delta$

$-\frac{7}{60} \sech \tanh \Delta - \frac{1}{10} \sech \tanh^6 \Delta - \frac{1}{10} \sech \tanh \Delta$

$-\frac{8}{15} \sech \tanh \Delta - \frac{1}{5} \sech^5 \tanh^2 \Delta - \frac{4}{15} \sech^3 \tanh^2 \Delta$

$-\frac{8}{15} \sech \tanh \Delta - 6 \alpha \xi^2 - 10 \frac{4}{15} \sech \tanh^2 \Delta$
\[-\frac{1}{2} \Delta \text{sech}^7 \Delta \tanh \Delta + \frac{1}{10} \Delta \text{sech}^5 \Delta \tanh^2 \Delta + \frac{2}{15} \Delta \text{sech}^3 \Delta \tanh^2 \Delta\]

\[+ \frac{4}{15} \Delta \text{sech} \Delta \tanh \Delta + \frac{1}{15} \cos(2\Delta) \Delta \text{sech}^3 \Delta \tanh^4 \Delta + \frac{4}{5} \Delta \text{sech}^3 \Delta \tanh^4 \Delta\]

\[-\alpha \eta^2 \Delta \text{sech} \tanh^2 \Delta + \frac{11}{7} \Delta \text{sech}^7 \Delta \tanh^2 \Delta + \frac{66}{7} \Delta \text{sech}^5 \Delta \tanh^2 \Delta\]

\[+ \frac{88}{7} \Delta \text{sech}^3 \Delta \tanh^2 \Delta + \frac{176}{7} \Delta \text{sech} \Delta \tanh \Delta - \frac{33}{7} \Delta \text{sech} \Delta \tanh^2 \Delta\]

\[+ \frac{165}{16} \Delta \text{sech}^9 \Delta \tanh \Delta - \frac{165}{112} \Delta \text{sech}^7 \Delta \tanh^2 \Delta - \frac{99}{56} \Delta \text{sech}^5 \Delta \tanh^2 \Delta\]

\[-\frac{33}{14} \Delta \text{sech}^3 \Delta \tanh^2 \Delta - \frac{33}{14} \Delta \text{sech} \Delta \tanh \Delta - \frac{44}{21} \Delta \text{sech} \Delta \tanh^2 \Delta\]

\[+ \frac{55}{14} \Delta \text{sech}^7 \Delta \tanh^2 \Delta - \frac{11}{14} \Delta \text{sech}^5 \Delta \tanh \Delta - \frac{22}{21} \Delta \text{sech}^3 \Delta \tanh^2 \Delta\]

\[-\frac{44}{21} \Delta \text{sech} \Delta \tanh \Delta + \frac{24}{35} \Delta \text{sech} \Delta \tanh^2 \Delta - \frac{9}{7} \Delta \text{sech}^7 \Delta \tanh^2 \Delta\]

\[+ \frac{9}{35} \Delta \text{sech}^5 \Delta \tanh^2 \Delta + \frac{24}{35} \Delta \text{sech} \Delta \tanh \Delta + \frac{3}{7} \Delta \text{sech} \Delta \tanh^2 \Delta\]

\[+ \frac{27}{16} \Delta \text{sech}^9 \Delta \tanh \Delta - \frac{9}{4} \Delta \text{sech}^7 \Delta \tanh^2 \Delta - \frac{27}{112} \Delta \text{sech}^5 \Delta \tanh^2 \Delta\]

\[+ \frac{9}{56} \Delta \text{sech}^5 \Delta \tanh^2 \Delta + \frac{3}{14} \Delta \text{sech}^3 \Delta \tanh^2 \Delta + \frac{3}{7} \Delta \text{sech} \Delta \tanh \Delta\]

\[+ \frac{9}{70} \cos(2\Delta) \Delta \text{sech}^3 \Delta \tanh^6 \Delta + \frac{27}{35} \Delta \text{sech}^3 \Delta \tanh^6 \Delta\]

\[-\eta^2 \frac{5}{2} \Delta \text{sech}^5 \Delta \tanh \Delta + \frac{5}{2} \Delta \text{sech} \Delta \tanh \Delta + 2 \Delta \text{sech} \Delta \tanh^2 \Delta\]

