CHAPTER - 2

A NEW VIEW ON $\gamma^*$-OPEN SETS

The concept of $b$-open set was introduced by Andrijevic [2]. In 2002, Min [31] used the idea of semi convergence of filters to introduce a new class of sets called $\gamma$-open sets and notions of $\gamma$-closure, $\gamma$-interior and $\gamma$-continuity and investigated some properties. The topological aspect of function spaces was first considered by Fox [11], Lambrinos [26], Porter [36,37] and other topologists introduced different set-open topologies on function spaces like bounded-open topology, open-open topology, regular-open topology etc.

In this chapter, new topology on function spaces with the help of $\gamma^*$-open set and open set named as $\gamma^*$-open open topology are introduced. Some interesting properties and characterizations of them are investigated. Different types of continuous like functions, relation among different types of continuity are introduced. Interrelations among them are discussed with relevant examples. Finally, different types of topologies on the corresponding function spaces are also introduced.
2.1. $\gamma^*$-OPEN TOPOLOGY FOR FUNCTION SPACES

In this section, the concept of $\gamma^*$-open set is introduced and some of its properties are discussed.

**Definition 2.1.1**

A subset $M(x)$ is called a b-neighbourhood of a point $x \in X$ if there exists a b-open set $S$ such that $x \in S \subseteq M(x)$.

**Note 2.1.1** : The family of all b-open sets in a topological space $(X, T)$ is denoted by $BO(X)$.

**Definition 2.1.2**

Let $S(x) = \{A : A \in BO(X) : x \in A\}$ and let

$$S_x = \{A \subseteq X : \text{there exists } \mu \subseteq S(x) \text{ such that } \mu \text{ is finite and } \cap \mu \subseteq A\}.$$  
Then $S_x$ is called the b-neighbourhood filter at $x$.

**Definition 2.1.3**

For any filter $\Gamma$ on $X$, we say that $\Gamma$ b-converges to $x$ iff $\Gamma$ is finer than the b-neighbourhood filter at $x$.

**Definition 2.1.4**

A subset $U$ of $X$ is called a $\gamma^*$-open set if whenever a filter $\Gamma$ b-converges to $x$, $x \in U$ and $U \in \Gamma$. The complement of a $\gamma^*$-open set is called a $\gamma^*$-closed set.

**Definition 2.1.5**

Let $(X, T)$ be a topological space. The intersection of all $\gamma^*$-closed sets containing $A$ is called the $\gamma^*$-closure of $A$, denoted by $cl_{\gamma^*}(A)$. A subset $A$ is $\gamma^*$-closed if $A = cl_{\gamma^*}(A)$. The family of all $\gamma^*$-open sets are
denoted by $T^{\gamma^*}$. Clearly $T^{\gamma^*}$ is a topology on $X$. In a topological space $(X, T)$ it is always true that $T \subseteq BO(X) \subseteq T^{\gamma^*}$.

**Definition 2.1.6**

A point $x \in X$ is said to be a $\gamma^*$-interior point of $A$ if there exists a $\gamma^*$-open set $U$ containing $x$ such that $U \subseteq A$. The set of all $\gamma^*$-interior points of $A$ is said to be $\gamma^*$-interior of $A$ and is denoted by $\text{int}_{\gamma^*}(A)$.

**Proposition 2.1.1**

Let $(X, T)$ be a topological space. For a subset $A$ of a space $X$, $\text{int}_{\gamma^*}(X - A) = X - \text{cl}_{\gamma^*}(A)$.

**Proof:**

Proof is simple.

**Definition 2.1.7**

A net $\{x_\lambda : \lambda \in \Lambda\}$ is said to $\gamma^*$-converge to $x \in X$ if $\{x_\lambda : \lambda \in \Lambda\}$ is eventually in each $\gamma^*$-open set containing $x$, denoted by $x_\lambda \xrightarrow{\gamma^*} x$.

**Definition 2.1.8**

Let $(X, T)$ and $(Y, S)$ be any two topological spaces. Let $\mathcal{U}$ and $\mathcal{V}$ be the collections of subsets of $X$ and $Y$ respectively. Let $F \subseteq Y^X$ be a collection of functions from $X$ into $Y$. We define, for $U \in \mathcal{U}$ and $V \in \mathcal{V}$, $(U, V) = \{ f \in F : f(U) \subseteq V \}$. Let $S(U, V) = \{ (U, V) : U \in \mathcal{U}, V \in \mathcal{V} \}$. If $S(U, V)$ is a subbasis for a topology $T(U, V)$ on $F$, then $T(U, V)$ is called a $\gamma^*$-set-set topology.

**Definition 2.1.9**

Let $\mathcal{U}$ be the collection of all $\gamma^*$-open sets in $X$ and $\mathcal{V}$ be the collection of all open sets in $Y$, then $S_{\gamma^* oo} = S(U, V)$ where $U \in \mathcal{U}$ and $V \in \mathcal{V}$. 

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$V \in \mathcal{V}$ is the subbasis for a topology, $T_{\gamma^{oo}}$, on any $F \subseteq Y^X$, is called the \( \gamma^* \)-open open topology.

**Proposition 2.1.2**

Let $F \subseteq C(X, Y)$ where $C(X, Y)$ is the set of all continuous functions from the topological space $(X, T)$ into $(Y, S)$. If $(Y, S)$ is $T_i$, for $i = 0, 1, 2$; then $(F, T_{\gamma^{oo}})$ is $T_i$, for $i = 0, 1, 2$.

**Proof**

Let us prove the case for $i = 2$. Let $f, g \in F$ be such that $f \neq g$. Then there exists some $x \in X$ such that $f(x) \neq g(x)$. If $Y$ is $T_2$, then there exist disjoint open sets $U$ and $V$ in $Y$ such that $f(x) \in U$ and $g(x) \in V$. Since $f$ and $g$ are continuous, there exist open sets $M$ and $N$ in $X$ with $x \in M \cap N$, such that $f(M) \subseteq U$ and $g(N) \subseteq V$. Since each open set is a $\gamma^*$-open set, $M$ and $N$ are $\gamma^*$-open. Hence, $f \in (M, U)$, $g \in (N, V)$ and $(M, U) \cap (N, V) = \emptyset$. Hence, $(F, T_{\gamma^{oo}})$ is $T_2$.

**Definition 2.1.10**

A topology $T^*$ on $F \subseteq Y^X$ is called $\gamma^*$ admissible topology for $F$ provided the evaluation map $E : (F, T^*) \times (X, T) \rightarrow (Y, S)$ defined by $E(f, x) = f(x)$ is continuous.

**Definition 2.1.11**

Let $(X, T)$ and $(Y, S)$ be any two topological spaces. A mapping $f : (X, T) \rightarrow (Y, S)$ is said to be $\gamma^*$-continuous if the inverse image of every open set of $(Y, S)$ is $\gamma^*$-open in $(X, T)$. The set of all $\gamma^*$-continuous mapping from $(X, T)$ into $(Y, S)$ is denoted by $\gamma^*C(X, Y)$. 
**Proposition 2.1.3**

Let \((X, T)\) and \((Y, S)\) be any two topological spaces. If \(F \subseteq \gamma^*C(X, Y)\), then \(T_{\gamma^{\infty}}\) is \(\gamma^*\)admissible for \(F\).

**Proof**

Let \(F \subseteq \gamma^*C(X, Y)\). Let \(V \in (Y, S)\) and \((f, p) \in E^{-1}(V)\). Then \(f(p) \in V\).

Since \(f\) is \(\gamma^*\)-continuous, there exists a \(\gamma^*\)-open set \(U \in T\) such that \(p \in U\) and \(f(U) \subseteq V\). Hence \((f, p) \in (U, V) \times U\). If \((g, y) \in (U, V) \times U\) then \(g(U) \subseteq V\) and \(y \in U\). So \(g(y) \in V\). Hence, \((U, V) \times U \subseteq E^{-1}(V)\). Therefore \(T_{\gamma^{\infty}}\) is admissible for \(F\).

**Definition 2.1.12**

Let \((X, T)\) and \((Y, S)\) be any two topological spaces. A mapping \(f : (X, T) \rightarrow (Y, S)\) is said to be \(\gamma^*\)-homeomorphism if it is a bijection so that the image and the inverse image of \(\gamma^*\)-open sets are \(\gamma^*\)-open.

The collection of all \(\gamma^*\)-homeomorphisms from \(X\) into \(Y\) is denoted by \(\gamma^*H(X, Y)\).

**Notation 2.1.1**

The set of all self \(\gamma^*\)-homeomorphisms on a topological space \((X, T)\) denoted by \(\gamma^*H(X)\).

**Remark 2.1.2**

The sets of the form \((U, V)\) where both \(U\) and \(V\) are \(\gamma^*\)-open sets in \(X\) form a subbasis for \((\gamma^*H(X), T_{\gamma^{\infty}})\). Let \((G, \circ)\) be a group such that \((G, T)\) is a topological space; then \((G, T)\) is a topological group provided the two maps
(i). \( m : G \times G \to G \), defined by \( m(g_1, g_2) = g_1 \circ g_2 \).

(ii). \( \varphi : G \to G \) defined by \( \varphi(g) = g^{-1} \) are continuous. If only the first map is continuous, then we call \((G, T)\) a quasi-topological group. Note that \( \gamma^*H(X) \) with the binary operation \( \circ \), composition of functions, and identity element \( e \), is a group.

**Proposition 2.1.4**

Let \((X, T)\) be a topological group and let \( G \) be a subgroup of \( \gamma^*H(X) \). Then \((G, T_{\gamma^{**}})\) is a topological group.

**Proof**

Let \((X, T)\) be a topological group and \( G \) be a subgroup of \( \gamma^*H(X) \). To prove that the two maps \( m : G \times G \to G \) defined by \( m(g_1, g_2) = g_1 \circ g_2 \) and \( \varphi : G \to G \) defined by \( \varphi(g) = g^{-1} \) are continuous.

