Chapter 1

Introduction

1.Introduction

1.1 Difference Equations

Difference equations are equations that allow us to compute the value of functions recursively from a set of values. They usually describe the evolution of certain phenomena over a course of time. There are many problems described by difference equations in digital simulations, sampled data, control systems, economics, biology, social systems, computer science and other problems in science where the system is discrete in nature. For the basic theory of difference equations and its applications one can refer the monographs by Agarwal et al. [1, 2, 4], Gyori and Ladas [26], Kelley and Peterson [38] and Zhang and Geo [97].

Before giving a formal definition to difference equation, let us introduce the following notation which are used in this thesis.

$$N_0 = \{0, 1, 2, \ldots\}$$

and

$$N(n_0) = \{n_0, n_0 + 1, n_0 + 2, \ldots\}, \quad n_0 \in N_0.$$

A difference equation of order m is defined as a functional relation of the form

$$F(n, x_n, x_{n+1}, \ldots, x_{n+m}) = 0, \quad n \in N_0 \tag{1.1.1}$$

or

$$\Delta^m x_n + f(n, x_n, \Delta x_n, \cdots, \Delta^{m-1} x_n) = 0, \quad n \in N_0 \tag{1.1.2}$$

where Δ is the forward difference operator defined by $\Delta x_n = x_{n+1} - x_n$ and in general $\Delta^i x_n = \Delta(\Delta^{i-1} x_n)$, $i = 2, 3, \cdots, m$. A function x_n defined on N_0 is said to be a solution of the difference equation (1.1.1) or (1.1.2) if the values of x_n reduce the equation (1.1.1) or (1.1.2) to an identity over N_0.

As in the case of differential equations, the difference equations can also be classified as ordinary, delay, advanced and neutral type difference equations.

A difference equation of the form

$$\Delta^m x_n + f(n, x_n, x_{n+1}, \cdots, x_{n+m}) = 0, \quad n \in N_0, \tag{1.1.3}$$

is called an *ordinary difference equation*.

An equation of the form

$$\Delta^m x_n + f(n, x_{n-A}) = 0, \quad n \in N_0, \tag{1.1.4}$$

is called a *delay difference equation* if A is a positive integer and *advanced type* if A is a negative integer.

An equation of the form

$$\Delta^m (x_n + p_n x_{n-k}) + f(n, x_{n-A}) = 0, \quad n \in N_0, \tag{1.1.5}$$

where k and A are integers, is called a difference equation of *neutral type*.

The origin of the modern theory of difference equations may be traced back to the fundamental work of Poincare at the end of the nineteenth century, see [1], and the references cited therein. Through the course of the twentieth century, a large number of papers dealing with various qualitative and quantitative aspects of the solutions of the different types of difference equations appeared, see [1, 2, 4, 26, 38, 97], and the references given therein. In the qualitative theory of difference equations oscillatory behavior of solutions play an important role. A nontrivial solution of a difference equation is said to be oscillatory if it is neither eventually positive nor eventually negative and nonoscillatory otherwise. These types of solutions occur in many physical phenomena such as, vibrating mechanical systems, electrical circuits and in population dynamics.

1.2 Motivation

The literature on the oscillation and asymptotic behavior of solutions of different types of difference equations have grown to such an extent that it is impossible to mention all the authors who had contributed on this subject. For example Agarwal et al. [1-8], Aktas et al. [9], Brayton et al. [10], Cheng et al. [12, 13], Dahiya et al. [14], Dosla et al. [15], Dosly et al. [16-18], Gao et al. [19], Georgiou et al. [20], Grace et al. [21-23], Graef et al. [24, 25], Gyori et al. [26], Hartman [28], Hooker and Patula [29], Jiang et al. [31-36], Karpuz et al. [37], Ladas et al. [40], Lalli et al. [41, 42], Li et al. [30, 44], Liu et al. [45], Morshedy et al. [46], Ocalan et al. [47], Parhi et al, [48, 49], Peng et al. [50], Philos [51], Saker et al. [52-59, 67], Sandra Pinelas [39, 60], Smith et al. [63], Stavroulakis et al. [62, 64-66], Tang at el. [68-71], Thandapani et al. [72-85], Tripathy et al. [86, 87], Wang at el. [89], Wei [90], Yu et al. [91], Zafer [92-94], B.G.Zhang [95] and G.Zhang et al. [96, 97] have done extensive work on this topic.

From the review of literature it is known that compared to the study of ordinary difference equations, the study of neutral type difference equations has received less attention. This motivated us to study the oscillatory behavior of solutions of various types of neutral type difference equations. The study of neutral type difference equations is both an interesting and useful area of research since such type of equations arise in population dynamics when maturation and gestation are included, in cobweb models in economics where demand depends on current price but supply depends on the price at an earlier time and in electric networks containing lossless transmission lines.

Keeping in view of the importance of the subject and in the light of the above trend, the author has obtained some significant results on the following topics:

1. **First order nonlinear neutral difference equations.**
2. **Second order neutral difference equations.**
3. **Second order neutral delay and advanced difference equations.**
4. **Third order nonlinear neutral difference equations.**
5. **Third order neutral difference equations with distributed delay.**

1.3 Summary of the Thesis

This thesis consists of six chapters including this introductory chapter.