\[-\frac{3}{4} \Delta \text{sech}^5 \Delta \tanh \Delta + \frac{1}{2} \Delta \text{sech}^3 \Delta \tanh \Delta - 2 \log \cosh \Delta \Delta \text{sech} \Delta \tanh \Delta\]

\[+ \frac{3}{4} \Delta \text{sech} \Delta \tanh \Delta - \frac{5}{4} \Delta \text{sech} \Delta \tanh^2 \Delta - \Delta \text{sech}^5 \Delta \tanh^2 \Delta + \frac{5}{4} \Delta \text{sech} \Delta \tanh^5 \Delta\]

\[+ \frac{5}{4} \Delta \text{sech} \Delta \tanh \Delta - \eta^3 \Delta \text{sech} \Delta \tanh^2 \Delta - \eta^3 \Delta \text{sech} \Delta \tanh^2 \Delta - \frac{1}{15} \cos(2\Delta) \Delta \text{sech}^3 \Delta \tanh^4 \Delta\]

\[+ \frac{4}{15} \Delta \text{sech}^3 \Delta \tanh^4 \Delta + \frac{7}{60} \Delta \text{sech} \Delta \tanh^2 \Delta + \frac{1}{4} \Delta \text{sech}^7 \Delta \tanh \Delta\]

\[+ \frac{3}{8} \Delta \text{sech}^5 \Delta \tanh \Delta - \frac{1}{20} \Delta \text{sech}^5 \Delta \tanh^2 \Delta + \frac{7}{120} \Delta \text{sech}^3 \Delta \tanh^2 \Delta\]

\[+ \frac{7}{60} \Delta \text{sech} \Delta \tanh \Delta + \frac{1}{10} \Delta \text{sech} \Delta \tanh^6 \Delta + \frac{1}{10} \Delta \text{sech} \Delta \tanh \Delta\]
\( + \frac{8}{15} \text{sech} \Delta \tanh \Delta + \frac{2}{15} \text{sech} \Delta \tanh^2 \Delta - \frac{1}{4} \Delta \text{sech}^7 \Delta \tanh \Delta \)

\( + \frac{1}{20} \text{sech}^5 \Delta \tanh^2 \Delta + \frac{1}{15} \text{sech}^3 \Delta \tanh^2 \Delta + \frac{2}{15} \text{sech} \Delta \tanh \Delta \)

\( + \frac{2}{15} \text{sech}^3 \Delta \tanh^4 \Delta + \frac{1}{30} \cos(2\Delta) \text{sech}^3 \Delta \tanh^4 \Delta - \alpha \xi^4 - \frac{1}{4} \text{sech}^2 \Delta \)

\( \tanh^2 \Delta + \frac{1}{4} \Delta \text{sech}^6 \Delta \tanh \Delta + \frac{161}{6} \alpha \eta^5 + \frac{35}{3} \eta^3 + \frac{20}{3} \alpha \xi^2 \eta^3 + 2 \alpha \xi^2 - \frac{10}{3} \eta \alpha \eta^4 \)

\( \text{sech}^7 \Delta - \frac{139}{2} \alpha \eta^5 + \frac{15}{4} \eta^5 - \frac{3}{5} \xi^2 + \frac{b_3 \epsilon^2}{4} - \frac{1}{8} \frac{\eta^2}{\eta^2} - \frac{5 \eta^2}{8} \alpha \eta^4 \)

\( - 6 b_2 + \frac{12}{e^2} \eta \text{sech}^5 \Delta - \frac{103}{2} \alpha \eta^5 + \frac{55}{8} \alpha \eta^2 + \frac{9 \alpha}{16 \eta} \text{sech}^9 \Delta + \frac{32}{2} + \frac{b_3 \epsilon^2}{4} \)

\( - \frac{b_3 \epsilon^4 \xi^2}{8} \eta \text{sech} \Delta \tanh^4 \Delta - \frac{b_3 \epsilon^4}{3} - 4 \eta^3 \text{sech} \Delta \tanh^6 \Delta - \frac{5}{2} - \frac{5}{24} b_3 \epsilon^4 \eta^3 \)

