Chapter 5

The CIPH on frame wavelet spaces

The set $\mathcal{W}$ of all one-dimensional orthonormal wavelets on $\mathbb{R}$ forms a subset of the unit ball of the space $L^2(\mathbb{R})$. Thus $\mathcal{W}$ is a topological space with the topology induced from that of $L^2(\mathbb{R})$. The topological property of $\mathcal{W}$ and certain subsets of $\mathcal{W}$ have drawn attention of several workers in the field of wavelets during the past one decade [DDGH, DDL, S4, W7]. Such a study has also been carried over to higher dimensional orthonormal wavelets.

Recently, Dubey and Vyas in their paper entitled “Wavelets and the complete invariance property”, appeared in Math. Vesnik 62 (2010), 183–188, have studied the topological notion of the complete invariance property of $\mathcal{W}$ and certain subsets of $\mathcal{W}$. They noticed a free action of the unit circle $S^1$ on $\mathcal{W}$ and obtained each orbit isometric to $S^1$. Employing Theorem 1.3.4 they proved that the set of all one-dimensional orthonormal wavelets, the set of all MRA wavelets and the set of all MSF wavelets on $\mathbb{R}$ have the complete invariance property with respect to homeomorphism.

In this Chapter, we study the complete invariance property with respect to homeomorphism over the spaces $W \subset \prod_{1 \leq j \leq L} L^2(\mathbb{R}^n)$, containing all orthonormal multiwavelets on $\mathbb{R}^n$ in $L$-tuple form, $W_T \subset \prod_{1 \leq j \leq L} L^2(\mathbb{R}^n)$, containing all tight frame multiwavelets on $\mathbb{R}^n$ in $L$-tuple form, $\mathcal{SW}_n = \{ \vartheta = (\eta_1, ..., \eta_n) : (\eta_1, ..., \eta_n) \text{ is a super-wavelet of length } n \text{ for } L^2(\mathbb{R}) \} \otimes n$ and $\mathcal{SW}^{NT}_n = \{ \vartheta = (\eta_1, ..., \eta_n) : (\eta_1, ...,$
\( \eta_n \) is a normalized tight super frame wavelet of length \( n \) for \( L^2(\mathbb{R})^{\oplus n} \). In case, the action of \( S^1 \) over \( W, W_T \) and \( SW_1^{NT_n} \) we obtain that the action is free but orbits are not isometric to \( S^1 \). Observing this fact, we have proved that Theorem 1.3.4 is also true for orbits isometric to a circle of finite radius.

## 5.1 Frames and frame wavelets

For a generic countable (or finite) index set \( J \) such as \( \mathbb{N}, \mathbb{Z}, \mathbb{N} \cup \mathbb{N}, \mathbb{Z} \times \mathbb{Z} \) etc., a collection of elements \( \Phi = \{ \phi_j : j \in J \} \) in a separable Hilbert space \( H \) is called a frame if there exist constants \( A \) and \( B, \) \( 0 < A \leq B < \infty \) such that

\[
A ||f||^2 \leq \sum_{j \in J} | < f, \phi_j > |^2 \leq B ||f||^2, \quad \text{for all } f \in H.
\] (5.1)

The optimal constants (maximal for \( A \) and minimal for \( B \)) are called the frame bounds. \( A \) is called a lower frame bound and \( B \) is called an upper frame bound of the frame. The frame \( \{ \phi_j : j \in J \} \) is called a tight frame if \( A = B, \) and is called normalized tight frame if \( A = B = 1. \) Any orthonormal basis in a Hilbert space is a normalized tight frame. Notice that for a nonzero element \( \phi_k \) of a frame \( \Phi \) in \( \mathbb{H}, \) the following inequality holds:

\[
||\phi_k|| \leq \sqrt{B}, \quad \text{for all } k \in J.
\]

This follows by noting that

\[
||\phi_k||^4 = | < \phi_k, \phi_k > |^2 \leq \sum_{j \in J} | < \phi_k, \phi_j > |^2 \leq B ||\phi_k||^2.
\]

This shows that the elements of a frame need not be normal but they must have an upper bound. It is well known that the collection \( \Phi \subset \mathbb{H} \) is referred as a Bessel family if there exists \( B > 0 \) so that

\[
\sum_{j \in J} | < f, \phi_j > |^2 \leq B ||f||^2, \quad \text{for all } f \in \mathbb{H}.
\]
In view of inequality (5.1), we can say that the constants $A$ and $B$ remain same if we add the zero element to the collection \{\phi_j : j \in J\}. Thus a frame is not necessarily a basis for the Hilbert space $\mathbb{H}$. Therefore, we can enlarge frame $\Phi$ by adding some elements but constants $A$ and $B$ are not necessarily same. In case of normalized tight frame $\Phi$, the only way to enlarge $\Phi$ in such a way that it remains a normalized tight frame is to add zero vectors. This can be seen as follows:

Let \{\phi_j : j \in J\} be a normalized tight frame for $H$. Assume \{\phi_j : j \in \Lambda\} is a subset of \{\phi_j : j \in J\} which is also a normalized tight frame for $H$. We may assume $\Lambda \subset J$. If $j \notin \Lambda$, then

$$||\phi_j||^2 = \sum_{k \in J} |\langle \phi_j, \phi_k \rangle|^2 = \sum_{k \in \Lambda} |\langle \phi_j, \phi_k \rangle|^2.$$ 

This shows that

$$\sum_{k \notin \Lambda} |\langle \phi_j, \phi_k \rangle|^2 = 0,$$

and hence we have

$$\phi_j = 0, \quad \text{for all } j \in J - \Lambda.$$ 

If \{\phi_j : j \in J\} is a tight frame and $||\phi_{j_0}|| \geq 1$ for some $j_0 \in J$, then we have

$$A||\phi_{j_0}||^2 = \sum_{j \in J} |\langle \phi_{j_0}, \phi_j \rangle|^2$$

$$= ||\phi_{j_0}||^4 + \sum_{j \neq j_0} |\langle \phi_{j_0}, \phi_j \rangle|^2,$$

and hence

$$A = \left\{ ||\phi_{j_0}||^2 + \sum_{j \neq j_0} \frac{1}{||\phi_{j_0}||^2} |\langle \phi_{j_0}, \phi_j \rangle|^2 \right\} \geq 1.$$ 

In this situation, if $A = 1$, then $||\phi_{j_0}|| = 1$ and $\langle \phi_{j_0}, \phi_j \rangle$ must be zero for all $j \in J$. Therefore, if the elements of a tight frame all have norm 1 and $A = 1$, the frame is an orthonormal basis for $H$.

**Definition 5.1.1 [W6].** Let $A$ be an $n \times n$ expansive matrix such that $AZ^n \subset Z^n$. Then a finite set $\Psi = \{\psi^1, \ldots, \psi^L\} \subset L^2(\mathbb{R}^n)$ is called an orthonormal multiwavelet if the collection $\mathcal{A}(\Psi) = \{\psi^l_{j,k} : j \in \mathbb{Z}, k \in \mathbb{Z}^n, l = 1, \ldots, L\}$ is an orthonormal
basis for \( L^2(\mathbb{R}^n) \), where for \( \psi \in L^2(\mathbb{R}^n) \) we use the convention
\[
\psi_{j,k} = |\det A|^\frac{1}{2} \psi(A^j \cdot -k).
\]

If a multiwavelet \( \Psi \) consists of a single element \( \psi \) then we say that \( \psi \) is a wavelet. By an expansive matrix \( A \), we mean a square matrix, the modulus of whose eigenvalues are greater than 1.

If the collection \( \mathcal{A}(\Psi) = \{ \psi_{l,j,k} : j \in \mathbb{Z}, k \in \mathbb{Z}^n, l = 1, \ldots, L \} \) is a normalized tight frame, then the set \( \Psi = \{ \psi_1, \ldots, \psi_L \} \subset L^2(\mathbb{R}^n) \) is called a normalized tight frame multiwavelet. Similarly, \( \Psi \) is called a tight frame multiwavelet when above collection \( \mathcal{A}(\Psi) \) is a tight frame and a frame multiwavelet when above collection \( \mathcal{A}(\Psi) \) is a frame.

The following result establishes a characterization of normalized tight frame multiwavelet.

**Theorem 5.1.2 [B5].** Suppose \( \Psi = \{ \psi_1, \ldots, \psi_L \} \subset L^2(\mathbb{R}^n) \). Then the collection \( \mathcal{A}(\Psi) = \{ \psi_{l,j,k} : j \in \mathbb{Z}, k \in \mathbb{Z}^n, l = 1, \ldots, L \} \) with a dilation \( A \) is a normalized tight frame if and only if
\[
\begin{align*}
(i) & \quad \sum_{l=1}^{L} \sum_{j \in \mathbb{Z}} |\hat{\psi}^l(B^j \xi)|^2 = 1, \quad \text{for a.e. } \xi \in \mathbb{R}^n, \text{ where } B \text{ is the transpose of } A, \\
(ii) & \quad t_q(\xi) \equiv \sum_{l=1}^{L} \sum_{j \geq 0} \hat{\psi}^l(B^j \xi) \overline{\hat{\psi}^l(B^j(\xi + q))} = 0, \quad \text{for } q \in \mathbb{Z}^n \setminus B\mathbb{Z}^n \text{ and a.e. } \xi \in \mathbb{R}^n.
\end{align*}
\]

In particular \( \Psi \) is a multiwavelet if and only if above conditions hold and \( ||\psi^l|| = 1 \) for all \( l = 1, \ldots, L \).

The Fourier transform of \( f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \) is defined by
\[
\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \langle \xi, x \rangle} dx, \quad \xi \in \mathbb{R}^n,
\]
where \( \langle \xi, x \rangle \) denotes the real inner product.

Since \( L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \) is a dense subset of \( L^2(\mathbb{R}^n) \), this definition extends uniquely to \( L^2(\mathbb{R}^n) \).
The applications of wavelet theory to signal processing and image processing are now well known. Probably the main reason for the success of the wavelet theory was the introduction of the concept of multiresolution analysis (MRA), which provided the right framework to construct orthogonal wavelet bases with good localization properties.