Let \((U, V)\) be a subbasic open set in \( T_{\gamma^{**}} \) such that both \( U \) and \( V \) are \( \gamma^* \)-open sets. Let \((f, g) \in m^{-1}((U, V))\). Then \( f \circ g(U) \subseteq V \) and \( g(U) \subseteq f^{-1}(V) \). So \((f, g) \in (g(U), V) \times (U, g(U)) \in T_{\gamma^{**}} \times T_{\gamma^{**}} \). But \((g(U), V)) \times U \), \( g(U)) \subseteq m^{-1}((U, V)) \). Hence \( m \) is continuous.

Now, the inverse map \( \varphi : G \to G \) is bijective and \( \varphi^{-1} = \varphi \). To show that \( \varphi \) is continuous, it is sufficient to show that \( \varphi \) is an open map. Let \((U, V)\) be a subbasic open set in \( T_{\gamma^{**}} \) where \( U \) and \( V \) are both \( \gamma^* \)-open sets. To show that, \( \varphi \) is a \( \gamma^* \)-homeomorphism, we have to show that \( \varphi((U, V)) = ((X \setminus V, X \setminus U)) \). If \( C \) and \( D \) are \( \gamma^* \)-closed sets, then \( \text{int}_\gamma C \) and \( \text{int}_\gamma D \) are \( \gamma^* \)-open sets. Since \((X \setminus V), (X \setminus U)\) are \( \gamma^* \)-closed sets, \( \text{int}_\gamma (X \setminus V), \text{int}_\gamma (X \setminus U)\) are \( \gamma^* \)-open sets. Since \( G \) is a set of \( \gamma^* \)-homeomorphisms,
\((X \setminus V, X \setminus U) = (\text{int}_\gamma(X \setminus V), \text{int}_\gamma(X \setminus U)) \in T_{\gamma^{*\text{oo}}}. Therefore \varphi(U, V) is an open set in \(T_{\gamma^{*\text{oo}}} \). Hence \(\varphi \) is open.

**Proposition 2.1.5**

Let \(\{ h_\nu : \nu \in \mathcal{U} \} \) be a net in the group \(\gamma^{*}H(X)\) of self \(\gamma^{*}\)-homeomorphisms of a topological space \((X, T)\). Then \(h_\nu \to h\) in \(T_{\gamma^{*\text{oo}}} \) iff \(h_\nu(x_\lambda) \to h(x) \) whenever \(x_\lambda \to x\).

**Proof**

Suppose that \(h_\nu \to h\) in \(T_{\gamma^{*\text{oo}}} \) and let \(x_\lambda \to x\) in \(X\). Suppose \(V\) is \(\gamma^{*}\)-open and \(h(x) \in V\). Then there exists a \(\gamma^{*}\)-open set \(U\) in \(X\) containing \(x\) such that \((U, V)\) is a subbasic open set in \((\gamma^{*}H(X), T_{\gamma^{*\text{oo}}} \) containing \(h\). Then there exists \(\nu_0 \in \mathcal{U}\) such that \(h_\nu \in (U, V)\) for every \(\nu \geq \nu_0\). That is, \(h_\nu(U) \subseteq V\), for every \(\nu \geq \nu_0\). Since \(x_\lambda \to x\), for every \(\gamma^{*}\)-open set \(U\) containing \(x\), there exists \(\lambda_0 \in \Lambda\) such that \(x_\lambda \in U\), for every \(\lambda \geq \lambda_0\). Hence, for every \(\nu \geq \nu_0, \lambda \geq \lambda_0\); \(h_\nu(x_\lambda) \in \mathcal{U}\). Also \(h(x) \in V\). Hence \(\{h_\nu(x_\lambda) ; \nu \in \mathcal{U}, \lambda \in \Lambda\} \) \(\gamma^{*}\)-converges to \(h(x)\) wherever \(x_\lambda \to x\).

Conversely, if possible let \(h_\nu \not\to h\) in \(T_{\gamma^{*\text{oo}}} \). Then there exists a neighbourhood \((U, V)\) where \(U, V\) both are \(\gamma^{*}\) open sets of \(X\) containing \(h\) such that for any \(\nu' \in \mathcal{U}\) and there exists a \(\nu \in \mathcal{U}\) such that \(h_\nu \notin (U, V)\) for \(\nu \geq \nu'\). That is, \(h_\nu(U) \not\subseteq V \) for \(\nu \geq \nu'\). From every \(\gamma^{*}\)-open set \(U\) containing \(x\), choose an element \(x_U\) with \(h_\nu(x_U) \in V\). Let \(\gamma^{*}N_x\) be the \(\gamma^{*}\)-neighbourhood system at \(x\). Now, \(h \in (U, V)\) implies that \(h (U) \subseteq V\). Hence, for any \((\nu', U) \in \nu \times \gamma^{*}N_x\), there exists a \((\nu, U) \in \nu \times \gamma^{*}N_x\) such
that \( h_\nu(x_U) \not\in h(U) \), for \( (\nu, U) \geq (\nu', U) \). Hence \( h_\nu(x_U) \not\rightarrow h(x) \).

Contrapositively, whenever \( h_\nu(X_U) \not\rightarrow h(x) \) for \( x_U \not\rightarrow x \), \( h_\nu \rightarrow h \) in \( T_\gamma \).

2.2 DIFFERENT TYPES OF CONTINUOUS-LIKE MAPPINGS

In this section, different types of continuous like mappings and some of their properties are discussed.

**Definition 2.2.1**

Let \((X, T)\) and \((Y, S)\) be any two topological spaces. A mapping \( f : (X, T) \to (Y, S) \) is \( \gamma^* \)-continuous if the inverse image of every open set of \((Y, S)\) is \( \gamma^* \)-open in \((X, T)\).

The set of all \( \gamma^* \)-continuous mapping from \((X, T)\) into \((Y, S)\) is denoted by \( \gamma^* \mathcal{C}(X, Y) \).

**Example 2.2.1**

Consider \( X = \{0, 1, 2, 3\} \), \( Y = \{a, b, c, d\} \), with topologies

\[
T = \{\emptyset, X, \{0\}, \{0, 1\}\} \text{ and }
\]

\[
S = \{\emptyset, Y, \{b, c, d\}, \{a, b, d\}, \{b, d\}\} \text{ respectively}.
\]

Define a map \( f : (X, T) \to (Y, S) \) by \( f(0) = b \), \( f(1) = a \), \( f(2) = c \), \( f(3) = d \). Then the inverse image of each open set of \((Y, S)\) under \( f \) is \( \gamma^* \)-open in \((X, T)\). Hence \( f : (X, T) \to (Y, S) \) is a \( \gamma^* \)-continuous function.

**Proposition 2.2.1**

Let \((X, T)\) and \((Y, S)\) be any two topological spaces. A mapping \( f : (X, T) \to (Y, S) \) is \( \gamma^* \)-continuous at a point \( x \in (X, T) \) iff for every net \( \{x_\lambda : \lambda \in \Lambda\} \) in \((X, T)\) \( \gamma^* \)-converging to \( x \), the net \( \{f(x_\lambda) : \lambda \in \Lambda\} \) converges to \( f(x) \) in \((Y, S)\).
Proof

Assume that $f$ is $\gamma^*$-continuous at $x \in X$. Let $\{ x_\lambda : \lambda \in \Lambda \}$ be a net in $X$, $\gamma^*$-converges to $x$. Let $V$ be an open set in $Y$ containing $f(x)$. Now, there exists a $\gamma^*$-open set $U$ containing $x$ in $X$ such that $f(U) \subseteq V$. Now, the net $\{ x_\lambda : \lambda \in \Lambda \} \gamma^*$-converges to $x$ implies that there exists a $\lambda_0 \in \Lambda$ such that $x_\lambda \in U$ for all $\lambda \geq \lambda_0$. Hence, for all $\lambda \geq \lambda_0$, $f(x_\lambda) \in V$. Therefore, $\{ f(x_\lambda) : \lambda \in \Lambda \}$ lies eventually in $V$, and hence it converges to $f(x)$.

Conversely, let $f$ be not $\gamma^*$-continuous at $x$. Then there exists an open set $W$ containing $f(x)$ in $Y$ such that for every $\gamma^*$-open set $U$ containing $x \in X$, there exists an element $x_U$ with $x_U \in U$ but $f(x_U) \notin W$. Let $\gamma^*\mathcal{N}_x$ be the $\gamma^*$-neighbourhood system at $x$. So, the net $\{ x_U : U \in \gamma^*\mathcal{N}_x \}$ is a net in $(X, T)\gamma^*$-converges to $x$, but the net $\{ f(x_U) : U \in \gamma^*\mathcal{N}_x \}$ in $Y$ does not lie eventually in $W$ and consequently it cannot converge to $f(x)$ which is a contradiction. Hence the proof.

Definition 2.2.2

A net $\{ f_\mu : \mu \in M \}$ in $\gamma^*C(X, Y)$ $\gamma^*$-continuously converges to $f \in \gamma^* C (X, Y)$ if for every net $\{ x_\lambda : \lambda \in \Lambda \}$ in $X$ which $\gamma^*$-converges to $x$ in $X$, the net $\{ f_\mu(x_\lambda) : (\lambda, \mu) \in \Lambda \times M \}$ converging to $f(x)$ in $Y$, where the direction $\geq$ in $\Lambda \times M$ is given as follows: for $(\lambda_1, \mu_1), (\lambda_2, \mu_2) \in \Lambda \times M$, $(\lambda_1, \mu_1) \geq (\lambda_2, \mu_2)$ if $\lambda_1 \geq \lambda_2$ and $\mu_1 \geq \mu_2$.

Proposition 2.2.2

A net $\{ f_\mu : \mu \in M \}$ in $\gamma^*C(X, Y)$ $\gamma^*$-continuously converges to $f \in \gamma^*C(X, Y)$ iff for every $x \in X$ and for every open neighbourhood $V$ of
f(x) in Y, there exist a \( \mu_o \in M \) and a \( \gamma^* \)-open neighbourhood U of x in X such that \( f_{\mu}(U) \subseteq V \), for all \( \mu \geq \mu_o \).