In Chapter 2, we consider the first order nonlinear neutral difference equation of the form

$$\Delta(x_n + px_{n-k}) + q_n x^{\alpha}_{n-A} = 0, \quad n \in N_0, \qquad (1.3.1)$$

where $\{q_n\}$ is a positive real sequence, $0 \le p < \infty$, k, A are positive integers and α is a ratio of odd positive integers. Section 2.1 provides necessary introduction. In Section 2.2, we present some sufficient conditions for the oscillation of all solutions of equation (1.3.1) and in Section 2.3, we present some examples to illustrate the main results. The results presented in this chapter extend and generalize some of the known results given in [1, 10, 35, 40, 42, 51, 62, 66, 70, 71, 91].

Chapter 3 deals with the second order neutral difference equation of the form

$$\Delta(r_n\Delta(x_n + p_n x_{n-k})) + q_n x^{\alpha}_{n-A} = 0, \quad n \in N_0 \qquad (1.3.2)$$

where $\{r_n\}$ is a positive real sequence with $\sum_{n=n_0}^{\infty}\frac{1}{r_n} = \infty$, $\{p_n\}$ is a real sequence with $0 \le p_n \le p < \infty$, $\{q_n\}$ is a positive real sequence, k, A are positive integers and

α is a ratio of odd positive integers. Section 3.1 presents necessary introduction and motivation. In Section 3.2, we establish some sufficient conditions for the oscillation of all solutions of equation (1.3.2), and in Section 3.3, some examples are provided to illustrate the main results. The results presented in this chapter generalize and improve that of in [96].

In Chapter 4, we study the oscillatory behavior of solutions of the second order neutral delay difference equation of the form

$$\Delta(r_n\Delta(x_n + p_n x_{n-k})) + q_n x_{n-A} + v_n x^{\alpha}_{n-m} = 0, \quad n \in \mathbb{N}_0 \qquad (1.3.3)$$

and the advanced difference equation of the form

$$\Delta(r_n\Delta(x_n + p_n x_{n+k})) + q_n x_{n+A} + v_n x^{\alpha}_{n+m} = 0, \quad n \in \mathbb{N}_0 \qquad (1.3.4)$$

where $\{r_n\}$ is a positive real sequence with $\sum_{n=n_0}^{\infty} \frac{1}{r_n} = \infty$, $\{p_n\}$ is a real sequence with $0 \le p_n \le p < \infty$, $\{q_n\}$, $\{v_n\}$ are positive real sequences, k, A, m are positive integers and α is a ratio of odd positive integers. In Section 4.1, necessary introduction and motivation are provided, and in Section 4.2, we discuss the oscillatory behavior of all solutions of equations (1.3.3) and (1.3.4), and in Section 4.3, some examples are provided to illustrate the main results obtained in Section 4.2. The results of this chapter extend and generalize some of the known results in [2, 7, 8, 31, 32, 43, 44, 78, 87, 88, 96].

In Chapter 5, we establish some oscillation criteria for third order nonlinear neutral difference equation of the form

$$\Delta\left(r_n(\Delta^2(x_n \pm p_n x_{\sigma(n)}))^{\alpha}\right) + f(n, x_{\tau(n)}) = 0, \quad n \in \mathbb{N}_0 \qquad (1.3.5)$$

where $\{r_n\}$ is a positive real sequence with $\sum_{n=n_0}^{\infty} \frac{1}{r_n^{1/\alpha}} = \infty$, $\{p_n\}$ is a real sequence with $-\mu \le p_n \le 1$ for $\mu \in (0, 1)$, $\{\sigma(n)\}$ is a nonnegative sequence of integers with

$\sigma(n) \leq n$ such that $\lim_{n\to\infty}\sigma(n) = \infty$, $\{\tau(n)\}$ is a nonnegative sequence of integers with $\tau(n) \leq n$ such that $\lim_{n\to\infty}\tau(n) = \infty$, $f : \mathbb{N}_0 \times \mathbb{R} \to [0,\infty)$ and there is a nonnegative real sequence $\{q_n\}$ such that $\frac{f(n,u)}{u^{\alpha}} \geq Lq_n$, for $u = 0$ where $L > 0$ and α is a ratio of odd positive integers. Section 5.1 provides necessary introduction and review of literature. In Section 5.2, we present some theorems regarding oscillatory and asymptotic behavior of solutions of the equation (1.3.5), and in Section 5.3, some examples are provided in support of our main results. The results obtained in this chapter generalize and complement to that of in [22, 79, 82, 83, 85].

Finally, in Chapter 6, we study the oscillatory behavior of third order neutral difference equation of the form

$$\Delta\left(r_n\Delta^2\left(x_n+\sum_{s=a}^{b} p_{n,s}x_{n+s-\tau}\right)\right)+\sum_{s=c}^{d} q_{n,s}f(x_{n+s-\sigma}) = 0,\quad n \in \mathbb{N}_0 \qquad (1.3.6)$$

where $\{r_n\}$ is a positive real sequence with $\sum_{n=n_0}^{\infty}\frac{1}{r_n} = \infty$, $\{q_{n,s}\}$ and $\{p_{n,s}\}$ are nonnegative real sequences with $0 \leq p_n \equiv \sum_{s=a}^{b} p_{n,s} \leq P < 1$, $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $\frac{f(u)}{u} \geq L > 0$ for $u = 0$ and $a, b, c, d \in \mathbb{N}_0$ with $a \leq b$ and $c \leq d$. Section 6.1 provides necessary introduction and motivation. In Section 6.2, we present some preliminary lemmas, and in Section 6.3, we establish some sufficient conditions which ensure that all solutions of equation (1.3.6) are either oscillatory or converging to zero. Some examples are given to illustrate the main results in Section 6.4. The results obtained in this chapter generalize and complement to that of in [3, 21, 24, 45, 57, 58].