CHAPTER - 5

ON UNIFORM REGULAR SPACES AND

UNIFORM NORMAL SPACES VIA $b$-OPEN SETS

Andre weil [1] formulated the concept of uniform space which is a generalization of a metric space. Andrijevic [2] introduced a new class of generalized open sets called $b$-open sets. In this chapter, a new class of topological spaces called uniform topological space is introduced. The concepts of uniform $b$-regular spaces, uniform $b$-normal spaces and uniform $b$-$T_{\frac{1}{2}}$ spaces are introduced. Interesting properties and characterizations are established.
5.1 PROPERTIES OF b-REGULAR AND b-NORMAL SPACES.

In this section, the concepts of b-open symmetric member, uniform b-regular spaces, uniform b-normal spaces are introduced. Some of their properties are established.

Notation 5.1.1

The nonempty set \( X \times X \) is denoted by \( X = X \times X \)

Definition 5.1.1

A uniform topology on a nonempty set \( X \) is a collection \( \mathcal{U} \) of subsets containing the diagonal \( \Delta x \) in \( X \), which satisfies the following axioms:

(i). \( \phi, X \in \mathcal{U} \)

(ii). \( U \cap V \in \mathcal{U} \) for any \( U, V \in \mathcal{U} \)

(iii). \( U \cup V \in \mathcal{U} \), for any arbitrary family \( \{ V_i : i \in J \} \subseteq \mathcal{U} \)

(iv). If \( U \subseteq V \) and \( U \in \mathcal{U} \), then \( V \in \mathcal{U} \)

(v). If \( U \in \mathcal{U} \), then \( U^{-1} = \{(x, y) : (y, x) \in U\} \in \mathcal{U} \)

(vi). For any \( U \in \mathcal{U} \), then there is some \( W \in \mathcal{U} \), such that \( W \circ W \subseteq U \).

In this case, the pair \( (X, \mathcal{U}) \) is called a uniform topological space. The members of \( \mathcal{U} \) are called open symmetric members. The complement of a open symmetric member is called closed symmetric member.
**Definition 5.1.2**

Let \((X, \mathcal{U})\) be a uniform topological space and \(A\) be a symmetric member. The symmetric closure of \(A\) is defined as \(\text{symcl}(A) = \bigcap \{ B : B \text{ is closed symmetric member and } A \subseteq B \}\).

**Definition 5.1.3**

Let \((X, \mathcal{U})\) be a uniform topological space and \(A\) be a symmetric member. The symmetric interior of \(A\) is defined as \(\text{symint}(A) = \bigcap \{ G : G \text{ is open symmetric member and } A \supseteq B \}\).

**Definition 5.1.4**

Let \((X, \mathcal{U})\) be a uniform topological space. A subset \(A\) is said to be \(b\)-open symmetric member if \(A \subseteq \text{symcl}(\text{symint}(A)) \cup \text{symint}(\text{symcl}(A))\). The complement of \(b\)-open symmetric member is called \(b\)-closed symmetric member.

**Definition 5.1.5**

Let \((X, \mathcal{U})\) be a uniform topological space. A set \(G\) in \(X\) is said to be \(b\)-open symmetric neighbourhood of \((x_1, x_2)\) if there exists a \(b\)-open symmetric set \(V\) such that \((x_1, x_2) \in V \subseteq G\).

**Definition 5.1.6**

A uniform topological space \((X, \mathcal{U})\) is said to be uniform \(b\)-regular if for every \(b\)-closed symmetric member \(F\) and each \((x_1, x_2) \notin F\), there exist disjoint open symmetric members \(U\) and \(V\) such that \(F \subseteq U\) and \((x_1, x_2) \in V\).
Proposition 5.1.1

Let \((X, \mathcal{U})\) be a uniform topological space. Then the following statements are equivalent:

(i) \((X, \mathcal{U})\) is a uniform b-regular space.

(ii) For each \((x_1, x_2) \in X\) and b-open symmetric neighborhood \(W\) of \((x_1, x_2)\), there exists an open symmetric neighborhood \(V\) of \((x_1, x_2)\) such that \(\text{symcl}(V) \subseteq W\).