**Proof**

Let \( \{ x_\lambda : \lambda \in \Lambda \} \) be a net in X \( \gamma^* \)-converging to x and let V be a neighbourhood of f(x). Suppose that the condition holds. Then there exists a \( \gamma^* \)-open neighbourhood U of x and a \( \mu_o \in M \) such that \( f_{\mu}(U) \subseteq V \), for all \( \mu \geq \mu_o \). Since \( x_\lambda \xrightarrow{\gamma^*} x \), there exists a \( \lambda_o \in \Lambda \) such that \( x_\lambda \in U \), for all \( \lambda \geq \lambda_o \). Hence, \( f_{\mu}(x_\lambda) \in V \), for all \( (\lambda, \mu) \geq (\lambda_o, \mu_o) \). That is \( f_{\mu}(x_\lambda) \rightarrow f(x) \) and thus the net \( \{ f_{\mu} : \mu \in M \} \) in \( \gamma^*C(X, Y) \) \( \gamma^* \)-continuously converges to \( f \in \gamma^*C(X, Y) \).

Conversely, let \( x \in X \) and let V be an open neighbourhood of f(x) such that for every \( \mu \in M \) and every \( \gamma^* \)-open neighbourhood U of x \( \in X \), there exists a \( \mu' \geq \mu, \mu' \in M \) such that \( f_{\mu'}(U) \not\subseteq V \). Then for every \( \gamma^* \)-open neighbourhood U of x, choose a point \( x_U \in U \) such that \( f_{\mu'}(x_U) \not\in V \). Hence the net \( \{ x_U : U \in \gamma^*N_x \} \) \( \gamma^* \)-converges to x but the net \( \{ f_{\mu}(x_U) : (U, \mu) \in (\gamma^*N_x \times M) \} \) does not converge to f(x) in Y.

**Definition 2.2.3**

Let P(X) be the set of all subsets of X. If \( \Lambda \) is a directed set, then \( \gamma^*\lim_{\Lambda} (A_\lambda) \), where \( A_\lambda \subseteq X \), is called the \( \gamma^* \)-upper limit of the net \( \{ A_\lambda : \lambda \in \Lambda \} \) in P(X). That is, the set of all points \( x \in X \) such that for every \( \lambda_o \in \Lambda \) and for every \( \gamma^* \)-open neighbourhood U of x in X, there exists a \( \lambda \in \Lambda \) for which \( \lambda \geq \lambda_o \) and \( A_\lambda \cap U \neq \phi \).
**Proposition 2.2.3**

If a net \( \{ f_\mu : \mu \in M \} \) in \( \gamma^*C(X, Y) \) \( \gamma^* \)-continuously converges to \( f \in \gamma^*C(X, Y) \), then \( \gamma^*\lim_M (f_\mu^{-1}(K)) \subseteq f^{-1}(K) \) for every closed subset \( K \) of \( Y \).

**Proof**

Let \( \{ f_\mu : \mu \in M \} \) be a net in \( \gamma^*C(X, Y) \) which \( \gamma^* \)-continuously converges to \( f \) and \( K \) be an arbitrary closed subset of \( Y \). Let \( x \in \gamma^*\lim_M (f_\mu^{-1}(K)) \) and let \( W \) be an open neighbourhood of \( f(x) \) in \( Y \). Since the net \( \{ f_\mu : \mu \in M \} \) is \( \gamma^* \) continuously converges to \( f \), there exists a \( \gamma^* \)-open neighbourhood \( V \) of \( x \) in \( X \) and a \( \mu_0 \in M \) such that \( f_\mu(V) \subseteq W \), for all \( \mu \geq \mu_0 \). Otherwise there exists a \( \mu \in M, \mu \geq \mu_0 \) such that \( V \cap f_\mu^{-1}(K) \neq \phi \). Hence, \( f_\mu(V) \cap K \subseteq W \cap K \neq \phi \). That is, \( f(x) \in \text{cl} (K) = K \). Thus \( x \in f^{-1}(K) \). Hence, the theorem is proved.

**Proposition 2.2.4**

Let \( \{ f_\mu : \mu \in M \} \) be a net in \( \gamma^*C(X, Y) \) such that \( \gamma^*\lim_M (f_\mu^{-1}(K)) \subseteq f^{-1}(K) \) for every closed subset \( K \) of \( Y \). Then the net \( \{ f_\mu : \mu \in M \} \) \( \gamma^* \)-continuously converges to \( f \in \gamma^*C(X, Y) \).

**Proof**

Let \( \{ f_\mu : \mu \in M \} \) be a net in \( \gamma^*C(X, Y) \) and \( f \in \gamma^*C(X, Y) \) be such that the given condition holds for every closed subset \( K \) of \( Y \). Let \( x \in X \) and \( W \) be an open neighbourhood of \( f(x) \) in \( Y \). Let \( K = Y \setminus W \). Since \( x \notin f^{-1}(K) \), \( x \notin \gamma^*\lim_M (f_\mu^{-1}(K)) \). Hence there exists a \( \mu_0 \in M \) and a \( \gamma^*-open
neighbourhood $U$ of $x$ in $X$ such that $f^{-1}_\mu(K) \cap U = \emptyset$ for $\mu \geq \mu_0$. Then $U \subseteq X \setminus f^{-1}_\mu(K) = f^{-1}_\mu(Y \setminus K) \subseteq f^{-1}_\mu(W)$. Thus $f^{-1}_\mu(U) \subseteq W$ for all $\mu \geq \mu_0$. That is, the net $\{ f_\mu : \mu \in M \}$ $\gamma^*$-continuously converges to $f \in \gamma^*C(X, Y)$.

**Definition 2.2.4**

Let $(X, T)$ and $(Y, S)$ be any two topological spaces. A mapping $f : (X, T) \to (Y, S)$ is called almost $\gamma^*$-continuous at $x \in X$ if for every open neighbourhood $V$ of $f(x)$ in $Y$, there exists a $\gamma^*$-open neighbourhood $U$ of $x$ in $X$ such that $f(U) \subseteq \text{int}(\text{cl}(V))$. A mapping $f : (X, T) \to (Y, S)$ is said to be almost $\gamma^*$-continuous on $X$ if $f$ is almost $\gamma^*$-continuous at each point $x$ in $X$. The set of all almost $\gamma^*$-continuous mapping $f : (X, T) \to (Y, S)$ is denoted by $A\gamma^*C(X, Y)$.

**Example 2.2.3**

Consider $X = \{0, 1, 2, 3\}$, $Y = \{a, b, c, d\}$ with topologies $T = \{\emptyset, X, \{0\}, \{0, 2\}, \{1, 2\}, \{2\}, \{0, 1, 2\}\}$ and $S = \{\emptyset, Y, \{a, b\}, \{a, b, d\}, \{d\}, \{b, d\}, \{b\}\}$ respectively.

Define a map $f : (X, T) \to (Y, S)$ by $f(0) = d$, $f(1) = c$, $f(2) = b$, $f(3) = a$.

Then $f : (X, T) \to (Y, S)$ is an almost $\gamma^*$-continuous mapping.

**Proposition 2.2.5**

A mapping $f : (X, T) \to (Y, S)$ is almost $\gamma^*$-continuous at a point $x \in X$ iff for every net $\{x_\lambda : \lambda \in \Lambda\}$ in $X$ is $\gamma^*$-converging to $x$, the net $\{f(x_\lambda) : \lambda \in \Lambda\}$ weakly $\theta^*$-converges to $f(x)$ in $Y$.

**Proof:** Similar to the proof of Proposition 2.2.1
**Definition 2.2.5**

A net \{ f_\mu : \mu \in M \} in \(A^*C(X, Y)\) almost \(\gamma^*\)-continuously converges to \(f \in A^*C(X, Y)\) if for every net \(\{x_\lambda : \lambda \in \Lambda\}\) in \(X\) which \(\gamma^*\)-converges to \(x\) in \(X\), the net \(\{f(x_\lambda) : (\lambda, \mu) \in (\Lambda \times M)\}\) weakly \(\theta\)- converges to \(f(x)\) in \(Y\).

**Proposition 2.2.6**

A net \{ f_\mu : \mu \in M \} in \(A^*C(X, Y)\) almost \(\gamma^*\)-continuously converges to \(f \in A^*C(X, Y)\) iff for every \(x \in X\) and for every open neighbourhood \(V\) of \(f(x)\) in \(Y\), there exist a \(\mu_0 \in M\) and a \(\gamma^*\)-open neighbourhood \(U\) of \(x\) in \(X\) such that \(f_\mu(U) \subseteq \text{int} (\text{cl}(V))\), for all \(\mu \geq \mu_0\).

**Proof:** Similar to the proof of Proposition 2.2.2

**Proposition 2.2.7**

If a net \(\{f_\lambda : \lambda \in \Lambda\}\) in \(A^*C(X, Y)\) almost \(\gamma^*\)-continuously converges to \(f \in A^*C(X, Y)\), then \(\gamma^*\lim_{\Lambda} (f_\lambda^{-1}(K)) \subseteq f^{-1}(K)\) for every \(\delta\)-closed subset \(K\) of \(Y\).

**Proof:** Similar to the proof of Proposition 2.2.3

**Proposition 2.2.8**

Let \(\{f_\lambda : \lambda \in \Lambda\}\) be a net in \(A^*C(X, Y)\) such that \(\gamma^*\lim_{\Lambda} (f_\lambda^{-1}(K)) \subseteq f^{-1}(K)\) for every closed subset \(K\) of \(Y\). Then the net \(\{f_\lambda : \lambda \in \Lambda\}\) almost \(\gamma^*\)-continuously converges to \(f \in A^*C(X, Y)\).

