Chapter 3

Second Order Neutral Difference Equations

3.Second Order Neutral Difference Equations

3.1 Introduction

In this chapter, we deal with the second order neutral difference equation of the form

$$\Delta(r_n\Delta(x_n + p_n x_{n-k})) + q_n x^{\alpha}_{n-\ell} = 0, \quad n \in \mathbb{N}_0, \tag{3.1.1}$$

subject to the following conditions:

(C_1) $\{q_n\}$ and $\{r_n\}$ are positive real sequences with $\sum_{n=n_0}^{\infty} \frac{1}{r_n} = \infty$;

(C_2) $\{p_n\}$ is a nonnegative real sequence with $0 \le p_n \le p < \infty$;

(C_3) k and ℓ are positive integers and α is a ratio of odd positive integers.;

Let $\theta = \max\{k, \ell\}$. By a solution of equation (3.1.1) we mean a nontrivial real sequence $\{x_n\}$ defined for all $n \ge n_0 - \theta$ and satisfying the equation (3.1.1) for all $n \ge n_0$. We assume that such solutions exist for the equation (3.1.1).

In [1, 2, 7, 8, 11, 12, 16, 17, 18, 23, 29, 30, 31, 32, 34, 36, 37, 39, 41, 43, 44, 46, 50, 52, 53, 55, 67, 72, 76, 77, 78, 80, 81, 84, 86, 87, 88, 95, 96, 97], the authors discuss the oscillatory behavior of second order neutral type difference equations. In [96], the author considered the following difference equation

$$\Delta^2(x_n + p x_{n-k}) + q_n x^{\alpha}_{n-\ell} = 0, \quad n = 0, 1, 2, \ldots, \tag{3.1.2}$$

and obtained the following theorem.

Theorem 3.1.1. *Assume that $\alpha \in (0, 1)$ and $p \in [0, 1)$. Suppose further that*

$$\sum_{n=0}^{\infty} q_n(n - k - \ell)^{\alpha} = \infty,$$

then every solution of equation (3.1.2) *oscillates. While* $\alpha > 1$, $p \in [0, 1)$, *and there exists* $\lambda > (k + A)^{-1} \log \alpha$ *such that*

$$\liminf_{n \to \infty} \left[q_n (n - k - A)^{\alpha} exp(-e^{\lambda n}) \right] > 0,$$

then all solutions of the equation (3.1.2) *are oscillatory.*

A close look at the theorem shows that the conclusion of the theorem is true only when $0 \le p < 1$. Motivated by this observation, in this chapter, we obtain sufficient conditions for the oscillation of all solutions of equation (3.1.1) when $0 \le p_n \le p < \infty$ and $\sum_{n=n_0}^{\infty} \frac{1}{r_n} = \infty$. So our results extend Theorem 3.1.1, when $r_n \equiv 1$.

In Section 3.2, we establish some sufficient conditions for the oscillation of all solutions of equation (3.1.1). In Section 3.3, some examples are provided to illustrate the main results.

3.2 Oscillation Theorems

In the section, we establish some new oscillation criteria for the equation (3.1.1). We use the following notation throughout this chapter without further mention:

$$z_n = x_n + p_n x_{n-k}, \tag{3.2.1}$$

$$Q_n = \min\{q_n, q_{n-k}\} \quad \text{for all} \quad n \in \mathbb{N}_0, \tag{3.2.2}$$

and

$$Q_n^* = \begin{cases} Q_n \left(\sum_{s=n_1}^{n-A-1} \frac{1}{r_s} \right)^{\alpha}, & \text{if } 0 < \alpha \le 1; \\ 2^{1-\alpha} Q_n \left(\sum_{s=n_1}^{n-A-1} \frac{1}{r_s} \right)^{\alpha}, & \text{if } \alpha \ge 1. \end{cases} \tag{3.2.3}$$

We begin with the following lemma.

Lemma 3.2.1. *If* $\{x_n\}$ *is a positive solution of* (3.1.1), *then the corresponding* $\{z_n\}$ *satisfies*

$$z_n > 0, \quad r_n \Delta z_n > 0, \quad \Delta(r_n \Delta z_n) < 0 \tag{3.2.4}$$

eventually.