Proof

\((i) \Rightarrow (ii)\) Let \(W\) be any b-open symmetric neighborhood of \((x_1, x_2)\). Then there exists a b-open symmetric member \(G\) such that \((x_1, x_2) \in G \subseteq W\). Since \(X - G\) is a b-closed symmetric member and \((x_1, x_2) \notin X - G\), by hypothesis, there exist disjoint open symmetric members \(U\) and \(V\) such that \(X - G \subseteq U\) and therefore \((x_1, x_2) \in V\) and so \(V \subseteq X - U\). Now, \(\text{cl}(V) \subseteq \text{cl}(X - U) = X - U\) and \(X - G \subseteq U\). This implies that \(X - U \subseteq G \subseteq W\). Therefore, \(\text{cl}(V) \subseteq W\).

\((ii) \Rightarrow (i)\) Let \(F\) be any b-closed symmetric member and \((x_1, x_2) \notin F\). Then, \((x_1, x_2) \in X - F\) and \(X - F\) is b-open symmetric member and so \(X - F\) is a b-open symmetric neighborhood of \((x_1, x_2)\). By hypothesis, there exists an open symmetric neighborhood \(V\) of \((x_1, x_2)\) such that \((x_1, x_2) \in V\) and \(\text{cl}(V) \subseteq X - F\), which implies that \(F \subseteq X - \text{cl}(V)\). Then, \(X - \text{cl}(V)\) is an open symmetric member containing \(F\) and \(V \cap (X - \text{cl}(V)) = \emptyset\). Therefore, \((X, \mathcal{U})\) is a uniform b-regular space.
Definition 5.1.7

A uniform topological space \((X, \mathcal{U})\) is said to be a uniform b-T\(_{1/2}\) space if every b-open symmetric member \(A\) in \((X, \mathcal{U})\) is symmetric open.

Definition 5.1.8

A uniform topological space \((X, \mathcal{U})\) is said to be a uniform normal space iff for each disjoint pair of closed symmetric members \(A\) and \(B\), there exist disjoint open symmetric members \(U\) and \(V\) such that \(A \subseteq U\) and \(B \subseteq V\).

Proposition 5.1.2

Let \((X, \mathcal{U})\) be a uniform b-T\(_{1/2}\) space. Then, the following statements are equivalent:

(i). \((X, \mathcal{U})\) is uniform normal.

(ii). For every pair of disjoint closed symmetric members \(A\) and \(B\), there exist b-open symmetric members \(U\) and \(V\) such that \(A \subseteq U\), \(B \subseteq V\) and \(U \cap V = \emptyset\).

Proof

(i) \(\Rightarrow\) (ii) Let \(A\) and \(B\) be disjoint closed symmetric members of \((X, \mathcal{U})\). By hypothesis, there exist disjoint open symmetric members (resp. b-open members) \(U\) and \(V\) such that \(A \subseteq U\), \(B \subseteq V\). Hence \((X, \mathcal{U})\) is uniform normal.
(ii) ⇒ (i) Let A and B be disjoint closed symmetric members of \((X, \mathcal{U})\). By assumption, there exist \(b\)-open symmetric members \(G\) and \(H\) such that \(A \subseteq G\), \(B \subseteq H\) and \(G \cap H = \emptyset\). A \(\subseteq \text{symint}(G)\), \(B \subseteq \text{symint}(H)\). Further \(\text{symint}(G) \cap \text{symint}(H) = \text{symint}(G \cap H) = \emptyset\). Hence the proof.

**Proposition 5.1.3**

A uniform topological space \((X, \mathcal{U})\) is uniform \(b\)-regular iff for each \(b\)-closed symmetric member \(F\) of \((X, \mathcal{U})\) and each \((x_1, x_2) \in X - F\), there exist open symmetric members \(U\) and \(V\) of \((X, \mathcal{U})\) such that \((x_1, x_2) \in U, F \subseteq V\) and \(\text{symcl}(U) \cap \text{symcl}(V) = \emptyset\).

**Proof**

Let \(F\) be a \(b\)-closed symmetric member of \((X, \mathcal{U})\) and let \((x_1, x_2) \notin F\). Since \((X, \mathcal{U})\) is uniform \(b\)-regular, there exist open symmetric members \(U_0\) and \(V\) of \((X, \mathcal{U})\) such that \((x_1, x_2) \in U_0, F \subseteq V\) and \(U_0 \cap V = \emptyset\) which implies that \(U_0 \cap \text{symcl}(V) = \emptyset\). Since \(\text{symcl}(V)\) is closed symmetric, it is \(b\)-closed symmetric and \((x_1, x_2) \notin \text{symcl}(V)\). Since \((X, \mathcal{U})\) is uniform \(b\)-regular, there exist open symmetric members \(G\) and \(H\) of \((X, \mathcal{T})\) such that \((x_1, x_2) \in G, \text{symcl}(V) \subseteq H\) and \(G \cap H = \emptyset\) which implies that \(\text{symcl}(G) \cap H = \emptyset\). Let \(U = U_0 \cap G\), then \(U\) and \(V\) are open symmetric members of \((X, \mathcal{U})\) such that \((x_1, x_2) \in U, F \subseteq V\) and \(\text{symcl}(U) \cap \text{symcl}(V) = \emptyset\).
Proposition 5.1.4