**Proof:** Similar to the proof of Proposition 2.2.4

**Definition 2.2.6**

Let \((X, T)\) and \((Y, S)\) be any two topological spaces. A mapping \(f : (X, T) \rightarrow (Y, S)\) is called almost strongly \(\theta\)-\(\gamma^*\)-continuous at \(x \in X\) if
for every open neighbourhood $V$ of $f(x)$ in $Y$, there exists a $\gamma^*$-open neighbourhood $U$ of $x$ in $X$ such that $f(\text{cl}_{\gamma^*}(U)) \subseteq \text{int}(\text{cl}(V))$. A mapping $f : (X, T) \to (Y, S)$ is said to be almost strongly $\theta-\gamma^*$-continuous on $X$ if $f$ is almost strongly $\theta-\gamma^*$-continuous at each point $x$ in $X$. The set of all almost strongly $\theta-\gamma^*$-continuous mapping $f : (X, T) \to (Y, S)$ is denoted by $\text{AS}\theta\gamma^*C(X, Y)$.

**Example 2.2.4**

Consider $X = \{0, 1, 2, 3\}$, $Y = \{a, b, c, d\}$ with topologies

$T = \{\emptyset, X, \{0\}, \{0, 2\}, \{1, 2\}, \{2\}, \{0, 1, 2\}\}$

$S = \{\emptyset, Y, \{a\}, \{a, b\}, \{a, b, d\}, \{d\}, \{a, d\}\}$ respectively.

Define a map $f : (X, T) \to (Y, S)$ by $f(0) = c$, $f(1) = a$, $f(2) = b$, $f(3) = d$.

Then $f : (X, T) \to (Y, S)$ is an almost strongly $\theta-\gamma^*$-continuous mapping.

**Definition 2.2.7**

A net $\{x_\lambda : \lambda \in \Lambda\}$ in $X$ $\theta-\gamma^*$-converges to $x \in X$ if for each $\gamma^*$-open neighbourhood $U$ of $x$, there exists a $\lambda_0 \in \Lambda$ such that $x_\lambda \in \text{cl}_{\gamma^*}(U)$ for all $\lambda \geq \lambda_0$.

**Proposition 2.2.9**

Let $(X, T)$ and $(Y, S)$ be any two topological space. A mapping $f : (X, T) \to (Y, S)$ is almost strongly $\theta-\gamma^*$-continuous at a point $x \in X$ iff for every net $\{x_\lambda : \lambda \in \Lambda\}$ in $X$ $\theta-\gamma^*$-converging to $x$, then the net $\{f(x_\lambda) : \lambda \in \Lambda\}$ weakly $\theta$-converges to $f(x)$ in $Y$.

**Proof**: Similar to the proof of Proposition 2.2.1
Definition 2.2.8

A net \( \{ f_\mu : \mu \in M \} \) in \( AS0\gamma^*C(X, Y) \) almost strongly \( 0-\gamma^* \)-continuously converges to \( f \in AS0\gamma^*C(X, Y) \) if for every net \( \{ x_\lambda : \lambda \in \Lambda \} \) in \( X \) which \( 0-\gamma^* \)-converges to \( x \) in \( X \), the net \( \{ f_\mu(x_\lambda) : (\lambda, \mu) \in (\Lambda \times M) \} \) weakly \( 0 \)-converges to \( f(x) \) in \( Y \).

Proposition 2.2.10

A net \( \{ f_\mu : \mu \in M \} \) in \( AS0\gamma^*C(X, Y) \) almost strongly \( 0-\gamma^* \)-continuously converges to \( f \in AS0\gamma^*C(X, Y) \) iff for every \( x \in X \) and for every open neighbourhood \( V \) of \( f(x) \) in \( Y \), there exist a \( \mu_0 \in M \) and a \( \gamma^* \)-open neighbourhood \( U \) of \( x \) in \( X \) such that \( f_\mu(\overline{\gamma^*}(U)) \subseteq \text{int}(\overline{\text{cl}(V)}) \) for all \( \mu \geq \mu_0 \).

Proof: Similar to the proof of Proposition 2.2.2

Definition 2.2.9

Let \( P(X) \) be the set of all subsets of \( X \). If \( \Lambda \) is directed set, then \( 0-\gamma^* \lim_{\Lambda} (A_\lambda) \), where \( A_\lambda \subseteq X \), is called the \( 0-\gamma^* \)-upper limit of the net \( \{ A_\lambda : \lambda \in \Lambda \} \) in \( P(X) \). That is, the set of all points \( x \in X \) such that for every \( \lambda_0 \in \Lambda \) and for every \( \gamma^* \)-open neighbourhood \( U \) of \( x \) in \( X \), there exists a \( \lambda \in \Lambda \) for which \( \lambda \geq \lambda_0 \) and \( A_\lambda \cap \overline{\gamma^*}(U) \neq \emptyset \).

Proposition 2.2.11

If a net \( \{ f_\lambda : \lambda \in \Lambda \} \) in \( AS0\gamma^*C(X, Y) \) almost strongly \( 0-\gamma^* \)-continuously converges to \( f \in AS0\gamma^*C(X, Y) \), then \( 0-\gamma^* \lim_{\Lambda} (f_\lambda^{-1}(K)) \subseteq f^{-1}(K) \) for every \( \delta \)-closed subset \( K \) of \( Y \).

Proof: Similar to the proof of Proposition 2.2.3
Proposition 2.2.12
Let \{ f_\lambda : \lambda \in \Lambda \} be a net in \text{AS}\gamma^*\text{C}(X, Y) such that
\[ 0^- \gamma^* \lim_{\lambda} (f_\lambda)^{-1}(K) \subseteq f_\lambda^{-1}(K) \] for every closed subset K of Y. Then the net \{f_\lambda : \lambda \in \Lambda\} almost strongly \(0^-\gamma^*\)-continuously converges to \(f \in \text{AS}\gamma^*\text{C}(X, Y)\).

Proof: Similar to the proof of Proposition 2.2.4

Definition 2.2.10
Let \((X, T)\) and \((Y, S)\) be any two topological spaces. A mapping \(f : (X, T) \rightarrow (Y, S)\) is weakly \(\gamma^*\)-continuous at a point \(x \in X\) if for every open neighbourhood \(V\) of \(f(x)\) in \(Y\), there exists a \(\gamma^*\)-open neighbourhood \(U\) of \(x\) in \(X\) such that \(f(U) \subseteq \text{cl}(V)\). A mapping \(f : (X, T) \rightarrow (Y, S)\) is said to be weakly \(\gamma^*\)-continuous on \(X\) if \(f\) is weakly \(\gamma^*\)-continuous at each point \(x \in X\). The set of all weakly \(\gamma^*\)-continuous mapping \(f : (X, T) \rightarrow (Y, S)\) is denoted by \(\text{W}\gamma^*\text{C}(X, Y)\).

Example 2.2.4
Consider \(X = \{0, 1, 2, 3\}\), \(Y = \{a, b, c, d\}\) with topologies
\[ T = \{\emptyset, X, \{0\}, \{0, 1\}, \{0, 3\}, \{2, 3\}, \{3\}, \{0, 1, 3\}, \{0, 2, 3\}\} \] and
\[ S = \{\emptyset, Y, \{a, b\}, \{c\}, \{a, b, c\}\} \] respectively.
Define a map \(f : (X, T) \rightarrow (Y, S)\) by \(f(0) = a, f(1) = b, f(2) = c, f(3) = d\).

Then \(f : (X, T) \rightarrow (Y, S)\) is a weakly \(\gamma^*\)-continuous mapping.

Definition 2.2.11
A net \(\{x_\lambda : \lambda \in \Lambda\}\) in \(X\) is \(\theta\)-converges to \(x \in X\) if for each neighbourhood \(U\) of \(x\), there exists a \(\lambda_0 \in \Lambda\) such that \(x_\lambda \in \text{cl}(U)\) for all \(\lambda \geq \lambda_0\).
**Proposition 2.2.13**

Let \((X, T)\) and \((Y, S)\) be any two topological spaces. A map \(f : (X, T) \rightarrow (Y, S)\) is an weakly \(\gamma^*\)-continuous at a point \(x \in X\) iff for every net \(\{x_\lambda : \lambda \in \Lambda\}\) in \(X\) \(\gamma^*\)-converges to \(x\) in \(X\), the net \(\{f(x_\lambda) : \lambda \in \Lambda\}\) in \(Y\) is \(\theta\)-converges to \(f(x)\) in \(Y\).

**Proof:** Similar to the proof of Proposition 2.2.1

**Definition 2.2.12**

A net \(\{f_\mu : \mu \in M\}\) in \(W\gamma^*C(X, Y)\) weakly \(\gamma^*\)-continuously converges to \(f \in W\gamma^*C(X, Y)\) if for every net \(\{x_\lambda : \lambda \in \Lambda\}\) in \(X\) which \(\gamma^*\)-converges to \(x\) in \(X\), the net \(\{f_\mu(x_\lambda) : (\lambda, \mu) \in \Lambda \times M\}\) \(\theta\)-converges to \(f(x)\) in \(Y\).

**Proposition 2.2.14**

A net \(\{f_\mu : \mu \in M\}\) in \(W\gamma^*C(X, Y)\) weakly \(\gamma^*\)-continuously converges to \(f \in W\gamma^*C(X, Y)\) if and only if for every \(x \in X\) and for every open neighbourhood \(V\) of \(f(x)\) in \(Y\), there exist a \(\mu_o \in M\) and a \(\gamma^*\)-open neighbourhood \(U\) of \(x\) in \(X\) such that \(f_\mu(U) \subseteq cl(V)\) for all \(\mu \geq \mu_o\).

**Proof:** Similar to the proof of Proposition 2.2.2

**Proposition 2.2.15**

If a net \(\{f_\lambda : \lambda \in \Lambda\}\) in \(W\gamma^*C(X, Y)\) weakly \(\gamma^*\)-continuously converges to \(f \in W\gamma^*C(X, Y)\), then \(\gamma^*\lim_\lambda (f_\lambda^{-1}(K)) \subseteq f^{-1}(K)\) for every \(\theta\)-closed subset \(K\) of \(Y\).

**Proof:** Similar to the proof of Proposition 2.2.3
Proposition 2.2.16

Let \( \{ f_\lambda : \lambda \in \Lambda \} \) be a net in \( W_{\gamma^*}C(X, Y) \) such that

\[
\gamma^*\lim_{\lambda} (f_\lambda^{-1}(K)) \subseteq f^{-1}(K)
\]

for every closed subset \( K \) of \( Y \). Then the net \( \{ f_\lambda : \lambda \in \Lambda \} \) weakly \( \gamma^* \)-continuously converges to \( f \in W_{\gamma^*}C(X, Y) \).

