Chapter 2

First Order Nonlinear Neutral Difference Equations

2.First Order Nonlinear Neutral Difference Equations

2.1 Introduction

In this chapter, we consider the first order nonlinear neutral difference equation of the form

$$\Delta(x_n + px_{n-k}) + q_n x_{n-A}^{\alpha} = 0, \quad n \in \mathrm{N}_0, \tag{2.1.1}$$

subject to the following conditions:

(C_1) $\{q_n\}$ is a positive real sequence;

(C_2) k and A are positive integers and $0 \leq p < \infty$;

(C_3) α is a ratio of odd positive integers.

Let $\theta = \max\{k, A\}$. By a solution of equation (2.1.1) we mean a nontrivial real sequence $\{x_n\}$ defined for all $n \geq n_0 - \theta$ and satisfying equation (2.1.1) for all $n \geq n_0$. We assume that such solutions exist for the equation (2.1.1).

In recent years, there has been much research concerning the oscillation of delay difference equations and neutral difference equations, see for example [1, 10, 19, 20, 25, 33, 35, 42, 47, 49, 62, 66, 69, 70, 71, 73, 90, 91], and the references cited therein.

In [35, 40, 51, 62, 66, 71], the authors studied the oscillatory behavior of all solutions of equation (2.1.1) when $\alpha = 1$ and $p \equiv 0$. When $p \equiv 0$ the oscillatory behavior of all solutions of equation (2.1.1) are studied in [1, 70]. In [1, 10, 42, 91], the authors studied the oscillatory behavior of all solutions of equation (2.1.1) when $\alpha = 1$. Motivated by this observation, in this chapter, we obtain some sufficient conditions for the oscillation of all solutions of equation (2.1.1) for different values of α.

In Section 2.2, we present some sufficient conditions for the oscillation of all solutions of equation (2.1.1) and in Section 2.3, we present some examples to illustrate the main results.

2.2 Oscillation Theorems

In this section, we present some sufficient conditions for the oscillation of all solutions of equation (2.1.1). Throughout this chapter we use the following notation without further mention:

$$z_n = x_n + px_{n-k}, \tag{2.2.1}$$

and

$$Q_n = \min\{q_n, q_{n-k}\} \text{ for all } n \in \mathbb{N}_0. \tag{2.2.2}$$

Lemma 2.2.1. *If $A \geq 0, B \geq 0$ and $0 < \alpha \leq 1$, then*

$$A^\alpha + B^\alpha \geq (A + B)^\alpha. \tag{2.2.3}$$

Proof. If $A = 0$ or $B = 0$, then the result is obvious.

Assume that $A > 0$ and $B > 0$. Define a function $g(A, B)$ by

$$g(A, B) = A^\alpha + B^\alpha - (A + B)^\alpha.$$

Fix A. Then, since $0 < \alpha \leq 1$, we have

$$\frac{dg(A, B)}{dB} = \alpha B^{\alpha-1} - \alpha(A + B)^{\alpha-1} = \alpha \left[B^{\alpha-1} - (A + B)^{\alpha-1} \right] \geq 0.$$

Thus, g is nondecreasing with respect to B, which yields $g(A, B) \geq 0$. The proof of the lemma is complete. □

Lemma 2.2.2. *If $A \geq 0, B \geq 0$ and $\alpha > 1$, then*

$$A^\alpha + B^\alpha \geq \frac{1}{2^{\alpha-1}}(A + B)^\alpha. \tag{2.2.4}$$

Proof. If $A = 0$ or $B = 0$, then the result is obvious.

Assume that $A > 0$ and $B > 0$. Define a function g by $g(u) = u^\alpha$, $u \in (0, \infty)$. Then $g''(u) = \alpha(\alpha - 1)u^{\alpha-2} > 0$. Thus, g is a convex function. By the definition of convex function, we have

$$g\left(\frac{A+B}{2}\right) \leq \frac{g(A)+g(B)}{2}.$$

This completes the proof. □

Lemma 2.2.3. *Let $\{q_n\}$ be a positive real sequence, A be a positive integer and α be a ratio of odd positive integers. Then the difference inequality*

$$\Delta x_n + q_n x^\alpha_{n-A} \leq 0 \tag{2.2.5}$$

has an eventually positive solution if and only if the difference equation

$$\Delta x_n + q_n x^\alpha_{n-A} = 0 \tag{2.2.6}$$

has an eventually positive solution.