Let \((X, \mathcal{U})\) be a uniform topological space. Then the following statements are equivalent:

(i). \((X, \mathcal{U})\) is uniform \(b\)-regular

(ii). For each \((x_1, x_2) \in X\) and \(b\)-open symmetric neighborhood \(W\) of \((x_1, x_2)\), there exists an open symmetric neighborhood \(V\) of \((x_1, x_2)\) such that \(\text{symcl}(V) \subseteq W\).

(iii). For each \((x_1, x_2) \in X\) and for each \(b\)-closed symmetric member \(F\) not containing \((x_1, x_2)\), there exists an open symmetric neighborhood \(V\) of \((x_1, x_2)\) such that \(\text{sym}(\text{cl}(V)) \cap F = \emptyset\).

Proof

(i) \(\iff\) (ii)  Proof is simple.

(ii) \(\Rightarrow\) (iii) Let \((x_1, x_2) \in X\) and let \(F\) be a \(b\)-closed symmetric member not containing \((x_1, x_2)\). Then \(X - F\) is a \(b\)-open symmetric neighbourhood of \((x_1, x_2)\) and by hypothesis, there exists an open symmetric neighborhood \(V\) of \((x_1, x_2)\) such that \(\text{symcl}(V) \subseteq X - F\) and hence \(\text{symcl}(V) \cap F = \emptyset\).

(iii) \(\Rightarrow\) (ii) Let \((x_1, x_2) \in X\) and \(b\)-open symmetric neighborhood \(W\) of \((x_1, x_2)\). Then there exists \(b\)-open symmetric member \(G\) such that \((x_1, x_2) \in G \subseteq W\). Since \(X - G\) is a \(b\)-closed symmetric member and \((x_1, x_2) \notin X - G\), by hypothesis there exist an open symmetric
neighborhood \( V \) of \((x_1, x_2)\) such that \( \text{symcl}(V) \cap (X - G) = \emptyset \). That is, \( \text{symcl}(V) \subset G \subset W. \)

**Definition 5.1.9**

Let \((X, \mathcal{U})\) and \((Y, \mathcal{V})\) be any two uniform topological spaces and let \( f : (X, \mathcal{U}) \rightarrow (Y, \mathcal{V}) \) be a mapping. Then

(i). \( f \) is a uniform continuous mapping if \( f^{-1}(A) \) is a closed symmetric member in \((X, \mathcal{U})\) for each closed symmetric member \( A \) in \((Y, \mathcal{V})\).

(Equivalently, \( f^{-1}(B) \) is an open symmetric member in \((X, \mathcal{U})\) for each open symmetric member \( B \) in \((Y, \mathcal{V})\)).

(ii). \( f \) is a uniform open mapping if \( f(A) \) is an open symmetric member in \((X, \mathcal{U})\) and for each open symmetric member \( A \) in \((Y, \mathcal{V})\).

(Equivalently, \( f(B) \) is a closed symmetric member in \((X, \mathcal{U})\) for each closed symmetric member \( B \) in \((Y, \mathcal{V})\)).

(iii). \( f \) is a uniform \( b \)-irresolute mapping if \( f^{-1}(A) \) is a \( b \)-closed symmetric member in \((X, \mathcal{U})\) for each \( b \)-closed symmetric member \( A \) in \((Y, \mathcal{V})\).

(Equivalently, \( f^{-1}(B) \) is a \( b \)-open symmetric member in \((X, \mathcal{U})\) for each \( b \)-open symmetric member \( B \) in \((Y, \mathcal{V})\)).