**Proof**: Similar to the proof of Proposition 2.2.4

Definition 2.2.13

Let \((X, T)\) and \((Y, S)\) be any two topological spaces. A mapping \( f : (X, T) \to (Y, S) \) is called weakly \( \theta_{\gamma^*} \)-continuous at \( x \in X \), if for every open neighbourhood \( V \) of \( f(x) \) in \( Y \), there exists a \( \gamma^* \)-open neighbourhood \( U \) of \( x \) in \( X \) such that \( f(\text{int}_{\gamma^*}(\text{cl}_{\gamma^*}(U))) \subseteq \text{cl}(V) \). A mapping \( f : (X, T) \to (Y, S) \) is said to be weakly \( \theta_{\gamma^*} \)-continuous on \( X \) if \( f \) is weakly\( \theta_{\gamma^*} \)-continuous at each point \( x \) in \( X \). The set of all weakly \( \theta_{\gamma^*} \)-continuous mapping \( f : (X, T) \to (Y, S) \) is denoted by \( W_{\theta_{\gamma^*}}C(X, Y) \).

Example 2.2.5

Consider \( X = \{ 0, 1, 2, 3 \} \), \( Y = \{ a, b, c, d \} \) with topologies

\[
T = \{ \emptyset, X, \{ 0 \}, \{ 1 \}, \{ 0, 1 \}, \{ 0, 3 \}, \{ 0, 1, 3 \} \} \text{ and }
\]

\[
S = \{ \emptyset, Y, \{ a, c \}, \{ c \}, \{ a, d \}, \{ a \}, \{ a, c, d \}, \{ b, c \}, \{ a, b, c \} \}
\]

respectively.

Define a map \( f : (X, T) \to (Y, S) \) by \( f(0) = a, f(1) = b, f(2) = c, f(3) = d \).

Then \( f : (X, T) \to (Y, S) \) is a weakly \( \theta_{\gamma^*} \)-continuous mapping.
**Definition 2.2.14**

A net \( \{ x_\lambda : \lambda \in \Lambda \} \) in \( X \) is weakly \( \theta-\gamma^* \)-converges to \( x \in X \), if for each \( \gamma^* \)-open neighbourhood \( U \) of \( x \), there exists a \( \lambda_0 \in \Lambda \) such that \( x_\lambda \in \operatorname{int}_{\gamma^*}(\operatorname{cl}_{\gamma^*}(U)) \) for all \( \lambda \geq \lambda_0 \).

**Proposition 2.2.17**

Let \( (X, T) \) and \( (Y, S) \) be any two topological spaces. A map \( f : (X, T) \to (Y, S) \) is weakly \( \theta-\gamma^* \)-continuous at a point \( x \in X \) iff for every net \( \{ x_\lambda : \lambda \in \Lambda \} \) in \( X \) weakly \( \theta-\gamma^* \)-converging to \( x \) in \( X \), the net \( \{ f(x_\lambda) : \lambda \in \Lambda \} \) in \( Y \) \( \theta \)-converges to \( f(x) \) in \( Y \).

**Proof:** Similar to the proof of Proposition 2.2.1

**Definition 2.2.15**

A net \( \{ f_\mu : \mu \in M \} \) in \( W\theta\gamma^*C(X, Y) \) weakly \( \theta-\gamma^* \)-continuously converges to \( f \in W\theta\gamma^*C(X, Y) \) if for every net \( \{ x_\lambda : \lambda \in \Lambda \} \) in \( X \) which weakly \( \theta-\gamma^* \)-converges to \( x \) in \( X \), the net \( \{ f_\mu(x_\lambda) : (\lambda, \mu) \in (\Lambda \times M) \} \) \( \theta \)-converges to \( f(x) \) in \( Y \).

**Proposition 2.2.18**

A net \( \{ f_\mu : \mu \in M \} \) in \( W\theta\gamma^*C(X, Y) \) weakly \( \theta-\gamma^* \)-continuously converges to \( f \in W\theta\gamma^*C(X, Y) \) iff for every \( x \in X \) and for every open neighbourhood \( V \) of \( f(x) \) in \( Y \), there exist a \( \mu_0 \in M \) and a \( \gamma^* \)-open neighbourhood \( U \) of \( x \) in \( X \) such that \( f_\mu \left( \operatorname{int}_{\gamma^*}(\operatorname{cl}_{\gamma^*}(U)) \right) \subseteq \operatorname{cl}(V) \) for all \( \mu \geq \mu_0 \).

**Proof:** Similar to the proof of Proposition 2.2.2
**Definition 2.2.16**

Let $P(X)$ be the set of all subsets of $X$. If $\Lambda$ is a directed set, then $w^\theta \gamma^* \lim_{\Lambda} (A_{\lambda})$, where $A_{\lambda} \subseteq X$, is called the weak $\theta^\gamma^*$ upper limit of the net $\{ A_{\lambda} : \lambda \in \Lambda \}$ in $P(X)$. That is, the set of all points $x \in X$, such that for every $\lambda_0 \in \Lambda$ and for every $\gamma^*$-open neighbourhood $U$ of $x$ in $X$, there exists a $\lambda \in \Lambda$ for which $\lambda \geq \lambda_0$ and $A_{\lambda} \cap \text{int}_{\gamma^*}(\text{cl}_{\gamma^*}(U)) \neq \phi$.

**Proposition 2.2.19**

If a net $\{ f_{\lambda} : \lambda \in \Lambda \}$ in $W\theta^\gamma^* C(X, Y)$ weakly $\theta^\gamma^*$-continuously converges to $f \in W\theta^\gamma^* C(X, Y)$, then $w^\theta \gamma^* \lim_{\Lambda} (f_{\lambda}^{-1}(K)) \subseteq f^{-1}(K)$ for every $\theta$-closed subset $K$ of $Y$.

**Proof**: Similar to the proof of Proposition 2.2.3

**Proposition 2.2.20**

Let $\{ f_{\lambda} : \lambda \in \Lambda \}$ be a net in $W\theta^\gamma^* C(X, Y)$ such that $w^\theta \gamma^* \lim_{\Lambda} (f_{\lambda}^{-1}(K)) \subseteq f^{-1}(K)$ for every closed subset $K$ of $Y$. Then the net $\{ f_{\lambda} : \lambda \in \Lambda \}$ weakly $\theta^\gamma^*$-continuously converges to $f \in W\theta^\gamma^* C(X, Y)$.

**Proof**: Similar to the proof of Proposition 2.2.4

**Definition 2.17**

Let $(X, T)$ and $(Y, S)$ be any two topological spaces. A mapping $f : (X, T) \to (Y, S)$ is called super $\gamma^*$-continuous at a point $x \in X$, if for every open neighbourhood $V$ of $f(x)$ in $Y$, there exists a $\gamma^*$-open neighbourhood $U$ of $x$ in $X$ such that $f(\text{int}_{\gamma^*}(\text{cl}_{\gamma^*}(U))) \subseteq V$. A mapping $f : (X, T) \to (Y, S)$ is said to be super $\gamma^*$-continuous on $X$ if $f$ is super $\gamma^*$-continuous at each point $x$ in $X$.
The set of all super $\gamma^*$-continuous mapping $f : (X, T) \rightarrow (Y, S)$ is denoted by $S\gamma^*C(X, Y)$.

**Example 2.2.6**

Consider $X = \{0, 1, 2, 3\}$, $Y = \{a, b, c, d\}$ with topologies $T = \{\emptyset, X, \{0\}, \{0, 3\}, \{1, 2, 3\}, \{3\}, \{1, 3\}\}$ and $S = \{\emptyset, Y, \{a\}, \{b, c, d\}\}$ respectively.

Define a map $f : (X, T) \rightarrow (Y, S)$ by $f(0) = a$, $f(1) = b$, $f(2) = c$, $f(3) = d$.

Then $f : (X, T) \rightarrow (Y, S)$ is an super $\gamma^*$-continuous function.

**Proposition 2.2.21**

Let $(X, T)$ and $(Y, S)$ be any two topological spaces. A mapping $f : (X, T) \rightarrow (Y, S)$ is super $\gamma^*$-continuous at a point $x \in X$ if and only if for every net $\{x_\lambda : \lambda \in \Lambda\}$ in $X$ weakly $\theta-\gamma^*$-converges to $x$ in $X$, the net $\{f(x_\lambda) : \lambda \in \Lambda\}$ converges to $f(x)$ in $Y$.

**Proof**: Similar to the proof of Proposition 2.2.1

**Definition 2.2.18**

A net $\{f_\mu : \mu \in M\}$ in $S\gamma^*C(X, Y)$ super $\gamma^*$-continuously converges to $f \in S\gamma^*C(X, Y)$ if for every net $\{x_\lambda : \lambda \in \Lambda\}$ in $X$ which weakly $\theta-\gamma^*$-converges to $x$ in $X$, the net $\{f_\mu(x_\lambda) : (\lambda, \mu) \in (\Lambda \times M)\}$ converges to $f(x)$ in $Y$.

**Proposition 2.2.22**

A net $\{f_\mu : \mu \in M\}$ in $S\gamma^*C(X, Y)$ super $\gamma^*$- continuously converges to $f \in S\gamma^*C(X, Y)$ iff for every $x \in X$ and for every open neighbourhood
V of f(x) in Y, there exist a $\mu_0 \in M$ and a $\gamma^*$-open neighbourhood $U$ of $x$ in $X$ such that $f_{\mu}(\text{int } \gamma^*(\text{cl } \gamma^*(U))) \subseteq V$ for all $\mu \geq \mu_0$.

**Proof:** Similar to the proof of Proposition 2.2.2

**Proposition 2.2.23**

If a net $\{ f_{\lambda} : \lambda \in \Lambda \}$ in $S\gamma^*C(X, Y)$ super $\gamma^*$-continuously converges to $f \in S\gamma^*C(X, Y)$, then $w-\theta-\gamma^*\lim_{\Lambda} (f_{\lambda}^{-1}(K)) \subseteq f^{-1}(K)$ for every closed subset $K$ of $Y$.