Proof. The proof can be found in [26, 40, 70]. □

Lemma 2.2.4. *Assume that $0 < \alpha < 1$ and A is a positive integer. Then every solution of equation* (2.2.6) *is oscillatory if and only if*

$$\sum_{n=n_0}^{\infty} q_n = \infty. \tag{2.2.7}$$

Proof. The proof of the lemma can be found in [70]. However for easy reference we present the proof of the lemma.

Sufficiency. Assume that (2.2.7) holds and suppose that $\{x_n\}$ is an eventually positive solution of (2.2.6). Then there exists an integer $n_1 > 0$ such that

$$x_{n-A} > 0 \text{ and } x_{n+1} - x_n \leq 0 \text{ for } n \geq n_1. \tag{2.2.8}$$

Clearly (2.2.6), (2.2.7) and (2.2.8) imply that $\lim_{n\to\infty} x_n = 0$ and

$$x_n^{1-\alpha} - x_{n+1}^{1-\alpha} \geq (1-\alpha)x_n^{-\alpha}(x_n - x_{n+1}) \geq (1-\alpha)q_n, \; n \geq n_1.$$

It follows that

$$x_{n_1}^{1-\alpha} \geq (1-\alpha)\sum_{n=n_1}^{\infty} q_n,$$

which is a contradiction to (2.2.7).

Necessity. Assume that (2.2.6) holds and suppose that (2.2.7) is not true. Then there exists an integer $N > A$ such that $\sum_{s=N-A}^{\infty} q_s \leq \frac{1}{2}$. Define a sequence $\{y_n\}$ as follows:

$$y_n = \frac{1}{2} + \sum_{i=n}^{\infty} q_i, \quad n \geq N - A.$$

Clearly, $\frac{1}{2} \leq y_n \leq 1$ for $n \geq N - A$ and

$$y_n \geq \frac{1}{2} + \sum_{i=n}^{\infty} q_i y_{i-A}^{\alpha}, \quad n \geq N.$$

From this, it is easy to see that the corresponding equation

$$x_n = \frac{1}{2} + \sum_{i=n}^{\infty} q_i x_{i-A}^{\alpha}, \quad n \geq N$$

has an eventually positive solution $\{x_n\}$. Clearly, $\{x_n\}$ is also an eventually positive solution of (2.2.6), which is a contradiction. This completes the proof. □

Lemma 2.2.5. *If* $\alpha = 1$ *and*

$$\liminf_{n\to\infty} \sum_{s=n-A}^{n-1} q_s > \left(\frac{A}{A+1}\right)^{A+1} \tag{2.2.9}$$

then every solution of equation (2.2.6) *is oscillatory.*

Proof. The proof of the lemma can be found in [1, 40]. However for easy reference we present the proof of the lemma.

Let $\{x_n\}$ be a nonoscillatory solution of equation (2.2.6), which we can assume to be positive eventually, and since $q_n \geq 0$ this solution $\{x_n\}$ is eventually decreasing. Therefore, using $x_n \leq x_{n-A}$ in (2.2.6), we obtain

$$q_n \leq 1 - \frac{x_{n+1}}{x_n}.$$

Now using arithmetic - geometric means inequality, we find

$$\begin{aligned} \frac{1}{A}\sum_{s=n-A}^{n-1} q_s &\leq 1 - \frac{1}{A}\sum_{s=n-A}^{n-1} \frac{x_{s+1}}{x_s} \\ &\leq 1 - \left(\frac{x_n}{x_{n-A}}\right)^{\frac{1}{fl}}. \end{aligned} \tag{2.2.10}$$