Definition 5.1.3 [CG]. A multiresolution analysis (MRA) of multiplicity $r$ associated to a dilation matrix $A$ is a sequence of closed subspaces $\{V_j\}_{j \in \mathbb{Z}}$ of $L^2(\mathbb{R}^n)$ which satisfy:

(a) $V_j \subset V_{j+1}$, for all $j \in \mathbb{Z}$,

(b) $f \in V_j$ if and only if $f(A(\cdot)) \in V_{j+1}$, for all $j \in \mathbb{Z}$,

(c) $\cap_{j \in \mathbb{Z}} V_j = \{0\}$,

(d) $\cup_{j \in \mathbb{Z}} V_j = L^2(\mathbb{R}^n)$,

(e) there exist functions $\varphi_1, \ldots, \varphi_r \in L^2(\mathbb{R}^n)$ such that the collection $\{\varphi_i(\cdot - k) : k \in \mathbb{Z}^n, 1 \leq i \leq r\}$ forms an orthonormal basis for $V_0$.

If these conditions are satisfied, then the vector function $\Phi = (\varphi_1, \ldots, \varphi_r)$ is referred to as a scaling vector for the MRA.

Definition 5.1.4 [B3]. An MSF (minimally supported frequency) multiwavelet (of order $L$) is an orthonormal multiwavelet $\Psi = \{\psi_1, \ldots, \psi_L\}$ such that $|\hat{\psi}_l| = \chi_{W_l}$ for some measurable set $W_l \subset \mathbb{R}^n$, $l = 1, \ldots, L$. An MSF multiwavelet of order 1 is simply referred to as an MSF wavelet.

The following theorem characterizes all MSF multiwavelets.

Theorem 5.1.5 [DLS]. A set $\Psi = \{\psi^1, \ldots, \psi^L\} \subset L^2(\mathbb{R}^n)$ such that $|\hat{\psi}_l| = \chi_{W_l}$ for $l = 1, \ldots, L$ is an orthonormal multiwavelet with the dilation matrix $A$ if and only if

$$\sum_{k \in \mathbb{Z}^n} \chi_{W_l}(\xi + k)\chi_{W_{l'}}(\xi + k) = \delta_{ll'} \quad \text{a.e.,} \quad \xi \in \mathbb{R}^n, \quad l, l' = i, \ldots, L,$$

and
\[
\sum_{i=1}^{L} \sum_{j \in \mathbb{Z}} \chi_{W_i}(B^j \xi) = 1, \quad \text{a.e., } \xi \in \mathbb{R}^n, \quad B = A^T.
\]

**Definition 5.1.6.** Let \( \Psi = \{\psi^1, \ldots, \psi^L\} \subset L^2(\mathbb{R}^n) \) be an orthonormal multiwavelet associated with a dilation \( A \). Then

\[
D_{\Psi}(\xi) = \sum_{i=1}^{L} \sum_{j \in \mathbb{N}} \sum_{k \in \mathbb{Z}^n} \left| \hat{\psi}^i \left( B^j (\xi + k) \right) \right|^2, \quad \text{for a.e., } \xi \in \mathbb{R}^n
\]

describes the dimension function \( D_{\Psi} \) for \( \Psi \), where \( B \) is the transpose of \( A \).

We have following results analogous to those as in the case of one dimension.

**Theorem 5.1.7 [B3].** An orthonormal multiwavelet \( \Psi \in L^2(\mathbb{R}^n) \) is an MRA multiwavelet if and only if \( D_{\Psi}(\xi) = 1 \), for a.e., \( \xi \in \mathbb{R}^n \).

### 5.2 Some results related to the CIPH

In the paper entitled “Fixed point sets of homeomorphisms of metric products” appeared in Proc. Amer. Math. Soc. 103 (1988), 1293–1298, J. R. Martin has proved that (Theorem 2.2) “If \( X \) is a metric space, then \( X \times S^{2n-1} \) and \( X \times \mathbb{R} \) have the CIPH” and also remarked that “A space \( X \) has the CIPH if it satisfies the following conditions:

(i) \( S^1 \) acts on \( X \) freely.

(ii) \( X \) possesses a bounded metric such that each orbit is isometric to \( S^1 \).”

Thus to examine the CIPH over a metric space \( X \) we need a free \( S^1 \) action on \( X \) having orbits isometric to the unit circle. This result does not provide any information about the CIPH over \( X \) in case, the radii of orbits are different from unity.

In this Section, we shall show that if \( S^1 \) acts freely on a metric space \( X \) and orbits are isometric to circles of finite radii, then \( X \) has the CIPH.

**Theorem 5.2.1.** A space \( X \) has the CIPH if it satisfies the following conditions:
(i) $S^1$ acts on $X$ freely.

(ii) $X$ possesses a bounded metric such that each orbit is isometric to a circle of radius $L$.