**Proof:** Similar to the proof of Proposition 2.2.3

**Proposition 2.2.24**

Let $\{ f_{\lambda} : \lambda \in \Lambda \}$ be a net in $S\gamma^*C(X, Y)$ such that $w-\theta-\gamma^*\lim_{\Lambda} (f_{\lambda}^{-1}(K)) \subseteq f^{-1}(K)$ for every closed subset $K$ of $Y$. Then the net $\{f_{\lambda} : \lambda \in \Lambda\}$ super $\gamma^*$-continuously converges to $f \in S\gamma^*C(X, Y)$.

**Proof:** Similar to the proof of Proposition 2.2.4

### 2.3 RELATIONS AMONG DIFFERENT TYPES OF $\gamma^*$-CONTINUITY

In this section, the relations among different types of $\gamma^*$-continuity are studied. Counter examples are also discussed.

**Proposition 2.3.1**

(i). Every super $\gamma^*$-continuous mapping is $\gamma^*$-continuous mapping.

(ii). Every $\gamma^*$-continuous mapping is weakly $\theta-\gamma^*$-continuous mapping.

(iii). Every super $\gamma^*$-continuous mapping is almost strongly $\theta-\gamma^*$-continuous mapping.

(iv). Every almost strongly $\theta-\gamma^*$-continuous mapping is almost $\gamma^*$-continuous mapping.
(v). Every $\gamma^*$-continuous mapping is weakly $\theta-\gamma^*$-continuous mapping.

(vi). Every weakly $\theta\gamma^*$-continuous mapping is weakly $\gamma^*$-continuous mapping.

(vii). Every $\gamma^*$-continuous mapping is almost $\gamma^*$-continuous mapping.

**Proof**: Proof is obvious from the definition

**Remark 2.3.1**

The reverse implications need not be true which is shown in the following examples.

**Example 2.3.1**

Consider $X = \{0, 1, 2, 3\}$, $Y = \{a, b, c, d\}$ with topologies

$T = \{\emptyset, X, \{0\}, \{0, 1\}, \{0, 2\}\}$ and

$S = \{\emptyset, Y, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$ respectively.

Define a map $f : (X, T) \to (Y, S)$ by $f(0) = a$, $f(1) = b$, $f(2) = c$, $f(3) = d$.

Then $f$ is a $\gamma^*$-continuous mapping. But not super $\gamma^*$-continuous, for $\{a, b\}$, $\{a, b, c\}$, $\{a, b, d\}$ there does not exist $\gamma^*$ open set.

**Example 2.3.2**

Consider $X = \{0, 1, 2, 3\}$, $Y = \{a, b, c, d\}$ with topologies

$T = \{\emptyset, X, \{0\}, \{1\}, \{0, 1\}, \{0, 3\}, \{0, 1, 3\}\}$ and

$S = \{\emptyset, Y, \{a, b\}, \{b\}, \{b, c, d\}\}$ respectively.

Define a map $f : (X, T) \to (Y, S)$ by $f(0) = a$, $f(1) = b$, $f(2) = c$, $f(3) = d$. 

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Then \( f \) is a weakly \( \theta\gamma\ast \)-continuous mapping. But not \( \gamma\ast \)-continuous, for \( \{ 1, 2, 3 \} \) is not a \( \gamma\ast \)-open set in \( X \).

**Example 2.3.3**

Consider \( X = \{ 0, 1, 2, 3 \} \), \( Y = \{ a, b, c, d \} \) with topologies
\[
T = \{ \emptyset, X, \{ 0 \}, \{ 0, 2 \}, \{ 0, 2, 3 \}, \{ 2 \}, \{ 0, 1 \} \},
\]
and \( S = \{ \emptyset, Y, \{ a \}, \{ b \} \} \) respectively.

Define a map \( f : (X, T) \to (Y, S) \) by \( f(0) = c, f(1) = b, f(2) = a, f(3) = d \).

Then \( f \) is a weakly \( \gamma\ast \)-continuous mapping. But not weakly \( \theta\gamma\ast \)-continuous mapping, for \( \{ a \}, \{ b \} \) there does not exist \( \gamma\ast \)-open set.

**Example 2.3.4**

Consider \( X = \{ 0, 1, 2, 3 \} \), \( Y = \{ a, b, c, d \} \) with topologies
\[
T = \{ \emptyset, X, \{ 0 \}, \{ 0, 2 \}, \{ 1, 2 \}, \{ 2 \}, \{ 0, 1 \} \} \text{ and}
\]
\[
S = \{ \emptyset, Y, \{ a \}, \{ b \}, \{ a, b \}, \{ b, c \}, \{ a, b, c \} \} \text{ respectively.}
\]

Define a map \( f : (X, T) \to (Y, S) \) by \( f(0) = a, f(1) = d, f(2) = b, f(3) = c \).

Then \( f \) is an almost \( \gamma\ast \)-continuous mapping. But not almost strongly \( \theta\gamma\ast \)-continuous mapping, for \( \{ b \}, \{ b, c \} \) there does not exist \( \gamma\ast \)-open set.

**Example 2.3.5**

Consider \( X = \{ 0, 1, 2, 3 \} \), \( Y = \{ a, b, c, d \} \) with topologies
\[
T = \{ \emptyset, X, \{ 0 \}, \{ 0, 2 \}, \{ 1, 2 \}, \{ 2 \}, \{ 0, 1 \} \} \text{ and}
\]
\[
S = \{ \emptyset, Y, \{ a \}, \{ a, b \}, \{ d \}, \{ a, b, d \}, \{ a, d \} \} \text{ respectively.}
\]
Define a map $f : (X, T) \rightarrow (Y, S)$ by $f(0) = c$, $f(1) = a$, $f(2) = b$, $f(3) = d$.

Then $f$ is an almost strongly $\theta$-$\gamma^*$-continuous mapping. But not a super $\gamma^*$-continuous mapping, for $\{a, d\}$ there does not exist $\gamma^*$-open set.

Example 2.3.6

Consider $X = \{0, 1, 2, 3\}$, $Y = \{a, b, c, d\}$ with topologies

$T = \{\emptyset, X, \{0\}, \{0, 2\}, \{1, 2\}, \{2\}, \{0, 1, 2\}\}$ and

$S = \{\emptyset, Y, \{a\}, \{a, b\}, \{d\}, \{a, b, d\}, \{a, d\}\}$ respectively.

Define a map $f : (X, T) \rightarrow (Y, S)$ by $f(0) = c$, $f(1) = a$, $f(2) = b$, $f(3) = d$. Then $f$ is an almost strongly $\theta$-$\gamma^*$-continuous mapping. But not $\gamma^*$-continuous mapping, for $\{a, d\}$ there does not exist $\gamma^*$-open set.

Example 2.3.7

Consider $X = \{0, 1, 2, 3\}$, $Y = \{a, b, c, d\}$ with topologies

$T = \{\emptyset, X, \{0\}, \{0, 2\}, \{1, 2\}, \{2\}, \{0, 1, 2\}\}$ and

$S = \{\emptyset, Y, \{a\}, \{a, c\}, \{a, b\}, \{b\}, \{a, b, d\}\}$ respectively.

Define a map $f : (X, T) \rightarrow (Y, S)$ by $f(0) = a$, $f(1) = d$, $f(2) = b$, $f(3) = c$.

Then $f$ is a $\gamma^*$-continuous mapping. But not an almost strongly $\theta$-$\gamma^*$-continuous mapping, for $\{b\}$ there does not exist $\gamma^*$-open set.

Example 2.3.8

Consider $X = \{0, 1, 2, 3\}$, $Y = \{a, b, c, d\}$ with topologies
\[ T = \{ \emptyset, X, \{ 0, 1 \}, \{ 0, 3 \}, \{ 0 \}, \{ 2, 3 \}, \{ 3 \}, \{ 0, 1, 3 \}, \{ 0, 2, 3 \} \} \]

and \( S = \{ \emptyset, Y, \{ a, b \}, \{ c \}, \{ a, b, c \} \} \) respectively.

Define a map \( f : (X, T) \to (Y, S) \) by \( f(0) = a, f(1) = b, f(2) = c, f(3) = d \).

Then \( f \) is a weakly \( \gamma^* \)-continuous mapping. But not an almost strongly \( \theta \)-\( \gamma^* \)-continuous mapping, for \( \{ c \} \) there does not exist \( \gamma^* \)-open set.

**Example 2.3.9**

Consider \( X = \{ 0, 1, 2, 3 \}, Y = \{ a, b, c, d \} \) with topologies

\[ T = \{ \emptyset, X, \{ 0 \}, \{ 0, 2 \}, \{ 1, 2 \}, \{ 2 \}, \{ 0, 1, 2 \} \} \]

and

\[ S = \{ \emptyset, Y, \{ a \}, \{ c \}, \{ a, c \}, \{ a, d \}, \{ a, c, d \}, \{ a, b, d \} \} \] respectively.

Define a map \( f : (X, T) \to (Y, S) \) by \( f(0) = d, f(1) = a, f(2) = b, f(3) = c \).

Then \( f \) is an almost \( \gamma^* \)-continuous mapping. But not a \( \gamma^* \)-continuous mapping, for \( \{ a \} \) there does not exist \( \gamma^* \)-open set.

**Example 2.3.10**

Consider \( X = \{ 0, 1, 2, 3 \}, Y = \{ a, b, c, d \} \) with topologies

\[ T = \{ \emptyset, X, \{ 0 \}, \{ 0, 2 \}, \{ 1, 2 \}, \{ 2 \}, \{ 0, 1, 2 \} \} \]

and

\[ S = \{ \emptyset, Y, \{ a, b \}, \{ a, b, d \}, \{ d \}, \{ b, d \}, \{ b \} \} \] respectively.

Define a map \( f : (X, T) \to (Y, S) \) by \( f(0) = d, f(1) = c, f(2) = b, f(3) = a \).