Setting $\beta = \frac{A^{fl}}{(A+1)^{fl+1}}$, from (2.2.10), we can choose a constant γ such that for n sufficiently large, $\beta < \gamma \leq \frac{1}{A}\sum_{s=n-A}^{n-1} q_s$. Therefore from (2.2.10) for all large n, $\left(\frac{x_n}{x_{n-fl}}\right)^{\frac{1}{fl}} \leq 1 - \gamma$, which in particular implies that $0 < \gamma < 1$. Now since $\max_{0 \leq \gamma \leq 1} \left((1-\gamma)\gamma^{\frac{1}{fl}}\right) = \beta^{\frac{1}{fl}}$, see [26], we have $1 - \gamma \leq \beta^{\frac{1}{fl}}\gamma^{-\frac{1}{fl}}$ for $0 < \gamma < 1$, and hence it follows that

$$\left(\frac{x_n}{x_{n-A}}\right)^{\frac{1}{fl}} \leq \beta^{\frac{1}{fl}}\gamma^{-\frac{1}{fl}}$$

or

$$\frac{\gamma}{\beta}x_n \leq x_{n-A}. \tag{2.2.11}$$

Now using (2.2.11) instead of $x_n \leq x_{n-A}$ in (2.2.6) and repeating the arguments, we find $\left(\frac{\gamma}{\beta}\right)^2 x_n \leq x_{n-A}$ for all large n. Thus, by induction, for every $k \in N(1)$ there exists an integer n_k such that for all $n \in N(n_k)$

$$\left(\frac{\gamma}{\beta}\right)^k x_n \leq x_{n-A}. \tag{2.2.12}$$

Next, for sufficiently large n,

$$\sum_{s=n-A}^{n} q_s \geq \sum_{s=n-A}^{n-1} q_s \geq A\gamma = M.$$

Since $\gamma > \beta$, we can choose k such that

$$\left(\frac{\gamma}{\beta}\right)^{k} > \left(\frac{2}{M}\right)^{2}. \tag{2.2.13}$$

For this specific value of k, we consider n sufficiently large, say, n_1 so that for all $n \geq n_1$ all the above inequalities are satisfied. Then, for each $n \geq n_1 + A$ there exists an integer N with $n - A \leq N \leq n$ so that

$$\sum_{s=n-A}^{N} q_s \geq \frac{M}{2} \quad \text{and} \quad \sum_{s=N}^{n} q_s \geq \frac{M}{2}.$$

From (2.2.6) and the nonincreasing nature of $\{x_n\}$, we have

$$\begin{aligned} -x_{n-A} &\leq \sum_{s=n-A}^{N} \Delta x_s = \sum_{s=n-A}^{N} (-q_s x_{s-A}) \\ &\leq -\left(\sum_{s=n-A}^{N} q_s\right) x_{N-A} \leq -\frac{M}{2} x_{N-A} \end{aligned}$$

and hence

$$\frac{M}{2} x_{N-A} \leq x_{n-A}. \tag{2.2.14}$$

Similarly, we find

$$-x_N \leq -\frac{M}{2} x_{n-A}$$

and so

$$\frac{M}{2} x_{n-A} \leq x_N. \tag{2.2.15}$$

Combining (2.2.12), (2.2.14) and (2.2.15), we get

$$\left(\frac{\gamma}{\beta}\right)^{k} \leq \frac{x_{N-A}}{x_N} \leq \left(\frac{2}{M}\right)^{2}.$$

This contradicts (2.2.13). The proof is now complete. □

Lemma 2.2.6. *Let $\alpha > 1$. If there exists a $\lambda > \frac{1}{A} \log \alpha$ such that*

$$\liminf_{n \to \infty} \left[q_n exp(-e^{\lambda n})\right] > 0, \tag{2.2.16}$$

then every solution of equation (2.2.6) *is oscillatory.*

Proof. The proof of the lemma can be found in [70]. However for easy reference we present the proof of the lemma.