(iv) \( f \) is a uniform \( b \)-closed mapping if \( f(A) \) is a \( b \)-closed symmetric member in \((X, \mathcal{U})\) for each symmetric \( b \)-closed symmetric member \( A \) in \((Y, \mathcal{V})\).
Proposition 5.1.5

Let \((X, \mathcal{U})\) and \((Y, \mathcal{V})\) be any two uniform topological spaces and let \(f : (X, \mathcal{U}) \rightarrow (Y, \mathcal{V})\) be a uniform b-closed, uniform continuous injective mapping. If \((Y, \mathcal{V})\) is uniform b-regular then \((X, \mathcal{U})\) is uniform b-regular.

Proof

Let \(F\) be any b-closed symmetric member of \((X, \mathcal{U})\) and let \((x_1, x_2) \notin F\). Since \(f\) is a uniform b-closed mapping, \(f(F)\) is a b-closed symmetric member in \((Y, \mathcal{V})\) and \(f(x_1, x_2) \notin f(F)\). Since \((Y, \mathcal{V})\) is uniform b-regular, there exist disjoint open symmetric members \(U\) and \(V\) such that \(f(x_1, x_2) \in U\), \(f(F) \subseteq V\). That is, \((x_1, x_2) \in f^{-1}(U)\), \(F \subseteq f^{-1}(V)\). Since \(f\) is a uniform continuous mapping, \(f^{-1}(U)\) and \(f^{-1}(V)\) are open symmetric members in \((X, \mathcal{U})\). Further, \(f^{-1}(U) \cap f^{-1}(V) = \emptyset\). Therefore, \((X, \mathcal{U})\) is uniform b-regular.

Proposition 5.1.6

Let \((X, \mathcal{U})\) and \((Y, \mathcal{V})\) be any two uniform topological spaces and let \(f : (X, \mathcal{U}) \rightarrow (Y, \mathcal{V})\) be a uniform b-irresolute mapping, uniform open bijective mapping. If \((X, \mathcal{U})\) is uniform b-regular, then \((Y, \mathcal{V})\) is uniform b-regular.

Proof

Let \(F\) be any b-closed symmetric member of \((Y, \mathcal{V})\) and let \((y_1, y_2) \notin F\). Since \(f\) is a uniform b-irresolute mapping, \(f^{-1}(F)\) is a b-
closed symmetric member in \((X, \mathcal{U})\). Since \(f\) is bijective, \(f(x_1, x_2) = (y_1, y_2)\). Then, \((x_1, x_2) \notin f^{-1}(F)\). Since \((X, \mathcal{U})\) is uniform b-regular, there exist disjoint open symmetric members \(U\) and \(V\) such that \((x_1, x_2) \in U, f^{-1}(F) \subseteq V\). Since \(f\) is a uniform open mapping, \(f(U)\) and \(f(V)\) are open symmetric members in \((Y, \mathcal{V})\) such that \((y_1, y_2) \in f(U), F \subseteq f(V)\) and \(f(U) \cap f(V) = \emptyset\). Therefore, \((Y, \mathcal{V})\) is a uniform b-regular space.

**Definition 5.1.11**

A uniform space \((X, \mathcal{U})\) is said to be a uniform b-normal space if for any pair of disjoint b-closed symmetric members \(A\) and \(B\), there exist disjoint open symmetric members \(U\) and \(V\) such that \(A \subseteq U\) and \(B \subseteq V\).

**Proposition 5.1.7**

Let \((X, \mathcal{U})\) be a uniform topological space. Then the following statements are equivalent:

(i). \((X, \mathcal{U})\) is a uniform b-normal space.

(ii). For each b-closed symmetric member \(F\) and for each b-open symmetric member \(U\) containing \(F\), there exist an open symmetric member \(V\) containing \(F\) such that \(\text{symcl}(V) \subseteq U\).

(iii). For each pair of disjoint b-closed symmetric members \(A\) and \(B\) in \((X, \mathcal{U})\), there exist an open symmetric member \(U\) containing \(A\) such that \(\text{symcl}(U) \cap B = \emptyset\).
(iv). For each pair of disjoint b-closed symmetric members \( A \) and \( B \) in \((X, \mathcal{U})\), there exist open symmetric members \( U \) containing \( A \) and \( V \) containing \( B \) such that \( \text{symcl}(U) \cap \text{symcl}(V) = \phi \).