Then \( f \) is an almost \( \gamma^* \)-continuous mapping. But not a weakly \( \theta \)-\( \gamma^* \)-continuous mapping, for \( \{ d \} \) there does not exist any \( \gamma^* \)-open set.

**Example 2.3.11**

Consider \( X = \{ 0, 1, 2, 3 \}, Y = \{ a, b, c, d \} \) with topologies
\( T = \{ \emptyset, X, \{ 0 \}, \{ 0, 2 \}, \{ 1, 2 \}, \{ 2 \}, \{ 0, 1, 2 \} \} \) and
\( S = \{ \emptyset, Y, \{ a, b \}, \{ a, b, d \}, \{ d \}, \{ b, d \}, \{ b \} \} \) respectively.

Define a map \( f : (X, T) \rightarrow (Y, S) \) by \( f(0) = d, f(1) = c, f(2) = b, f(3) = a. \)

Then \( f \) is an almost \( \gamma^* \)-continuous mapping. But not a weakly \( \gamma^* \)-continuous mapping, for \( \{ d \} \) there does not exist \( \gamma^* \)-open set.

**Remark 2.3.1**

From the above discussions, we have the following diagram

\( A \rightarrow B \) we mean \( A \) implies \( B \) but not conversely

\( A \leftrightarrow B \) we mean \( A \) implies \( B \) and \( B \) implies \( A \)

Also, every almost strongly \( \theta-\gamma^* \)-continuous mapping is weakly \( \theta-\gamma^* \)-continuous mapping and every weakly \( \theta-\gamma^* \)-continuous mapping is almost strongly \( \theta-\gamma^* \)-continuous mapping.
2.4 DIFFERENT TYPES OF TOPOLOGIES ON FUNCTION SPACES

In this section, different types of topologies on function spaces are introduced. Some interesting properties are also discussed.

**Definition 2.4.1**

Let X be a space and $F : X \times Y \to Z$ be a $\gamma^*$-continuous map (respectively, a super $\gamma^*$-continuous map, an almost $\gamma^*$-continuous map, an almost strongly $\theta$-$\gamma^*$-continuous map, a weakly $\gamma^*$-continuous map, a weakly $\theta$-$\gamma^*$-continuous map). Then by $F_x$ where $x \in X$, we denote the $\gamma^*$-continuous map (respectively, the super $\gamma^*$-continuous map, an almost $\gamma^*$-continuous map, an almost strongly $\theta$-$\gamma^*$-continuous map, a weakly $\gamma^*$-continuous map, a weakly $\theta$-$\gamma^*$-continuous map) of $Y$ into $Z$ for which $F_x(y) = F(x, y)$ for every $y \in Y$. $\hat{F}$ denotes the map of $X$ into the set $\gamma^*C(Y, Z)$ (respectively, into the set $S\gamma^*C(Y, Z)$, $A\gamma^*C(Y, Z)$, $A\theta\gamma^*C(Y, Z)$, $W\gamma^*C(Y, Z)$, $W\theta\gamma^*C(Y, Z)$) for which $\hat{F}(x) = F_x$ for every $x \in X$.

**Definition 2.4.2**

Let $G$ be a map of the space $X$ into the set $\gamma^*C(Y, Z)$ or $S\gamma^*C(Y, Z)$ or $A\gamma^*C(Y, Z)$ or $A\theta\gamma^*C(Y, Z)$ or $W\gamma^*C(Y, Z)$ or $W\theta\gamma^*C(Y, Z)$. Let $\tilde{G}$ be the map of the space $X \times Y$ into the space $Z$, for which $\tilde{G}(x, y) = G(x)(y)$ for every $(x, y) \in X \times Y$.

**Definition 2.4.3**

A topology $T$ on $\gamma^*C(Y, Z)$ (respectively, on $S\gamma^*C(Y, Z)$, $A\gamma^*C(Y, Z)$, $A\theta\gamma^*C(Y, Z)$, $W\gamma^*C(Y, Z)$, $W\theta\gamma^*C(Y, Z)$) is called $\gamma^*$-splitting
(respectively, super $\gamma^*$-splitting, almost $\gamma^*$-splitting, almost strongly $\theta^*$-$\gamma^*$-splitting, weakly $\gamma^*$-splitting, weakly $0^*\gamma^*$-splitting) if for every $X$, the $\gamma^*$-continuity (respectively the super $\gamma^*$-continuity, almost $\gamma^*$-continuity, almost strongly $0^*\gamma^*$-continuity, weakly $\gamma^*$-continuity, weakly $0^*\gamma^*$-continuity) of a map $F : X \times Y \to Z$ implies the $\gamma^*$-continuity (respectively, the super $\gamma^*$-continuity, almost $\gamma^*$-continuity, almost strongly $0^*\gamma^*$-continuity, weakly $\gamma^*$-continuity, weakly $0^*\gamma^*$-continuity) of the map $\hat{F} : X \to \gamma^*C(Y, Z)$ (respectively, of the map $\hat{F} : X \to S\gamma^*C(Y, Z)$, $\hat{F} : X \to A\gamma^*C(Y, Z)$, $\hat{F} : X \to AS\theta\gamma^*C(Y, Z)$, $\hat{F} : X \to W\gamma^*C(Y, Z)$, $\hat{F} : X \to W\theta\gamma^*C(Y, Z)$).

**Definition 2.4.4**

A topology $T$ on $\gamma^*C(Y, Z)$ (respectively, on $S\gamma^*C(Y, Z)$, $A\gamma^*C(Y, Z)$, $AS\theta\gamma^*C(Y, Z)$, $W\gamma^*C(Y, Z)$, $W\theta\gamma^*C(Y, Z)$) is called $\gamma^*$-jointly continuous (respectively, super $\gamma^*$-jointly continuous, almost $\gamma^*$-jointly continuous, almost strongly $0^*\gamma^*$-jointly continuous, weakly $\gamma^*$-jointly continuous, weakly $0^*\gamma^*$-jointly continuous) if for every space $X$, the $\gamma^*$-continuity (respectively, the super $\gamma^*$-continuity, almost $\gamma^*$-continuity, almost strongly $0^*\gamma^*$-continuity, weakly $\gamma^*$-continuity, weakly $0^*\gamma^*$-continuity) of a map $G : X \to \gamma^*C(Y, Z)$ (respectively, of a map $G : X \to S\gamma^*C(Y, Z)$, $G : X \to A\gamma^*C(Y, Z)$, $G : X \to AS\theta\gamma^*C(Y, Z)$, $G : X \to W\gamma^*C(Y, Z)$, $G : X \to W\theta\gamma^*C(Y, Z)$) implies the $\gamma^*$-continuity (respectively, the super
continuity, weakly \(\gamma^*-\)continuity, weakly \(\theta-\gamma^*-\)continuity) of the map \(\tilde{G} : X \times Y \to Z\).

**Definition 2.4.5**

Let \(C^*_\gamma C\) (respectively, \(C^*_S\gamma C\), \(C^*_\Lambda\gamma C\), \(C^*_\Lambda S\gamma C\), \(C^*_W\gamma C\), \(C^*_W\theta\gamma C\)) denote the class of all pairs \(\{f_\lambda : \lambda \in \Lambda\}, f\) where \(\{f_\lambda : \lambda \in \Lambda\}\) is a net \(\gamma^* C(Y, Z)\) (respectively, in \(S\gamma^* C(Y, Z)\), \(A\gamma^* C(Y, Z)\), \(A\theta\gamma^* C(Y, Z)\), \(W\gamma^* C(Y, Z)\), \(W\theta\gamma^* C(Y, Z)\)) which \(\gamma^*\)-continuously converges (respectively, super \(\gamma^*\)-continuously converges, almost \(\gamma^*\)-continuously converges, almost strongly \(\gamma^*\)-continuously converges, weakly \(\gamma^*\)-continuously converges, weakly \(\theta\gamma^*\)-continuously converges) to \(f \in \gamma^* C(Y, Z)\) respectively, to \(f \in S\gamma^* C(Y, Z)\), \(f \in A\gamma^* C(Y, Z)\), \(f \in A\theta\gamma^* C(Y, Z)\), \(f \in W\gamma^* C(Y, Z)\), \(f \in W\theta\gamma^* C(Y, Z)\).

**Definition 2.4.6**

If \(T\) is a topology on \(\gamma^* C(Y, Z)\) (respectively, on \(S\gamma^* C(Y, Z)\), \(A\gamma^* C(Y, Z)\), \(A\theta\gamma^* C(Y, Z)\), \(W\gamma^* C(Y, Z)\), \(W\theta\gamma^* C(Y, Z)\)), then \((C(T))_\gamma C\) (respectively, by \((C(T))_S\gamma C\), \((C(T))_\Lambda\gamma C\), \((C(T))_\Lambda S\gamma C\), \((C(T))_W\gamma C\), \((C(T))_W\theta\gamma C\)) denotes the class of all pairs \(\{f_\lambda : \lambda \in \Lambda\}, f\) where \(\{f_\lambda : \lambda \in \Lambda\}\) is a net \(\gamma^* C(Y, Z)\) (respectively, in \(S\gamma^* C(Y, Z)\), \(A\gamma^* C(Y, Z)\), \(A\theta\gamma^* C(Y, Z)\), \(W\gamma^* C(Y, Z)\), \(W\theta\gamma^* C(Y, Z)\)) which converges. (respectively, converges, weakly \(\theta\)-converges, weakly \(\theta\)-converges, \(\theta\)-converges, \(\theta\)-converges) to \(f \in \gamma^* C(Y, Z)\) (respectively, to \(f \in S\gamma^* C(Y, Z)\), \(f \in A\gamma^* C(Y, Z)\), \(f \in A\theta\gamma^* C(Y, Z)\), \(f \in W\gamma^* C(Y, Z)\), \(f \in W\theta\gamma^* C(Y, Z)\)) in the topology \(T\).
Proposition 2.4.1

A topology $T$ on $\gamma^*C(Y, Z)$ is $\gamma^*$-splitting iff $C^*_{\gamma^*C} \subseteq (C(T))_{\gamma^*C}$.