\( \text{sech}^3 \Delta \tanh \Delta - \frac{2}{3} \eta^2 \cos(2\Delta) \text{sech}^3 \Delta \tanh \Delta - \frac{8}{3} \eta^2 \text{sech}^3 \Delta \tanh^3 \Delta \)

\( - \xi_1 (\theta - \theta_0) - 2 b_1 + b_3 \epsilon^2 \xi^2 + \frac{b_3 \epsilon^4 \xi^4}{12} + \frac{12 \omega_0}{e^2} \eta \text{sech}^2 \Delta - \frac{\xi_1 \eta^2 \text{sech}^6 \Delta}{4} \)

\( - \frac{\alpha \xi^4 \text{sech}^4 \Delta}{6} + \frac{7 b_3 \epsilon^4}{12} - \frac{1}{\eta^2} \text{sech} \Delta + \frac{1}{6} \text{sech} \Delta + \frac{b_3 \epsilon^2}{2} - \frac{b_3 \epsilon^4 \xi^2}{4} \eta^2 \text{sech} \Delta \)

\( - \frac{b_3 \epsilon^4 \xi^5}{3} - 4 \epsilon^5 \text{sech} \Delta + \frac{75}{4} \alpha \xi^5 - \frac{25}{2} \eta^3 - 3 \alpha \xi^2 - 10 \alpha \xi^2 \eta^3 + 5 \Delta \text{sech}^7 \Delta \)

\( \tanh \Delta + \frac{15}{4} \eta^3 + \frac{45}{8} \eta^5 - \frac{849}{16} \alpha \eta^5 - \frac{\xi^2}{\eta^2} - \frac{3}{8} b_3 \epsilon^2 - \frac{3}{16} \epsilon^4 \xi^2 - \frac{15 \xi^2}{2} \eta^4 - \frac{21 \alpha \xi^4}{16} \)

\( - 9 b_2 \eta + \frac{18 \eta}{e^2} \Delta \text{sech}^5 \Delta \tanh \Delta + \frac{165}{16} \alpha \eta^2 + \frac{27 \alpha}{32 \eta} + \frac{309}{2} \alpha \eta^5 \Delta \text{sech}^9 \Delta \tanh \Delta \)

\( - \frac{9}{4} \xi^2 + \frac{3 b_3 \epsilon^2}{8} - \frac{3 b_3 \epsilon^4 \xi^2}{16} \Delta \text{sech} \Delta \tanh^5 \Delta + \frac{b_3 \epsilon^4}{2} - 6 \eta^3 \text{sech} \Delta \tanh^7 \Delta \)

\( + \frac{15 \eta^3}{4} - \frac{5 n^3 b_3 \epsilon^4}{16} \Delta \text{sech}^3 \Delta \tanh \Delta + 4 \eta^2 \text{sech}^3 \Delta \tanh^4 \Delta + \eta^2 \Delta \cos(2 \Delta) \)