Proof. Assume that $\{x_n\}$ is a positive solution of equation (3.1.1). Then $z_n > 0$ for all $n \geq n_1 \geq n_0$. From the equation (3.1.1), we have

$$\Delta(r_n \Delta z_n) = -q_n x^{\alpha}_{n-A} < 0.$$

Consequently, $r_n \Delta z_n$ is nonincreasing and thus either $r_n \Delta z_n > 0$ or $r_n \Delta z_n \leq 0$. If $r_n \Delta z_n \leq 0$, we have

$$r_n \Delta z_n \leq r_{n_1} \Delta z_{n_1} < 0 \quad \text{for} \quad n \geq n_1.$$

Dividing the last inequality by r_n and then summing the resulting inequality from n_1 to $n-1$, we obtain

$$z_n < z_{n_1} + r_{n_1} \Delta z_{n_1} \sum_{s=n_1}^{n-1} \frac{1}{r_s} \to -\infty \ \text{ as } \ n \to \infty,$$

which is a contradiction for the positivity of $\{z_n\}$. This completes the proof. □

Theorem 3.2.1. *Assume that the first order neutral difference inequality*

$$\Delta(y_n + p^{\alpha} y_{n-k}) + Q^{*}_{n} y^{\alpha}_{n-A} \leq 0, \tag{3.2.5}$$

has no positive solution, then every solution of equation (3.1.1) *is oscillatory.*

Proof. Let $\{x_n\}$ be a nonoscillatory solution of equation (3.1.1). Without loss of generality we may assume that $x_n > 0$ and $x_{n-k} > 0$ for all $n \geq n_1 \geq n_0 - \theta$. Then $z_n > 0$ and from the equation (3.1.1), we obtain

$$\Delta(r_n \Delta z_n) + q_n x^{\alpha}_{n-A} = 0, \tag{3.2.6}$$

and

$$p^{\alpha} \Delta(r_{n-k} \Delta z_{n-k}) + p^{\alpha} q_{n-k} x^{\alpha}_{n-k-A} = 0. \tag{3.2.7}$$

Combining (3.2.6) and (3.2.7), then using (3.2.2), we get

$$\Delta(r_n \Delta z_n + p^{\alpha} r_{n-k} \Delta z_{n-k}) + Q_n \left[x^{\alpha}_{n-A} + p^{\alpha} x^{\alpha}_{n-k-A} \right] \leq 0. \tag{3.2.8}$$

When $0 < \alpha \leq 1$ applying Lemma 2.2.1 in (3.2.8), we obtain

$$\Delta(r_n \Delta z_n + p^\alpha r_{n-k} \Delta z_{n-k}) + Q_n z_{n-A}^\alpha \leq 0. \tag{3.2.9}$$

Similarly when $\alpha \geq 1$ applying Lemma 2.2.2 in (3.2.8), we get

$$\Delta(r_n \Delta z_n + p^\alpha r_{n-k} \Delta z_{n-k}) + 2^{1-\alpha} Q_n z_{n-A}^\alpha \leq 0. \tag{3.2.10}$$

Since $y_n = r_n \Delta z_n > 0$ is decreasing, we have

$$z_n \geq y_n \sum_{s=n_1}^{n-1} \frac{1}{r_s}. \tag{3.2.11}$$

Substituting (3.2.11) in (3.2.9) and in (3.2.10) for $0 < \alpha \leq 1$ and $\alpha \geq 1$ respectively, we see that $\{y_n\}$ is a positive solution of the inequality

$$\Delta(y_n + p^\alpha y_{n-k}) + Q_n^* y_{n-A}^\alpha \leq 0,$$

which is a contradiction to (3.2.5). The proof is now complete. □

Theorem 3.2.2. *If the first order difference inequality*

$$\Delta w_n + \frac{1}{(1+p^\alpha)^\alpha} Q_n^* w_{n-A+k}^\alpha \leq 0, \tag{3.2.12}$$

has no positive solution, then every solution of equation (3.1.1) *is oscillatory.*

Proof. Let $\{x_n\}$ be a nonoscillatory solution of equation (3.1.1). Then proceeding as in the proof of Theorem 3.2.1, we have (3.2.5). Define a function w_n by

$$w_n = y_n + p^\alpha y_{n-k}.$$

Then $w_n > 0$. By using the monotonicity of $\{y_n\}$, we have

$$w_n \leq (1 + p^\alpha) y_{n-k}. \tag{3.2.13}$$

Substituting (3.2.13) in (3.2.5), we see that $\{w_n\}$ is a positive solution of the inequality (3.2.12). This contradiction completes the proof. □

Corollary 3.2.1. *Let* $A > k$ *and* $0 < \alpha < 1$ *in equation* (3.1.1). *If*

$$\sum_{n=n_0}^{\infty} Q_n^* = \infty \tag{3.2.14}$$

then every solution of equation (3.1.1) *is oscillatory.*

Proof. The proof follows by applying Lemmas 2.2.3 and 2.2.4 in Theorem 3.2.2 and hence the details are omitted. □