**Proof.** Let $(X, d)$ be a metric space with $d \leq 3\pi L$ and let $*: X \times S^1 \rightarrow X$ be the action satisfying conditions (i) and (ii). For a nonempty closed set $A$ in $X$ set $a(x) = \frac{1}{2\pi}d(x, A)$ and define $f: X \rightarrow X$ by

$$f(x) = x * e^{ia(x)}, \quad x \in X.$$ 

Since $0 < a(x) < 2\pi$ if $x \notin A$, it follows that $\text{Fix} f = A$. To see that $f$ is one-one suppose $f(y) = f(z)$. Then $y$ and $z$ must lie on the same orbit isometric to the circle $S^1_L$ and for some real number $t$, $y = z * e^{it}$ with $d(y, z) = |Lt| \leq \pi L$.

Thus $y * e^{ia(y)} = z * e^{ia(z)}$ and hence for some integer $n$,

$$t + a(y) - a(z) = 2\pi n.$$ 

By the triangle inequality applying over $y, z, A$ we have

$$2L|a(y) - a(z)| \leq |Lt| \leq \pi L,$$

and so

$$|a(y) - a(z)| \leq \frac{\pi}{2}.$$ 

Thus the equation

$$t + a(y) - a(z) = 2\pi n,$$

holds only for $n = 0$ and hence $t = 0$.

For the remaining portion see the proof of Theorem 2.2 [M2].

**Theorem 5.2.2.** A space $X$ has the CIPH if it satisfies the following conditions:

(i) $S^1$ acts on $X$ freely.

(ii) $X$ possesses a bounded metric such that each orbit is isometric to $S^1_L$ a circle of radius $r$ where $0 < r \leq L$ for some $L > 0$. 
\textbf{Proof.} Let \((X, d)\) be a metric space with \(d \leq 3\pi L\) and let \(*: X \times S^1 \rightarrow X\) be the action satisfying conditions (i) and (ii). For a nonempty closed set \(A\) in \(X\) set \(a(x) = \frac{1}{2\pi}d(x, A)\) and define \(f : X \rightarrow X\) by

\[ f(x) = x * e^{ia(x)}, \quad x \in X. \]

Since \(0 < a(x) < 2\pi\) if \(x \notin A\), it follows that \(\text{Fix} f = A\). To see that \(f\) is one - one suppose \(f(y) = f(z)\). Then \(y\) and \(z\) must lie on the same orbit isometric to the circle \(S^1_r\), \(0 < r \leq L\) and for some real number \(t\), \(y = z * e^{it}\) with \(d(y, z) = |rt| \leq \pi L\).

Thus \(y * e^{ia(y)} = z * e^{ia(z)}\) and hence for some integer \(n\),

\[ t + a(y) - a(z) = 2\pi n. \]

By the triangle inequality applying over \(y, z, A\) we have

\[ 2L|a(y) - a(z)| \leq |rt| \leq \pi L, \]

and so

\[ |a(y) - a(z)| \leq \frac{\pi}{2}. \]

Thus the equation

\[ t + a(y) - a(z) = 2\pi n, \]

holds only for \(n = 0\) and hence \(t = 0\).

Since \(f\) is an orbit wise one-one map and a homeomorphism of \(S^1_r\) into itself must be onto, it follows that \(f\) is onto. In order to conclude that \(f\) is a homeomorphism it suffices to show that \(f\) is a closed mapping. For the remaining portion see the proof of Theorem 2.2 [M2].

\section*{5.3 Frame wavelet spaces and the CIPH}

Let \(A\) be an expansive matrix and \(L \geq 1\) is an integer. The space \(W_{AL} = W = \{(\psi^1, \ldots, \psi^L) \in \prod_{1 \leq j \leq L} L^2(\mathbb{R}^n): \{\psi^1, \ldots, \psi^L\} \text{ is an orthonormal multiwavelet}\}\) is
a metric space with the natural metric $d$ defined by

$$d^2(\Psi, \Phi) = \sum_{l=1}^{L} ||\psi_l - \phi_l||^2,$$

where $\Psi = (\psi^1, \ldots, \psi^L)$ and $\Phi = (\phi^1, \ldots, \phi^L)$.

**Theorem 5.3.1.** The space $W$ has the CIPH.

**Proof.** On account of Theorem 5.1.2, it is straightforward to see that for $e^{i\theta} \in S^1$ and $\Psi = (\psi^1, \ldots, \psi^L) \in W$, $e^{i\theta} \cdot \Psi = (e^{i\theta} \cdot \psi^1, \ldots, e^{i\theta} \cdot \psi^L) \in W$.