\( \text{sech}^3 \Delta \tanh^4 \Delta - \frac{3}{2} \xi_1 (\theta - \theta_0) - 2 b_1 + b_3 \epsilon^2 \xi^2 + \frac{b_3 \epsilon^4 \xi^4}{12} + \frac{12 \omega_0}{e^2} \)
\[ \Delta \text{sech}^2 \Delta \tanh \Delta + \frac{3}{10} \xi \eta^2 \Delta \text{sech}^6 \Delta \tanh \Delta + \frac{1}{4} \alpha \varepsilon^4 \Delta \text{sech}^4 \Delta \tanh \Delta \]
\[ + \frac{3}{2 \eta} - \frac{7}{12} b_3 \varepsilon^4 - \frac{3 \eta}{8} - 6 \xi^2 + b_3 \varepsilon^2 - \frac{b_3 \varepsilon^4 \xi^2}{2} + \frac{3}{2} b_3 \varepsilon^4 \eta^3 - 4 \eta^3 \]
\[ \Delta \text{sech} \Delta \tanh \Delta + \frac{1}{12} \alpha \eta^5 - \frac{35}{6} \eta^3 - \alpha \xi^2 + \frac{5}{3} \alpha \xi^2 \eta^3 \]
\[ \text{sech} \Delta \text{tanh} \Delta \sinh \Delta + \frac{5}{4} \eta^3 - \frac{41}{2} \alpha \eta^5 - \frac{3}{4} \xi^2 + \frac{b_3 \varepsilon^2}{8} - \frac{b_3 \varepsilon^4 \xi^2}{16} - \frac{1}{16} \eta^2 - \frac{5 \xi^2}{2} \]
\[ - \frac{7}{16} \alpha \xi^4 - 3 b_2 + \frac{6}{\varepsilon^2} \eta \text{ sech} \Delta \tanh \Delta \sinh \Delta + \frac{103}{2} \alpha \eta^5 + 55 \alpha \eta^3 + \frac{9 \alpha}{32} \eta^2 \]
\[ \text{sech} \Delta \text{tanh} \Delta \sinh \Delta - \frac{3 \xi^2}{4} + \frac{b_2 \varepsilon^2}{8} - \frac{b_3 \varepsilon^4 \xi^2}{16} \tan^5 \Delta \sinh \Delta + \frac{b_2 \varepsilon^4}{6} - 2 \]
\[ \frac{\eta^3 \tanh \Delta \sinh \Delta + 5 \xi}{4} - \frac{5 b_3 \varepsilon^4}{48} \eta^3 \text{sech} \Delta \text{tanh} \Delta \sinh \Delta + \frac{4}{3} \eta^2 \text{sech} \Delta \]
\[ \tanh \Delta \sinh \Delta + \frac{\eta^2}{3} \cos(2 \Delta) \text{sech} \Delta \tan^4 \Delta \sinh \Delta - \xi_1 (\theta - \theta_0) - 2 b_1 \]
\[ + b_3 \varepsilon^2 \xi^2 + \frac{b_3 \varepsilon^4 \xi^4}{12} + \frac{12 w_0}{\varepsilon^2} \frac{3}{2 \eta} \text{sech} \Delta \tanh \Delta \sinh \Delta + \frac{\xi^4 \eta^2}{10} - \text{sech}^5 \Delta \tanh \Delta \sinh \Delta \]
\[ + \frac{\alpha}{2} \varepsilon^4 \text{sech}^3 \Delta \tanh \Delta \sinh \Delta + \frac{7}{12} - \frac{7 b_3 \varepsilon^4}{2 \eta^2} + \frac{b_3 \varepsilon^4 \eta^3}{6} - 2 \eta^3 - \frac{\eta}{8} - 6 \xi^2 + b_3 \varepsilon^2 \]
\[ - \frac{b_3 \varepsilon^4 \xi^2}{2} \tan^2 \Delta \sinh \Delta + \frac{1}{34} \xi \eta^2 - \frac{14}{3} \alpha \xi \eta^4 - 2 \xi^3 + 3 \varepsilon^4 \xi^3 \]
\[ + \frac{1}{3} b_3 \varepsilon^4 \eta^3 + \frac{5}{18} b_3 \varepsilon^4 \eta^2 + \frac{8}{3} \alpha \xi^3 \eta^2 - 2 b_3 \varepsilon^2 \xi + \frac{\theta - \theta_0}{2} - \text{sech} \Delta \tanh \Delta \]