Let $\{x_n\}$ be a nonoscillatory solution of equation (2.2.6), which we can assume to be eventually positive solution. Then there exists an integer $N \geq n_0$ such that

$$x_{n-A} > 0 \text{ for } n \geq N.$$

By (2.2.16), there exist an integer $N_1 \in N_0$ and $\lambda_0 \in (\frac{1}{A}\log \alpha, \lambda)$ such that

$$q_n \geq e^{\lambda_0 n} exp (e^{\lambda_0 n}), \quad n \geq N_1. \tag{2.2.17}$$

Set

$$\varphi(n) = e^{\lambda_0 n}, \quad p_n = \varphi(n) e^{\varphi(n)} \text{ and } k = \alpha e^{-\lambda_0 A}.$$

Then $0 < k < 1$. Let $y_n = -\log x_n$ for $n \geq N_1 - A$. Then $y_n > 0$ for $n \geq N_1 - A$, and from (2.2.6), we have

$$1 - e^{y_n - y_{n+1}} = p_n e^{y_n - \alpha y_{n-A}}, \quad n \geq N_1. \tag{2.2.18}$$

Consequently, we obtain

$$y_{n+1} - y_n \geq p_n e^{y_n - \alpha y_{n-A}}, \quad n \geq N_1. \tag{2.2.19}$$

We consider the following two possible cases for $y_n - \alpha y_{n-A}$.

Case(i). Suppose $y_n - \alpha y_{n-A} \leq 0$ holds eventually. Choose an integer $N_2 \geq N_1$ such that

$$y_n - \alpha y_{n-A} \leq 0, \quad n \geq N_2.$$

It follows that

$$\frac{y_n}{\varphi(n)} \leq \frac{\alpha\varphi(n-A)}{\varphi(n)} \frac{y_{n-A}}{\varphi(n-A)} = k\frac{y_{n-A}}{\varphi(n-A)}, \quad n \geq N_2.$$

Set $z_n = \frac{y_n}{\varphi(n)}$, $n \geq N_2 - A$. Then

$$z_n \leq k z_{n-A}, \quad n \geq N_2. \tag{2.2.20}$$

Set $M = \max\{z_{N_2}, z_{N_2+1}, \ldots, z_{N_2+A-1}\}$. Then (2.2.20) implies that

$$z_n \le M k^{[(n-N_2)/A]}, \quad n \ge N_2.$$

Here $[x]$ denotes the greatest integer less than or equal to x. Hence

$$\lim_{n\to\infty} z_n = 0. \tag{2.2.21}$$

This shows that there exists an integer $N_3 > N_2$ such that

$$y_n < \frac{1}{1+\alpha}\varphi(n), \quad n \ge N_3. \tag{2.2.22}$$

From (2.2.19) and (2.2.22), we have

$$\begin{aligned} \Delta y_n &\ge p_n e^{-(\alpha-1)y_{n-fl}} \\ &\ge p_n e^{-(\alpha-1)\varphi(n)/(1+\alpha)} \\ &\ge \lambda_0 e^{\lambda_0}\varphi(n) \\ &\ge \Delta\varphi(n). \end{aligned}$$

Summing the last inequality from N_3 to n, we obtain

$$y_n > \varphi(n+1) - \varphi(N_3) + y_{N_3}, \quad \text{for} \quad n \ge N_3,$$

which is a contradiction to (2.2.22).

Case(ii). Now assume $y_n - \alpha y_{n-A} > 0$ holds. Then there exists a sequence $\{n_i\}$ of integers with $N_1 < n_1 < n_2 < \ldots$ such that

$$y_{n_i} - \alpha y_{n_i - A} > 0, \quad i = 1, 2, \ldots$$

which, together with (2.2.18) implies that

$$1 > q_{n_i} e^{y_{n_i} - \alpha y_{n_i - fl}} > q_{n_i} > 1, \quad i = 1, 2, \ldots.$$