Proof

Let $T$ be a topology on $\gamma^*C(Y, Z)$ such that $C^*_{\gamma^*C} \subseteq (C(T))_{\gamma^*C}$. Let $X$ be an arbitrary space and let $F : X \times Y \to Z$ be a $\gamma^*$-continuous map. Consider the map $\hat{F} : X \to \gamma^*C_T(Y, Z)$. Let $\{ x_\lambda : \lambda \in \Lambda \}$ be a net in $X$ which $\gamma^*$-converges to $x$. Let $\{ y_\mu : \mu \in M \}$ be a net in $Y$ $\gamma^*$ converging to $y$. Since $F$ is $\gamma^*$-continuous and $\{ (x_\lambda, y_\mu) : (\lambda, \mu) \in \Lambda \times M \}$ in $X \times Y\gamma^*$-converges to $(x, y)$ in $X \times Y$, we get $\{ F(x_\lambda, y_\mu) : (\lambda, \mu) \in \Lambda \times M \}$ converges to $F(x, y)$. That is, $\{ F_{x_\lambda}(y_\mu) : \lambda \in \Lambda, \mu \in M \}$ converges to $F_X(y)$. Thus the net $\{ \hat{F}(x_\lambda) : \lambda \in \Lambda \}$ $\gamma^*$-continuously converges to $\hat{F}(x)$. Also $\{ \hat{F}(x_\lambda) : \lambda \in \Lambda \}$ converges to $\hat{F}(x)$. Thus the map $\hat{F}$ is $\gamma^*$ continuous and the topology $T$ is $\gamma^*$-splitting.

Conversely, let $T$ be a topology on $\gamma^*C(Y, Z)$ which is $\gamma^*$-splitting. Let $\{ (f_\lambda : \lambda \in \Lambda, f) \} \in C^*_{\gamma^*C}$. That is, $\{ (f_\lambda : \lambda \in \Lambda, f) \}$ is a net in $\gamma^*C(Y, Z)$ which $\gamma^*$-continuously converges to $f \in \gamma^*C(Y, Z)$. Set $X = \Lambda \times \{ \infty \}$ where $\infty$ is a symbol such that $\infty \geq \lambda$, for all $\lambda \in \Lambda$. Topologize $X$ by defining $\{ \lambda \}, \lambda \in \Lambda$ to be open and open neighbourhoods of $\infty$ the sets $\{ \lambda \in \Lambda : \lambda \geq \lambda_o \}$ for some $\lambda_o \in \Lambda$. These open sets are also $\gamma^*$-open. Let $F : X \times Y \to Z$ be defined by $F(\lambda, y) = f_\lambda(y)$ for $\lambda \neq \infty$ and $F(\infty, y) = f(y)$ for every $y \in Y$. The map $F$ is $\gamma^*$-continuous. Now, $\hat{F}(\lambda) = f_\lambda$ for $\lambda \neq \infty$ and $\hat{F}(\infty) = f$. Since $T$ is $\gamma^*$-splitting, $\hat{F} : X \to \gamma^*C_T(Y, Z)$ is $\gamma^*$-continuous. Hence for every open
neighbourhood \( V \) of \( f \) in \( \gamma^* C(Y, Z) \), there exists a \( \gamma^* \)-open neighbourhood \( U \) of \( \infty \) in \( X \) such that \( F(U) \subseteq V \). By the definition of the topology on \( X \), there exists a \( \lambda_o \in \Lambda \) such that \( \lambda \in U \), for all \( \lambda \geq \lambda_o \). Hence, \( f_\lambda \in V \), for all \( \lambda \geq \lambda_o \). That is, the net \{ \( f_\lambda \) : \( \lambda \in \Lambda \) \} converges to \( f \) in the topology \( T \). Thus \( C^\lambda \subseteq (C(T))^{\gamma^*} \).

**Remark 2.4.1**

Similar results of Proposition 2.4.1, can be developed for almost \( \gamma^* \)-splitting, almost strongly \( \theta \)-\( \gamma^* \)-splitting, weakly \( \gamma^* \)-splitting, weakly \( \theta \)-\( \gamma^* \)-splitting and super \( \gamma^* \)-splitting.

**Proposition 2.4.2**

A topology \( T \) on \( \gamma^* C(Y, Z) \) is \( \gamma^* \)-jointly continuous iff \( (C(T))^{\gamma^*} \subseteq C^{\gamma^*} \).

**Proof**

Let \( T \) be a topology on \( \gamma^* C(Y, Z) \) such that \( (C(T))^{\gamma^*} \subseteq C^{\gamma^*} \) we have to prove that \( T \) is \( \gamma^* \)-jointly continuous. Let \( X \) be an arbitrary space and let \( G : X \to \gamma^* C(Y, Z) \) be \( \gamma^* \)-continuous. We prove that \( \tilde{G} : X \times Y \to Z \) is \( \gamma^* \)-continuous. Let \{ \( (x_\lambda, y_\mu) \) : \( (\lambda, \mu) \in \Lambda \times M \) \} be a net in \( X \times Y \) which \( \gamma^* \)-converges to \( (x, y) \). Since \{ \( x_\lambda \) : \( \lambda \in \Lambda \) \} \( \gamma^* \)-converges to \( x \in X \), the net \{ \( G(x_\lambda) \) : \( \lambda \in \Lambda \) \} converges to \( G(x) \). Hence, by the given condition, the net \{ \( G(x_\lambda) \) : \( \lambda \in \Lambda \) \} \( \gamma^* \)-continuously converges to \( G(x) \). Since \{ \( y_\mu \) : \( \mu \in M \) \} \( \gamma^* \)-converges to \( y \), the net \{ \( G(x_\lambda)(y_\mu) \equiv \tilde{G}(x_\lambda, y_\mu) \) : \( (\lambda, \mu) \in \Lambda \times M \) \} converges to \( G(x)(y) \equiv \tilde{G}(x, y) \). Hence, \( T \) is \( \gamma^* \)-jointly continuous.
Conversely, let $T$ be a topology on $\gamma^*C(Y, Z)$ which is $\gamma^*$-jointly continuous. Let $(\{ f_\lambda : \lambda \in \Lambda \}, f) \in (C(T))_{\gamma^*C}$. Then $(f_\lambda : \lambda \in \Lambda)$ is a net in $\gamma^*C(Y, Z)$ which converges to $f \in \gamma^*C(Y, Z)$. Set $X = \Lambda \times \{ \infty \}$ where $\infty$ is a symbol such that $\infty \geq \lambda$, for all $\lambda \in \Lambda$. Topologize $X$ by defining $\{ \lambda \}$, $\lambda \in \Lambda$ to be open, and open neighbourhoods of $\infty$ the sets $\{ \lambda \in \Lambda : \lambda \geq \lambda_0 \}$ for some $\lambda_0 \in \Lambda$. These open sets are also $\gamma^*$-open. The map $G : X \to \gamma^*C_T(Y, Z)$ defined by $G(\lambda) = f_\lambda$, $\lambda \neq \infty$ and $G(\infty) = f$ is clearly $\gamma^*$-continuous. Since $T$ is $\gamma^*$-jointly continuous, $\tilde{G} : X \times Y \to Z$ is $\gamma^*$-continuous. To prove that $(\{ f_\lambda : \lambda \in \Lambda \}, f) \in C^*_{\gamma^*C}$. Let $\{ y_\mu : \mu \in M \}$ be a net in $Y$ $\gamma^*$-converging to $y \in Y$. Then the net $(f_\lambda(y_\mu) : (\lambda, \mu) \in \Lambda \times M)$ converges to $f(y)$. Also the net $\{ \lambda : \lambda \in \Lambda \}$ in $X$ $\gamma^*$-converges to $\infty$. Thus the net $(\lambda, y_\mu) : (\lambda, \mu) \in \Lambda \times M$ $\gamma^*$-converges to $(\infty, y)$. Since $\tilde{G}$ is $\gamma^*$-continuous the net $\{ \tilde{G}(\lambda, y_\mu) : (\lambda, \mu) \in \Lambda \times M \}$ converges to $\tilde{G}(\infty, y) \equiv G(\infty)(y) \equiv f(y)$. Hence, $(\{ f_\lambda : \lambda \in \Lambda \}, f) \in C^*_{\gamma^*C}$ and thus the theorem is proved.

**Remark 2.4.2**

Proposition 2.4.2 can be developed for almost $\gamma^*$-jointly continuous, almost strongly $\theta$-$\gamma^*$-continuous, weakly $\gamma^*$-continuous, weakly $\theta$-$\gamma^*$-jointly continuous, super $\gamma^*$-jointly continuous.

**Proposition 2.4.3**

A topology $T$ on $\gamma^*C(Y, Z)$ (respectively, on $S\gamma^*C(Y, Z)$, $A\gamma^*C(Y, Z)$, $AS\theta\gamma^*C(Y, Z)$, $W\gamma^*C(Y, Z)$, $W0\gamma^*C(Y, Z)$) is simultaneously $\gamma^*$-splitting and $\gamma^*$-jointly continuous (respectively, super $\gamma^*$-splitting and super
\( \gamma^* \)-jointly continuous, almost \( \gamma^* \)-splitting and almost \( \gamma^* \)-jointly continuous, almost strongly \( \theta - \gamma^* \)-splitting and almost strongly \( \theta - \gamma^* \)-jointly continuous, weakly \( \gamma^* \)-splitting and weakly \( \gamma^* \)-jointly continuous, weakly \( \theta - \gamma^* \)-splitting and weakly \( \theta - \gamma^* \)-jointly continuous) iff

\[
C^\gamma C = (C(T))_\gamma C \quad \text{(respectively, } C^\omega \omega C = (C(T))_\omega \omega C, \quad C^A A C = (C(T))_A A C, \quad C^w w C = (C(T))_w w C, \quad C^\omega \omega C = (C(T))_w \omega C.
\]

**Proof**: Similar to the proofs of Proposition 2.4.1, Remark 2.4.1, Proposition 2.4.2 and Remark 2.4.2.