Thus, the function $\eta : S^1 \times W \rightarrow W$ defined by

$$\eta(e^{i\theta}, \Psi) = e^{i\theta} \cdot \Psi, \quad e^{i\theta} \in S^1, \ \Psi \in W$$

is well defined. The following

(i) $\eta(1, \Psi) = \Psi$, and

(ii) $\eta(e^{i\theta_1}, \eta(e^{i\theta_2}, \Psi)) = \eta(e^{i(\theta_1 + \theta_2)}, \Psi)$,

are easily checked. This shows that the map $\eta$ is a free action of $S^1$ on $W$. For the continuity of $\eta$ at $(e^{i\theta}, \Psi)$, we simply observe that

$$||\eta(e^{i\theta}, \Psi) - \eta(e^{i\theta_1}, \Psi) ||_W = ||e^{i\theta} \cdot \Psi - e^{i\theta_1} \cdot \Psi_1 ||_W$$

$$= ||e^{i\theta} \cdot \Psi - e^{i\theta_1} \cdot \Psi + e^{i\theta_1} \cdot \Psi - e^{i\theta_1} \cdot \Psi_1 ||_W$$

$$\leq |e^{i\theta} - e^{i\theta_1}| ||\Psi||_W + ||\Psi - \Psi_1 ||_W$$

$$= |e^{i\theta} - e^{i\theta_1}| \sqrt{L} + ||\Psi - \Psi_1 ||_W,$$

where $(e^{i\theta_1}, \Psi_1) \in S^1 \times W$ and

$$||\Psi||_W = \sqrt{||\psi^1||^2 + \ldots + ||\psi^L||^2} = \sqrt{L}.$$

Choosing $\delta = \sqrt{\frac{L}{\sqrt{L}}}$, we get that $\eta$ is continuous.

The orbit of $\Psi$ is given by

$$\eta(S^1 \times \{\Psi\}) = \{ \eta(e^{i\theta}, \Psi) : \theta \in [0, 2\pi) \}.$$
which is isometric to $S^1_{\sqrt{L}}$, the circle of radius $\sqrt{L}$ via the map

$$\varphi : \eta(S^1 \times \{\Psi\}) \rightarrow S^1_{\sqrt{L}}$$

which sends $e^{i\theta} \cdot \Psi$ to $\sqrt{L} e^{i\theta}$, where $\sqrt{L} = ||\Psi||_W$. Thus from Theorem 5.2.1 it follows that the space $W$ has the CIPH.

Remark. Let $W^M = \{(\psi^1, \ldots, \psi^L) \in W : \{\psi^1, \ldots, \psi^L\} \text{ is an MRA multiwavelet}\}$ and $W^S = \{(\psi^1, \ldots, \psi^L) \in W : \{\psi^1, \ldots, \psi^L\} \text{ is an MSF multiwavelet}\}$. By noting that the dimension function of $\Psi$, $D(\Psi)$ is equal to $D(\eta(e^{i\theta}, \Psi))$, it follows from Theorem 5.1.7, that $\Psi$ is an MRA wavelet iff $\eta(e^{i\theta}, \Psi)$ is an MRA wavelet. Also, we note that $\Psi$ is an MSF wavelet iff $\eta(e^{i\theta}, \Psi)$ is an MSF wavelet. Thus $W^M$, $W^S$ and $W^M \cap W^S$ are invariant sets in $W$ with respect to the action of topological group $S^1$. The orbits of these invariant sets remain isometric to $S^1_{\sqrt{L}}$. Thus from Theorem 5.2.1 it follows that the spaces $W^M$, $W^S$ and $W^M \cap W^S$ has the CIPH.

Theorem 5.3.2. If $A$ is an expansive matrix and $L \geq 1$ is an integer, then the space $W_{NT} = \{(\psi^1, \ldots, \psi^L) \in \prod_{1 \leq j \leq L} L^2(\mathbb{R}^n) : \{\psi^1, \ldots, \psi^L\} \text{ is a normalized tight frame multiwavelet}\}$ has the CIPH.

Proof. In the proof of previous Theorem it has been shown that the function

$$\eta : S^1 \times W_{NT} \rightarrow W_{NT}$$

defined by $\eta(e^{i\theta}, \Psi) = (e^{i\theta} \cdot \psi^1, \ldots, e^{i\theta} \cdot \psi^L)$ is a free action, where $\Psi = (\psi^1, \ldots, \psi^L)$

$\in W_{NT}$.

Now, for the continuity of $\eta$ at $(e^{i\theta}, \Psi)$, we simply observe that

$$||\eta(e^{i\theta}, \Psi) - \eta(e^{i\theta_1}, \Psi_1)||_W = ||e^{i\theta} \cdot \Psi - e^{i\theta_1} \cdot \Psi_1||_W$$

$$\leq |e^{i\theta} - e^{i\theta_1}| ||\Psi||_W + ||\Psi - \Psi_1||_W$$

$$\leq |e^{i\theta} - e^{i\theta_1}| \sqrt{L} + ||\Psi - \Psi_1||_W,$$

where $(e^{i\theta_1}, \Psi_1) \in S^1 \times W_{NT}$ and

$$||\Psi||_W = \sqrt{||\psi^1||_2^2 + \ldots + ||\psi^L||_2^2} \leq \sqrt{L}.$$
Choosing \( \delta = \frac{\epsilon}{\sqrt{L}} \), we get that \( \eta \) is continuous.

The orbit of \( \Psi \) is given by

\[
\eta(S^1 \times \{ \Psi \}) = \left\{ \eta(e^{i\theta}, \Psi) : \theta \in [0, 2\pi) \right\},
\]

which is isometric to \( S^1_r \), the circle of radius \( 0 < r \leq \sqrt{L} \) via the map

\[
\varphi : \eta(S^1 \times \{ \Psi \}) \longrightarrow S^1_r
\]

which sends \( e^{i\theta} \cdot \Psi \) to \( r e^{i\theta} \), where \( r = ||\Psi||_W \). Thus from Theorem 5.2.2 it follows that the space \( W_{NT} \) has the CIPH.