This is a contradiction. The proof is complete. □

Theorem 2.2.1. *Let* $0 < \alpha \le 1$*. If the first order neutral difference inequality*

$$\Delta w_n + \frac{1}{(1+p^\alpha)^\alpha} Q_n w^\alpha_{n+k-A} \le 0, \quad n \ge n_0 \tag{2.2.23}$$

has no positive solution, then every solution of equation (2.1.1) *is oscillatory.*

Proof. Let $\{x_n\}$ be a nonoscillatory solution of equation (2.1.1). Without loss of generality we may assume that $x_n > 0$ and $x_{n-k} > 0$ for all $n \ge n_1 \ge n_0 - \theta$. Then $z_n > 0$ for all $n \ge n_1$. From the equation (2.1.1), we have

$$\Delta z_n + q_n x^\alpha_{n-A} = 0, \tag{2.2.24}$$

and

$$p^\alpha \Delta z_{n-k} + p^\alpha q_{n-k} x^\alpha_{n-k-A} = 0. \tag{2.2.25}$$

Combining (2.2.24) and (2.2.25), we get

$$\Delta(z_n + p^\alpha z_{n-k}) + Q_n \left[x^\alpha_{n-A} + p^\alpha x^\alpha_{n-k-A} \right] \le 0. \tag{2.2.26}$$

Applying Lemma 2.2.1 in inequality (2.2.26), we obtain

$$\Delta(z_n + p^\alpha z_{n-k}) + Q_n z^\alpha_{n-A} \le 0. \tag{2.2.27}$$

Let $w_n = z_n + p^\alpha z_{n-k}$. Then $w_n > 0$ and using the decreasing nature of $\{z_n\}$, we obtain

$$w_n \le (1 + p^\alpha) z_{n-k}$$

or

$$\frac{w_{n+k-A}}{(1+p^\alpha)} \le z_{n-A}. \tag{2.2.28}$$

Substituting (2.2.28) in (2.2.27), we obtain that $\{w_n\}$ is a positive solution of the inequality

$$\Delta w_n + \frac{1}{(1+p^\alpha)^\alpha} Q_n w^\alpha_{n+k-A} \le 0,$$

which is a contradiction to (2.2.23). The proof is now complete. □

Theorem 2.2.2. *Let $\alpha > 1$. If the first order neutral difference inequality*

$$\Delta w_n + \frac{1}{(1+p^\alpha)^\alpha} 2^{1-\alpha} Q_n w^\alpha_{n+k-\ell} \le 0, \quad n \ge n_0 \tag{2.2.29}$$

has no positive solution, then every solution of equation (2.1.1) *is oscillatory.*

Proof. Let $\{x_n\}$ be a nonoscillatory solution of equation (2.1.1). From the proof of Theorem 2.2.1, we have (2.2.26). Now applying Lemma 2.2.2 to (2.2.26), we obtain

$$\Delta(z_n + p^\alpha z_{n-k}) + 2^{1-\alpha} Q_n z^\alpha_{n-\ell} \le 0. \tag{2.2.30}$$

Let $w_n = z_n + p^\alpha z_{n-k}$. Then $w_n > 0$ and using the decreasing nature of $\{z_n\}$, we obtain

$$w_n \le (1 + p^\alpha) z_{n-k}$$

or

$$\frac{w_{n+k-\ell}}{(1+p^\alpha)} \le z_{n-\ell}. \tag{2.2.31}$$

Substituting (2.2.31) in (2.2.30), we get that $\{w_n\}$ is a positive solution of the inequality

$$\Delta w_n + \frac{1}{(1+p^\alpha)^\alpha} 2^{1-\alpha} Q_n w^\alpha_{n+k-\ell} \le 0,$$

which is a contradiction to (2.2.29). The proof is now complete. □

Corollary 2.2.1. *Let $\ell > k$ and $0 < \alpha < 1$ in equation* (2.1.1). *If*

$$\sum_{n=n_0}^{\infty} Q_n = \infty \tag{2.2.32}$$

then every solution of equation (2.1.1) *is oscillatory.*

Proof. From Lemmas 2.2.3 and 2.2.4, we see that the condition (2.2.32) implies that the inequality (2.2.23) has no positive solution and hence the proof follows from Theorem 2.2.1. This completes the proof. □