**Proof**

(i) \( \Rightarrow \) (ii)  Let \( F \) be a b-closed symmetric member and \( U \) be a b-open symmetric member containing \( F \). Then \( F \cap (X - U) = \phi \). By assumption, there exist open symmetric members \( V \) and \( W \) such that \( F \subseteq V \), \((X - U) \subseteq W \) and \( V \cap W = \phi \). This implies that \( \text{symcl}(V) \cap W = \phi \). Now, \( \text{symcl}(V) \cap (X - U) \subseteq \text{symcl}(V) \cap W = \phi \) and hence \( \text{symcl}(V) \subseteq U \).

(ii) \( \Rightarrow \) (iii)  Let \( A \) and \( B \) be disjoint b-closed symmetric members in \((X, \mathcal{U})\). Since \( A \cap B = \phi \), \( A \subseteq (X - B) \) and \((X - B)\) is b-open symmetric. By assumption, there exists an open symmetric member \( U \) containing \( A \) such that \( \text{symcl}(U) \subseteq (X - B) \). That is, \( \text{symcl}(U) \cap B = \phi \).

(iii) \( \Rightarrow \) (iv)  Let \( A \) and \( B \) be disjoint b-closed symmetric members in \((X, \mathcal{U})\). By assumption, there exists an symmetric open member \( U \) containing \( A \) such that \( \text{symcl}(U) \cap B = \phi \). Since \( \text{symcl}(U) \) is closed symmetric, it is b-closed symmetric and so \( B \) and \( \text{symcl}(U) \) are disjoint b-closed symmetric members in \((X, \mathcal{U})\). By assumption, there exists an open symmetric member \( V \) containing \( B \) such that \( \text{symcl}(U) \cap \text{symcl}(V) = \phi \).

(iv) \( \Rightarrow \) (i)  Let \( A \) and \( B \) be disjoint symmetric b-closed members in \((X, \mathcal{U})\). By assumption, there exists an open symmetric member \( U \)
containing A and V containing B such that \( \text{symcl}(U) \cap \text{symcl}(V) = \emptyset \).
That is, \( U \cap V = \emptyset \). Hence, \((X, \mathcal{U})\) is a uniform b-normal space.

**Proposition 5.1.8**

Let \((X, \mathcal{U})\) and \((Y, \mathcal{V})\) be any two uniform topological spaces and let \( f : (X, \mathcal{U}) \rightarrow (Y, \mathcal{V}) \) be a uniform b-closed, uniform continuous injective mapping. If \((Y, \mathcal{V})\) is uniform b-normal. Then \((X, \mathcal{U})\) is uniform b-normal.

**Proof**

Let A and B be disjoint b-closed symmetric members in \((X, \mathcal{U})\).
Since \( f \) is a uniform b-closed mapping, \( f(A) \) and \( f(B) \) are disjoint b-closed symmetric members of \((Y, \mathcal{V})\). Since \((Y, \mathcal{V})\) is uniform b-normal, there exist disjoint open symmetric members \( U \) and \( V \) such that \( f(A) \subseteq U, f(B) \subseteq V \). That is, \( A \in f^{-1}(U), B \subseteq f^{-1}(V) \) and \( f^{-1}(U) \cap f^{-1}(V) = \emptyset \).
Since \( f \) is a uniform continuous mapping \( f^{-1}(U) \) and \( f^{-1}(V) \) are open symmetric members in \((X, \mathcal{U})\). Therefore, \((X, \mathcal{U})\) is uniform b-normal.

**Proposition 5.1.9**

Let \((X, \mathcal{U})\) and \((Y, \mathcal{V})\) be any two uniform topological spaces and let \( f : (X, \mathcal{U}) \rightarrow (Y, \mathcal{V}) \) be a uniform b-irresolute mapping, uniform open bijective mapping. If \((X, \mathcal{U})\) is uniform b-normal, then \((Y, \mathcal{V})\) is uniform b-normal.
Proof

Let A and B be disjoint b-closed symmetric members of $(Y, \mathcal{V})$. Since $f$ is a uniform b-irresolute mapping, $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint b-closed symmetric members in $(X, \mathcal{U})$. Since $(X, \mathcal{U})$ is uniform b-normal, there exist disjoint open symmetric members $U$ and $V$ such that $f^{-1}(A) \subseteq U$, $f^{-1}(B) \subseteq V$. Since $f$ is a uniform open mapping $f(U)$ and $f(V)$ are open symmetric members in $(X, \mathcal{V})$ such that $A \subseteq f(U)$, $B \subseteq f(V)$ and $f(U) \cap f(V) = \emptyset$. Therefore, $(Y, \mathcal{V})$ is a uniform b-normal space.