In the case of tight frame, the frame bounds A and B are equal but need not to be 1. After a renormalization, we can assume \( A = B = 1 \). If we denote \( W_T = \{(\psi^1, \ldots, \psi^L) \in \prod_{1 \leq j \leq L} L^2(\mathbb{R}^n) : \{\psi^1, \ldots, \psi^L\} \text{ is a tight frame multiwavelet}\} \), then we have the following result.

**Theorem 5.3.3.** The space \( W_T \) has the CIPH.

**Theorem 5.3.4.** Let \( W_{B_0} = \{(\psi^1, \ldots, \psi^L) \in \prod_{1 \leq j \leq L} L^2(\mathbb{R}^n) : \{\psi^1, \ldots, \psi^L\} \text{ is a frame multiwavelet with the upper frame bound bounded by } B_0\} \). Then the space \( W_{B_0} \) has the CIPH.

**Proof.** Let \( \Psi = (\psi^1, \ldots, \psi^L) \in W_{B_0} \), that is the collection

\[
\mathcal{A}(\Psi) = \{\Psi^l_{j,k} : j \in \mathbb{Z}, \ k \in \mathbb{Z}^n \text{ and } l = 1, \ldots, L\},
\]

is a frame of \( L^2(\mathbb{R}^n) \).

Then

\[
A||f||_2^2 \leq \sum_{l=1}^{L} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^n} | < f, \ \psi^l_{j,k} > |^2 \leq B||f||_2^2,
\]

for all \( f \in L^2(\mathbb{R}^n) \) where \( A \) and \( B \) are frame bounds of the frame generated by \( \Psi \).
Chapter 5. The CIPH on frame wavelet spaces

Now, we show that $e^{i\theta} \cdot \Psi$ is an element of $W_{B_0}$. That is,

$$A ||f||_2^2 \leq \sum_{l=1}^{L} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^n} |<f, e^{i\theta} \psi_{j,k}^l>|^2 \leq B ||f||_2^2,$$

for all $f \in L^2(\mathbb{R}^n)$.

Note that

$$<f, e^{i\theta} \psi_{j,k}^l> = \int f e^{i\theta} \psi_{j,k}^l = e^{i\theta} \int f \psi_{j,k}^l.$$

Hence we have,

$$|<f, \psi_{j,k}^l>|^2 = |<f, e^{i\theta} \psi_{j,k}^l>|^2.$$

Thus the map

$$\eta : S^1 \times W_{B_0} \to W_{B_0}$$

defined by $\eta(e^{i\theta}, \Psi) = e^{i\theta} \cdot \Psi$ is well defined and describes a free action of $S^1$ on $W_{B_0}$.

For the continuity of $\eta$ at $(e^{i\theta}, \Psi)$ we simply observe that

$$||\eta(e^{i\theta}, \Psi) - \eta(e^{i\theta_1}, \Psi_1)||_W = ||e^{i\theta} \cdot \Psi - e^{i\theta_1} \cdot \Psi_1||_W$$

$$\leq |e^{i\theta} - e^{i\theta_1}| ||\Psi||_W + ||\Psi - \Psi_1||_W$$

$$\leq |e^{i\theta} - e^{i\theta_1}| \sqrt{LB_0} + ||\Psi - \Psi_1||_W,$$

where $(e^{i\theta_1}, \Psi_1) \in S^1 \times W_{B_0}$ and

$$||\Psi||_W = \sqrt{||\psi_1||_2^2 + \ldots + ||\psi_L||_2^2} \leq \sqrt{LB_0}.$$

Choosing $\delta = \frac{\epsilon}{\sqrt{LB_0} + 1}$, we get that $\eta$ is continuous.

The orbit of $\Psi$ is given by

$$\eta(S^1 \times \{\Psi\}) = \{\eta(e^{i\theta}, \Psi) : \theta \in [0, 2\pi)\},$$

which is isometric to $S^1_r$, the circle of radius $0 < r \leq \sqrt{LB_0}$ via the map

$$\varphi : \eta(S^1 \times \{\Psi\}) \to S^1_r.$$
which sends $e^{i\theta} \cdot \Psi$ to $re^{i\theta}$, where $r = ||\Psi||_W$. Thus from Theorem 5.2.2 it follows that the space $W_{B_0}$ has the CIPH.

### 5.4 Super-wavelets and the CIPH

One of the problems in networking is multiplexing, which consists of sending multiple signals or streams of information on a carrier at the same time in the form of a single, complex signal and then recovering the separate signals at the receiving end. To solve these type of problems, Han and Larson [HL] have introduced the notion of super-wavelet which has applications in signal processing, data compression and image analysis. The prefix “super” is used because they are orthonormal basis generators for a “super-space” of $L^2(\mathbb{R})$, namely the direct sum of finitely many copies of $L^2(\mathbb{R})$.

