Preface

The present thesis consisting of five chapters is devoted to the study of complete invariance property in topological spaces, hyperspaces and frame wavelet spaces. Also, the notion of $S$-equivariant complete invariance property over metric spaces and hyperspaces is considered.

In 1967, Robbins [Some complements to Brouwer’s fixed point theorem, Israel J. Math. 5 (1967), 225–226], obtained that for every nonempty closed set $F$ of the closed unit disc $B^2$ in Euclidean space $\mathbb{R}^2$, there is a selfmap $f$ on $B^2$ such that the set of all fixed points of $f$ ($\text{Fix} f$) is $F$. Dealing with this result, Robbins sowed the germ of the topological notion of the complete invariance property which got the formal definition by Ward [Fixed point sets, Pacific J. Math. 47 (1973), 553–565], as follows: A topological space $X$ is said to possess the complete invariance property (CIP) if each of its nonempty closed sets is $\text{Fix} f$, for some continuous selfmap $f$ on $X$. In case, $f$ can be chosen to be a homeomorphism, the space is said to possess the complete invariance property with respect to homeomorphism (CIPH) [J. R. Martin, Fixed point sets of homeomorphisms of metric products, Proc. Amer. Math. Soc. 103 (1988), 1293–1298]. These notions have been extensively studied by Schirmer, Martin, Nadler, Oversteegen, Tymchatyn, Weiss, Chigogidge, and Hofmann. They studied the preservation of these properties under various topological operations such as products, cones and wedge product. They obtained various spaces with or without these properties.

The $n$-cells, one dimensional Peano continuum, dendrites, locally compact metric groups, compact topological $n$-manifolds with or without boundary, an arbitrary product of the space of irrationals, a product of the real line with itself, the Hilbert cube and the Sorgenfrey line $K$ have the CIP. The spaces which do not possess the CIP are $K^n, n \geq 2$ where $K$ is the Sorgenfrey line, $S^1 \times Z_2^\lambda$ and $\mathbb{R} \times Z_2^\lambda$, where $Z_2 = \{0, 1\}$ is two point discrete group and $\lambda$ is uncountable cardinal, and the Stone-Čech compactification $\beta \mathbb{N}$ of $\mathbb{N}$.

We begin with the “Preliminaries and Introduction” as Chapter 1, in which we introduce the thesis after providing basic definitions and necessary results together with a brief account of the recent development in the direction of our work.
Spaces possessing the complete invariance property have been identified by the notion of uniform flow developed by Chigogidge, Hofmann and Martin in their paper entitled “Compact groups and fixed point sets” published in Trans. Amer. Math. Soc. 349 (1997), 4537–4554. They characterized compact metric spaces possessing complete invariance property with respect to homeomorphism by employing the structure theory on Lie groups. In Chapter 2, we restrict ourself to the passage of uniform flow of a metric space which is homeomorphic to a metric space possessing uniform flow. Having noticed a metric space homeomorphic to a metric space possessing uniform flow not getting a passage of uniform flow on it in a natural way, we provide a condition on the homeomorphism providing the desired passage of uniform flow.

In Chapter 3, $X$ denotes a metric space with a uniform flow $\varphi$. By inducing the uniform flow $\Phi$ on the hyperspace $2^X$ consisting of nonempty compact subsets of $X$ we obtain that $2^X$ enjoys the CIP. Also, when the uniform flow $\Phi$ is restricted to subspace $F_n(X)$ of $2^X$ consisting of nonempty subsets of $X$ containing atmost $n$ points, it has been noticed that it becomes a uniform flow on $F_n(X)$ which, in turn, shows that $F_n(X)$ enjoys the CIPH.

The notion of $S$-equivariant complete invariance property ($S$-ECIP) has been introduced by Azad and Srivastava in their paper entitled “On $S$-equivariant complete invariance property” [K. K. Azad and K. Srivastava, On $S$-equivariant complete invariance property, Journal of the Indian Math. Soc. 62 (1996), 2005–2009]. Besides many things, they have proved that if $(X, d)$ is a metric space and $S^1$ is the unit circle group, then the product $X \times S^1$ has $S$-ECIP. In Chapter 4, we describe $S$-ECIP on metric spaces and on hyperspaces. We have shown that a metric space on which $S^1$ acts freely such that the orbits are equidistant to each other, possesses $S$-ECIP, which is a general result in comparison to the above. Furthermore, it is obtained that the hyperspace $2^X$ of nonempty compact subsets of a metric space $X$ enjoys the notion of $S$-equivariant complete invariance property.

The set $\mathcal{W}$ of all one-dimensional orthonormal wavelets on $\mathbb{R}$ forms a subset of the unit ball of the space $L^2(\mathbb{R})$. Thus $\mathcal{W}$ is a topological space with the topology induced from that of $L^2(\mathbb{R})$. The topological property of $\mathcal{W}$ and certain subsets of $\mathcal{W}$ have drawn attention of several workers in the field of wavelets during the
past one decade. Such a study has also been carried over to higher dimensional orthonormal wavelets.

Recently, Dubey and Vyas in their paper entitled “Wavelets and the complete invariance property”, appeared in Math. Vesnik 62 (2010), 183–188, have studied the topological notion of the complete invariance property of $W$ and certain subsets of $W$. They noticed a free action of the unit circle $S^1$ on $W$ and obtained each orbit isometric to $S^1$. Employing result of Martin [Fixed point sets of homeomorphisms of metric products, Proc. Amer. Math. Soc. 103 (1988), 1293–1298] stated below: “A space $X$ has the CIPH if it satisfies the following conditions: (i) $S^1$ acts on $X$ freely. (ii) $X$ possesses a bounded metric such that each orbit is isometric to the unit circle.” they proved that the set of all one-dimensional orthonormal wavelets, the set of all MRA wavelets and the set of all MSF wavelets on $\mathbb{R}$ have the complete invariance property with respect to homeomorphism.

In Chapter 5, we study the complete invariance property with respect to homeomorphism over the spaces $W \subset \prod_{1 \leq j \leq L} L^2(\mathbb{R}^n)$, containing all orthonormal multiwavelets on $\mathbb{R}^n$ in $L$-tuple form, $W_T \subset \prod_{1 \leq j \leq L} L^2(\mathbb{R}^n)$, containing all tight frame multiwavelets on $\mathbb{R}^n$ in $L$-tuple form, $SW_n = \{ \vartheta = (\eta_1, ..., \eta_n) : (\eta_1, ..., \eta_n) \text{ is a super-wavelet of length } n \text{ for } L^2(\mathbb{R})^\oplus \}$ and $SW_{nT} = \{ \vartheta = (\eta_1, ..., \eta_n) : (\eta_1, ..., \eta_n) \text{ is a normalized tight super frame wavelet of length } n \text{ for } L^2(\mathbb{R})^\oplus \}$. In case, the action of $S^1$ over $W$, $W_T$ and $SW_{nT}$ we obtain that the action is free but orbits are not isometric to $S^1$. Observing this fact, we have proved that the result of Martin, mentioned above, is also true for orbits isometric to a circle of finite radius.

Towards the end follows a list of books and research papers referred to in the thesis. The contents of Chapter 2 have been published as a paper entitled “On uniform flow”, in Int. Math. Forum 8(35) (2013), 1703 - 1708. Chapter 3 constitutes the contents of our paper entitled “Hyperspaces and complete invariance property”, accepted for publication in the Scientiae Mathematicae Japonicae (SCMJ). Chapter 4 constitutes the contents of our paper entitled “Hyperspaces and the S-equivariant complete invariance property” accepted for publication in Kyungpook Mathematical Journal (KMJ).

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