Corollary 2.2.2. *Let* $A > k$ *and* $\alpha = 1$ *in equation* (2.1.1). *If*

$$\liminf_{n\to\infty} \sum_{s=n-A+k}^{n-1} Q_s > (1+p)\left(\frac{A-k}{A-k+1}\right)^{A-k+1} \tag{2.2.33}$$

then every solution of equation (2.1.1) *is oscillatory.*

Proof. From Lemmas 2.2.3 and 2.2.5, we see that the condition (2.2.33) implies that the inequality (2.2.23) has no positive solution and hence the proof follows from Theorem 2.2.1. This completes the proof. □

Corollary 2.2.3. *Let* $A > k$ *and* $\alpha > 1$ *in equation* (2.1.1). *If there exists a* $\lambda > 0$ *such that* $\lambda > \frac{1}{A-k}\log\alpha$ *and*

$$\liminf_{n\to\infty} \left[Q_n exp(-e^{\lambda n})\right] > 0 \tag{2.2.34}$$

then every solution of equation (2.1.1) *is oscillatory.*

Proof. From Lemmas 2.2.3 and 2.2.6, we see that the condition (2.2.34) implies that the inequality (2.2.29) has no positive solution and hence the proof follows from Theorem 2.2.2. This completes the proof. □

2.3 Examples

In this section, we present some examples to illustrate the main results.

Example 2.3.1. *Consider the neutral difference equation*

$$\Delta(x_n + 2x_{n-2}) + 6x_{n-4}^{1/3} = 0, \quad n \geq 1. \tag{2.3.1}$$

Here $p = 2$, $q_n = 6$, $k = 2$, $A = 4$, *and* $\alpha = \frac{1}{3}$ *It is easy to see that all conditions of Corollary 2.2.1 are satisfied. Hence every solution of equation* (2.3.1) *is oscillatory. In fact* $\{x_n\} = \{(-1)^{3n}\}$ *is one such solution of equation* (2.3.1) *since it satisfies the equation* (2.3.1).

Example 2.3.2. *Consider the neutral difference equation*

$$\Delta(x_n + 2x_{n-2}) + \frac{6n-5}{n-4}x_{n-4} = 0, \quad n \geq 5. \tag{2.3.2}$$

Here $p = 2$, $q_n = \frac{6n-5}{n-4}$, $k = 2$, $A = 4$, *and* $\alpha = 1$. *It is easy to see that all conditions of Corollary 2.2.2 are satisfied. Hence every solution of equation* (2.3.2) *is oscillatory. In fact* $\{x_n\} = \{n(-1)^n\}$ *is one such solution of equation* (2.3.2) *since it satisfies the equation* (2.3.2).

Example 2.3.3. *Consider the neutral difference equation*

$$\Delta(x_n + 3x_{n-2}) + \left(1 + \frac{1}{n}\right)e^{e^{2n}}x_{n-4}^3 = 0, \quad n \geq 1. \tag{2.3.3}$$

Here $p = 3$, $q_n = \left(1 + \frac{1}{n}\right)e^{e^{2n}}$, $k = 2$, $A = 4$, *and* $\alpha = 3$. *By choosing* $\lambda = 2$ *it is easy to see that all conditions of Corollary 2.2.3 are satisfied. Hence every solution of equation* (2.3.3) *is oscillatory.*

We conclude this chapter with the following remark.

Remark 2.3.1. *The results presented in this chapter extend and generalize some of the known results given in [1, 10, 35, 40, 42, 51, 62, 66, 70, 71, 91]. Further it would be interesting to extend the results of this chapter to the equation*

$$\Delta(x_n - p_n x_{n-k}) \pm q_n x_{n-A}^{\alpha} = e_n$$

where $\{p_n\}$, $\{q_n\}$ *and* $\{e_n\}$ *are real positive sequences and* α *is a ratio of odd positive integers.*