Corollary 3.2.2. *Let* $A > k$ *and* $\alpha = 1$ *in equation* (3.1.1). *If*

$$\liminf_{n\to\infty} \sum_{s=n-A+k}^{n-1} Q_s^* > (1+p)\left(\frac{A-k}{A-k+1}\right)^{A-k+1} \tag{3.2.15}$$

then every solution of equation (3.1.1) *is oscillatory.*

Proof. The proof follows by applying Lemmas 2.2.3 and 2.2.5 in Theorem 3.2.2 and hence the details are omitted. □

Corollary 3.2.3. *Let* $A > k$ *and* $\alpha > 1$ *in equation* (3.1.1). *If there exists a* $\lambda > 0$ *such that* $\lambda > \frac{1}{A-k}\log\alpha$ *and*

$$\liminf_{n\to\infty} \left[Q_n^* exp(-e^{\lambda n})\right] > 0 \tag{3.2.16}$$

then every solution of equation (3.1.1) *is oscillatory.*

Proof. The proof follows by applying Lemmas 2.2.3 and 2.2.6 in Theorem 3.2.2 and hence the details are omitted. □

Remark 3.2.1. *Let* $r_n \equiv 1$ *in equation* (3.1.1). *Then the condition of* (3.2.14) *and* (3.2.16) *are reduced to*

$$\sum_{n=n_0}^{\infty} Q_n^*(n-A)^{\alpha}(n-A-1)^{\alpha} = \infty, \tag{3.2.17}$$

and

$$\liminf_{n\to\infty} \left[Q_n^*(n-A)^{\alpha}(n-A-1)^{\alpha} exp(-e^{\lambda n})\right] > 0, \tag{3.2.18}$$

respectively. So our results improve and extend the results given in Theorem 3.1.1.

3.3 Examples

In this section, we present some examples to illustrate the main results.

Example 3.3.1. *Consider the neutral difference equation*

$$\Delta\left(\frac{1}{n}\Delta(x_n + 2x_{n-1})\right) + \frac{2(2n+1)}{n(n+1)}x_{n-2}^{1/3} = 0, \quad n \geq 1. \tag{3.3.1}$$

Here $r_n = \frac{1}{n}$, $p_n = 2$, $q_n = \frac{2(2n+1)}{n(n+1)}$, $k = 1$, $A = 2$, *and* $\alpha = \frac{1}{3}$. *It is easy to see that all conditions of Corollary 3.2.1 are satisfied. Hence every solution of equation* (3.3.1) *is oscillatory. In fact* $\{x_n\} = \{(-1)^{3n}\}$ *is one such solution of equation* (3.3.1) *since it satisfies the equation* (3.3.1).

Example 3.3.2. *Consider the neutral difference equation*

$$\Delta\left(\frac{1}{n}\Delta(x_n + 3x_{n-2})\right) + \frac{8(2n+1)}{n(n+1)}x_{n-3} = 0, \quad n \geq 1. \tag{3.3.2}$$

Here $r_n = \frac{1}{n}$, $p_n = 3$, $q_n = \frac{8(2n+1)}{n(n+1)}$, $k = 2$, $A = 3$, *and* $\alpha = 1$. *It is easy to see that all conditions of Corollary 3.2.2 are satisfied. Hence every solution of equation* (3.3.2) *is oscillatory. In fact* $\{x_n\} = \{(-1)^n\}$ *is one such solution of equation* (3.3.2) *since it satisfies the equation* (3.3.2).

Example 3.3.3. *Consider the neutral difference equation*

$$\Delta\left(\frac{1}{n}\Delta(x_n + 3x_{n-2})\right) + \frac{e^{e^n}}{n^6}x_{n-4}^3 = 0, \quad n \geq 1. \tag{3.3.3}$$

Here $r_n = \frac{1}{n}$, $p_n = 3$, $q_n = \frac{e^{e^n}}{n^6}$, $k = 2$, $A = 4$, *and* $\alpha = 3$. *Choose* $\lambda = 1$. *Then it is easy to see that all conditions of Corollary 3.2.3 are satisfied. Hence every solution of equation* (3.3.3) *is oscillatory.*

We conclude this chapter with the following remarks.

Remark 3.3.1. *Theorem 3.1.1 cannot be applied to any of the above examples, because it is applicable only if* $p < 1$.

Remark 3.3.2. *The results presented in the chapter generalize and improve that of in [96]. Further it would be interesting to obtain oscillation results to the equation*

$$\Delta\left(r_n \Delta(x_n + p_n x_{n+k})\right) + q_n x^{\alpha}_{n-A} = 0$$

where α is a ratio of odd positive integers and $\sum_{n=n_0}^{\infty} \frac{1}{r_n} < \infty$.