\[ + \frac{\delta_3 - 18 \xi \eta^2 - 14 \alpha \xi \eta^2 + \frac{3}{2} b_3 \varepsilon^4 \eta^2 + \frac{227}{15} \alpha \xi \eta^4 + 8 \alpha \xi^3 \eta^2 + \frac{2}{3} b_3 \varepsilon^4 \xi^3 - 2 \xi^3 }{2} \]
\[ - 2 b_3 \varepsilon^2 \xi + \frac{(\theta - \theta_0) T}{2} - \text{sech} \Delta + \frac{\delta_3 - \frac{1}{3} b_3 \varepsilon^4 \xi^2 - \frac{7}{2} \xi \eta^2 + 4 \xi^3 + 2 b_3 \varepsilon^2 \xi}{2} \]
\[ - \frac{1}{3} b_3 \varepsilon^4 \xi^3 - \frac{1}{24} b_3 \varepsilon^4 \xi^2 - (\theta - \theta_0) T \Delta \text{sech} \Delta + \frac{23}{3} \xi \eta^2 - 3 b_3 \varepsilon^4 \xi \eta^2 \]
\[ + \frac{14}{3} \alpha \xi \eta^2 - 2 \alpha \xi^3 \eta^2 - 2 \alpha \xi \eta^4 - \frac{5}{24} b_3 \varepsilon^4 \eta^2 \Delta \text{sech}^5 \Delta + 2 \xi \eta^2 + 2 \alpha \xi \eta^2 \]
\[- \frac{1}{6} b_3 \epsilon^4 \xi - \xi^3 - b_3 \epsilon^2 - \frac{1}{2} b_3 \epsilon^4 \xi^3 + \frac{1}{2} b_3 \epsilon^2 \xi - \frac{(\theta - \theta_0)\tau}{3} \Delta \text{sech}^3 \Delta \]
\[
+ \frac{-13}{3} \xi \eta^2 + \frac{1}{3} \alpha \xi \eta^2 - \frac{5}{36} \epsilon^4 \xi \eta^2 + \frac{4}{3} \alpha \xi^3 \eta^2 - \frac{59}{45} \alpha \xi \eta^4 \text{sech}^3 \Delta \tanh \Delta \]
\[
+ \frac{-20}{3} \xi \eta^2 - \frac{5}{9} b_3 \epsilon^4 \xi \eta^2 + \frac{16}{3} \alpha \xi^3 \eta^2 \text{sech} \Delta \tanh \Delta \frac{-112}{15} \alpha \xi \eta^4 \]
\[
\text{sech}^3 \Delta \tanh^3 \Delta \frac{-14}{3} \text{sech} \Delta + \frac{\eta^4}{\eta^2} \text{sech} \Delta \tanh \Delta - \frac{4}{3} \alpha \xi \eta^4 \cos(2\Delta) \]
\[
\text{sech} \Delta \tanh^3 \Delta - \frac{4}{5} \alpha \xi \eta^4 \text{sech} \Delta \tanh \Delta + \frac{-\delta_4}{2\eta} - \frac{1}{2} b_3 \epsilon^4 \xi \eta^2 + \frac{\xi \eta^2}{2} \sinh \Delta \]
\[
- \frac{26}{15} \alpha \xi \eta^4 \text{sech} \Delta \tanh \Delta + 2 \alpha \xi \eta^4 \text{sech} \Delta + 3 \alpha \xi \eta^4 \text{sech} \Delta \tanh^4 \Delta + \frac{\eta^4}{\eta^2} \Delta^2 \text{sech} \Delta \]
\[
+ \frac{\xi \eta^2}{2} - \frac{1}{24} b_3 \epsilon^4 \xi \eta^2 \text{sech} \Delta \tanh^6 \Delta - \alpha \xi \eta^4 \text{sech}^4 \Delta + \frac{-5}{2} \xi \eta^2 + \alpha \xi \eta^4 \]
\[
- \frac{5}{8} b_3 \epsilon^4 \eta^2 + 2 \alpha \xi^3 \eta^2 \text{sech}^4 \Delta \sinh \Delta + \frac{-1}{24} b_3 \epsilon \xi \eta^2 + \frac{\xi \eta^2}{2} \sinh^2(\Delta) \sinh \Delta \]
\[
- \frac{7}{3} \alpha \xi \eta^4 \text{sech} \Delta \sinh \Delta + \frac{-1}{2} b_3 \epsilon^2 \xi - \frac{1}{12} \text{sech}^2 \Delta \sinh \Delta , \quad (5.49) \]