**Definition 5.4.1.** Suppose that $\eta_1, \eta_2, ..., \eta_n$ are normalized tight frame wavelets for $L^2(\mathbb{R})$. We will call the $n$-tuple $(\eta_1, ..., \eta_n)$ a super-wavelet of length $n$ if

$$\{D^j T^k \eta_1 \oplus ... \oplus D^j T^k \eta_n : j, k \in \mathbb{Z}\}$$

is an orthonormal basis for $L^2(\mathbb{R}) \oplus ... \oplus L^2(\mathbb{R})$ (say, $L^2(\mathbb{R})^{\oplus n}$), where $D^j f(x) = 2^j f(2^j x)$ and $T^k f(x) = f(x-k)$ for $f \in L^2(\mathbb{R})$.

Han and Larson in their memoirs [HL] proved that for each $n$ ($n$ can be $\infty$), there is a super-wavelet of length $n$.

Following is a characterization of a super-wavelet of length $n$.

**Theorem 5.4.2.** Let $\eta_1, ..., \eta_n \in L^2(\mathbb{R})$. Then $(\eta_1, ..., \eta_n)$ is a super-wavelet of length $n$ if and only if the following equations hold:

(i) $\sum_{j \in \mathbb{Z}} \left| \hat{\eta}_i(2^j \xi) \right|^2 = 1$, for a.e. $\xi \in \mathbb{R}$, $i = 1, ..., n$,

(ii) $\sum_{j=0}^{\infty} \hat{\eta}_i(2^j \xi) \overline{\hat{\eta}_i(2^j (\xi + k))} = 0$, for a.e. $\xi \in \mathbb{R}$, $k \in 2\mathbb{Z} + 1$, $i = 1, ..., n$,

(iii) $\sum_{j \in \mathbb{Z}} \sum_{i=1}^{n} \left| \hat{\eta}_i(\xi + k) \right|^2 = 1$, for a.e. $\xi \in \mathbb{R}$,
(iv) $\sum_{j=0}^{\infty} \sum_{i=1}^{n} \hat{\eta}(2^j(\xi + k))\overline{\hat{\eta}(\xi + k)} = 0$, for a.e. $\xi \in \mathbb{R}, j \in \mathbb{N}$.

**Definition 5.4.3.** A super-wavelet $(\eta_1, ..., \eta_n)$ is said to be an MRA super-wavelet if every $\eta_i (i = 1, \ldots, n)$, is an MRA frame wavelet.

**Theorem 5.4.4.** Let $n \geq 2$ be an integer. Consider the set $SW_n$ defined by

$$SW_n = \{ \vartheta = (\eta_1, ..., \eta_n) : (\eta_1, ..., \eta_n) \text{ is a super-wavelet of length } n \text{ for } L^2(\mathbb{R})^\oplus n \}.$$

Then the space $SW_n$ has the CIPH.

**Proof.** Let $\vartheta = (\eta_1, ..., \eta_n)$ be an element of $SW_n$. From Theorem 5.4.2 it follows that $e^{i\theta} \cdot \vartheta = (e^{i\theta} \eta_1, ..., e^{i\theta} \eta_n)$ remains in $SW_n$, where $e^{i\theta} \in S^1$. Thus the map

$$\eta : S^1 \times SW_n \longrightarrow SW_n$$

defined by

$$\eta(e^{i\theta}, \vartheta) = e^{i\theta} \cdot \vartheta, \quad \vartheta \in SW_n, \quad e^{i\theta} \in S^1$$

is a free action.

The continuity of $\eta$ at $(e^{i\theta}, \vartheta)$ follows by noting that

$$||\eta(e^{i\theta}, \vartheta) - \eta(e^{i\theta_1}, \vartheta_1)|| = ||e^{i\theta} \cdot \vartheta - e^{i\theta_1} \cdot \vartheta_1||$$

$$\leq |e^{i\theta} - e^{i\theta_1}| ||\vartheta|| + ||\vartheta - \vartheta_1||$$

$$= |e^{i\theta} - e^{i\theta_1}| + ||\vartheta - \vartheta_1||,$$

where $(e^{i\theta_1}, \vartheta_1) \in S^1 \times SW_n$ and $||\vartheta|| = 1$.

The orbit of $\vartheta$ is given by

$$\eta(S^1 \times \{\vartheta\}) = \{ \eta(e^{i\theta}, \vartheta) : \theta \in [0, 2\pi) \},$$

which is isometric to $S^1$, via the map

$$\varphi : \eta(S^1 \times \{\vartheta\}) \longrightarrow S^1$$

which sends $e^{i\theta} \cdot \vartheta$ to $e^{i\theta}$. Thus from Theorem 1.3.4 it follows that the space $SW_n$ has the CIPH.
Remark. If $SW_n^M = \{ (\eta_1, \ldots, \eta_n) : (\eta_1, \ldots, \eta_n) \text{ is an MRA super-wavelet of length } n \text{ for } L^2(\mathbb{R})^{\oplus n} \}$, then for each $(\eta_1, \ldots, \eta_n) \in SW_n^M$, $(e^{i\theta} \eta_1, \ldots, e^{i\theta} \eta_n)$ is also an MRA super-wavelet. Thus $SW_n^M$ is an invariant set with respect to the action of $S^1$. Orbits of these invariant sets are isometric to the unit circle. Thus from Theorem 1.4.3 it follows that the space $SW_n^M$ has the CIPH.

Definition 5.4.5 [ZX]. Suppose that $(\eta_1, \eta_2, \ldots, \eta_n) \in L^2(\mathbb{R})^{\oplus n}$. We will call the $n$-tuple $(\eta_1, \ldots, \eta_n)$ a normalized tight super frame wavelet of length $n$ if

$$\{ D^j T^k \eta_1 \oplus \ldots \oplus D^j T^k \eta_n : j, k \in \mathbb{Z} \}$$

is a normalized tight frame for $L^2(\mathbb{R})^{\oplus n}$, where $D^j f(x) = 2^j f(2^j x)$ and $T^k f(x) = f(x - k)$ for $f \in L^2(\mathbb{R})$.

Theorem 5.4.6. Let $n \geq 2$ be an integer. Consider the set $SW_n^{NT}$ defined by $SW_n^{NT} = \{ \vartheta = (\eta_1, \ldots, \eta_n) : (\eta_1, \ldots, \eta_n) \text{ is a normalized tight super frame wavelet of length } n \text{ for } L^2(\mathbb{R})^{\oplus n} \}$. Then the space $SW_n^{NT}$ has the CIPH.

Proof. Let $\vartheta = (\eta_1, \ldots, \eta_n) \in SW_n^{NT}$. For $f = (f_1, \ldots, f_n) \in L^2(\mathbb{R})^{\oplus n}$ we have

$$||f||^2 = \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} | < f, D^j T^k \eta_1 \oplus \ldots \oplus D^j T^k \eta_n > |^2$$

$$= \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} | < f_1, D^j T^k \eta_1 > + \ldots + < f_n, D^j T^k \eta_n > |^2$$

$$= \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} | < f_1, D^j T^k e^{i\theta} \eta_1 > + \ldots + < f_n, D^j T^k e^{i\theta} \eta_n > |^2.$$

This shows that $e^{i\theta} \cdot \vartheta = (e^{i\theta} \eta_1, \ldots, e^{i\theta} \eta_n)$ remains in $SW_n^{NT}$, where $e^{i\theta} \in S^1$. Thus the map

$$\eta : S^1 \times SW_n^{NT} \longrightarrow SW_n^{NT}$$

defined by

$$\eta(e^{i\theta}, \vartheta) = e^{i\theta} \cdot \vartheta, \quad \vartheta \in SW_n^{NT}, \quad e^{i\theta} \in S^1$$

is a free action.

The continuity of $\eta$ at $(e^{i\theta}, \vartheta)$ follows by noting that
The CIPH on frame wavelet spaces

\[ ||\eta(e^{i\theta}, \vartheta) - \eta(e^{i\theta_1}, \vartheta_1)|| = ||e^{i\theta} \cdot \vartheta - e^{i\theta_1} \cdot \vartheta_1|| \]
\[ \leq |e^{i\theta} - e^{i\theta_1}| + ||\vartheta - \vartheta_1||, \text{ where } (e^{i\theta_1}, \vartheta_1) \in S^1 \times \mathcal{SW}_n^{NT} \]
and \[ ||\vartheta|| \leq 1. \]

The orbit of \( \vartheta \) is given by
\[ \eta(S^1 \times \{\vartheta\}) = \{\eta(e^{i\theta}, \vartheta) : \theta \in [0, 2\pi)\}, \]
which is isometric to \( S^1_r, 0 < r \leq 1 \), via the map
\[ \varphi : \eta(S^1 \times \{\vartheta\}) \longrightarrow S^1 \]
which sends \( e^{i\theta} \cdot \vartheta \) to \( e^{i\theta} \), where \( r = ||\vartheta||. \) Thus from Theorem 5.2.2 it follows that the space \( \mathcal{SW}_n^{NT} \) has the CIPH.

For a \( d \times d \), real expansive matrix \( A \), let \( D_A \) and \( T^k, (k \in \mathbb{R}^d) \) be the unitary operators on \( L_2(\mathbb{R}^d) \) defined by \( (D_Af)(x) = |A|^\frac{1}{2} f(Ax) \) and \( T^k f(x) = f(x - k) \) for \( f \in L^2(\mathbb{R}^d) \). Then we have the following

**Definition 5.4.7 [ZX].** Suppose that \( \eta_1, ..., \eta_n \) are \( A \)-dilation single normalized tight frame wavelets. We call the \( n \)-tuple \( (\eta_1, ..., \eta_n) \) an \( A \)-dilation normalized tight super frame wavelet of length \( n \) if
\[ \{ D_A^j T^k \eta_1 \oplus ... \oplus D_A^j T^k \eta_n : j \in \mathbb{Z}, k \in \mathbb{Z}^d \}, \]
is an \( A \)-dilation normalized tight frame for \( L_2(\mathbb{R}^d)^{\oplus n} \).

Thus from above definition analogous result to Theorem 5.4.6 holds in case of higher dimension as well.