where, \( \Delta = \eta(\theta - \theta_0) \). We have plotted the real part of the perturbed soliton Eq. (5.41) for various values of \( \eta \), ranging from \( \eta = 0.1 \sim 0.9 \) in Fig. (5.6). From the plot it is evident that when the amplitude of proton soliton increases, the perturbed real part of the solution, shows a notable decrease in the amplitude, but keeping the localized structure intact. We also have plotted the imaginary part of the solution of Eq. (5.46), for various values of the parameters. From the snapshots it is evident that the profile transforms from kink to anti-kink profile as shown in Fig. (5.7).

Surprisingly, the Fig. (5.8) which depicts the perturbed soliton \( \xi = q_0 + \lambda \xi_1 \), portrays the robust, coherent intact nature of the highly localized proton solitonic profile. Further when we increase the value of the perturbation parameter, the proton solitonic profile experiences an appreciable damping in the amplitude. The variation in the perturbation parameter \( \lambda \) which is proportional to length of the hydrogen bond, have an notable impact on the amplitude of
the proton solitons propagating along the peptide chain. If the length of the hydrogen bond is large, the proton soliton suffers larger damping with an appreciable loss in energy. Hence, it is evident that the hydrogen bond, and its length among the peptide group plays an important role in transporting the bio-energy in the form of proton soliton.

5.6 Conclusions

In this work, we have proposed a new model Hamiltonian to explain the proton dynamics in the polypeptide chains of protein molecule. More specifically, the inter-peptide proton transfer is considered with a symmetric double well potential where as the interspin interactions are neglected. Namely, the solitons in the present model are described by a set of coupled molecular dynamical equations which explain the dynamics of protons along the peptide chain. The propagation of proton soliton in the presence of nonlinear potential is governed by the NLS equation with higher order molecular interactions, excitations and effect of discreteness. To predict the velocity and amplitude of the soliton under perturbation, the perturbed NLS equation is solved numerically for soliton parameters obtained through multiple-scale analysis. It is observed that the soliton velocity and amplitude undergoes dramatic and curious changes in the presence of higher order terms. Also the perturbed soliton solutions are constructed and graphically illustrated. These proton solitons ensure that the energy transport along the molecular chain will be proportional to the soliton velocity. Thus, the present model is a good start for soliton energy transport mechanism in biological systems. Precisely, we conclude that these proton solitons appear to be more appropriate to describe the energy transfer in polypeptide chains of protein backbone as a mobile entity of such biological processes namely, muscle contraction and neuro-electric pulse transfer on the bio-membranes.
Figure 5.4: Velocity of the soliton.
Figure 5.5: Amplitude of the soliton.
Figure 5.6: Snapshots of real part of the perturbed soliton Eq. (5.41) for the parameters $\eta$. 
5.6 Conclusions

[Graphs and diagrams related to the conclusions]
Perturbed soliton excitations in hydrogen bonded polypeptide chain
Figure 5.7: Snapshots of imaginary part of the perturbed soliton Eq. (5.46) for the parameters $\eta$.

$\lambda = 10^{-15}\, m$

$\lambda = 10^{-14}\, m$
Perturbed soliton excitations in hydrogen bonded polypeptide chain

\[
\lambda = 10^{-13} \text{m}
\]

\[
\lambda = 1 \times 10^{-12} \text{m}
\]

\[
\lambda = 2 \times 10^{-12} \text{m}
\]

\[
\lambda = 3 \times 10^{-12} \text{m}
\]
\[ \lambda = 4 \times 10^{-12} \text{ m} \]

\[ \lambda = 5 \times 10^{-12} \text{ m} \]

\[ \lambda = 6 \times 10^{-12} \text{ m} \]

\[ \lambda = 7 \times 10^{-12} \text{ m} \]
Perturbed soliton excitations in hydrogen bonded polypeptide chain

\[ \lambda = 8 \times 10^{-12} \text{ m} \]

\[ \lambda = 9 \times 10^{-12} \text{ m} \]

\[ \lambda = 1 \times 10^{-11} \text{ m} \]

\[ \lambda = 2 \times 10^{-11} \text{ m} \]
$$\lambda = 3 \times 10^{-11} \text{ m}$$

$$\lambda = 4 \times 10^{-11} \text{ m}$$

$$\lambda = 5 \times 10^{-11} \text{ m}$$

$$\lambda = 6 \times 10^{-11} \text{ m}$$
\( \lambda = 7 \times 10^{-11} \text{ m} \)

\( \lambda = 8 \times 10^{-11} \text{ m} \)

\( \lambda = 9 \times 10^{-11} \text{ m} \)

\( \lambda = 10^{-10} \text{ m} \)

Figure 5.8: Perturbed soliton with varying the values of \( \lambda \).