Chapter 6

Third Order Neutral Difference Equations With Distributed Delay

6. Third Order Neutral Difference Equations With Distributed Delay

6.1 Introduction

In this final chapter, we study the oscillatory behavior of third order neutral difference equation of the form

$$\Delta\left(r_n\Delta^2\left(x_n + \sum_{s=a}^{b} p_{n,s}x_{n+s-\tau}\right)\right) + \sum_{s=c}^{d} q_{n,s}f(x_{n+s-\sigma}) = 0, \quad n \in \mathbb{N}_0 \qquad (6.1.1)$$

subject to the following conditions:

(C_1) $\{r_n\}$ is a positive real sequence with $\sum_{n=n_0}^{\infty} \frac{1}{r_n} = \infty$;

(C_2) $\{q_{n,s}\}$ and $\{p_{n,s}\}$ are nonnegative real sequences with $0 \leq p_n \equiv \sum_{s=a}^{b} p_{n,s} \leq P < 1$;

(C_3) $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $\frac{f(u)}{u} \geq L > 0$ for $u \neq 0$;

(C_4) $a, b, c, d \in \mathbb{N}_0$ with $a \leq b$ and $c \leq d$.

By a solution of equation (6.1.1), we mean a nontrivial real sequence $\{x_n\}$ satisfying equation (6.1.1) for all $n \in \mathbb{N}_0$. We consider only those solution $\{x_n\}$ of equation (6.1.1) which satisfy $\sup\{|x_n| : n \geq N\} > 0$ for all $N \in \mathbb{N}_0$. We assume that such solutions exist for the equation (6.1.1).

In recent years there is a great interest in studying the oscillatory behavior of third order difference equations, see for example [3, 5, 6, 9, 15, 21, 22, 24, 45, 48, 54, 56, 57, 58, 59, 61, 63, 75, 79, 82, 83, 85], and the references cited therein. Following this trend, in this chapter we obtain some sufficient conditions for the oscillation of all solutions of equation (6.1.1).

In Section 6.2, we present some preliminary lemmas, and in Section 6.3, we establish some sufficient conditions which ensure that all solutions of equation (6.1.1) are either oscillatory or converging to zero. Some examples are given to illustrate the main results in Section 6.4.

6.2 Preliminary Lemmas

In this section, we present some lemmas which will be useful in proving our main results. We write

$$z_n = x_n + \sum_{s=a}^{b} p_{n,s} x_{n+s-\tau}. \tag{6.2.1}$$

Lemma 6.2.1. *Let $\{x_n\}$ be a positive solution of equation* (6.1.1). *Then $\{z_n\}$ satisfies only the following two cases eventually*

(*I*) $z_n > 0, \quad \Delta z_n > 0, \quad \Delta^2 z_n > 0;$

(*II*) $z_n > 0, \quad \Delta z_n < 0, \quad \Delta^2 z_n > 0.$

Proof. The proof is similar to that of Lemma 5.2.1 and hence the details are omitted. □

Lemma 6.2.2. *Let $\{x_n\}$ be a positive solution of equation* (6.1.1)*, and let the corresponding function $\{z_n\}$ satisfies Case* (*II*) *of Lemma 6.2.1. If*

$$\sum_{n=n_0}^{\infty} \sum_{s=n}^{\infty} \left(\frac{1}{r_s} \sum_{t=s}^{\infty} \sum_{j=c}^{d} q_{t,j} \right) = \infty, \tag{6.2.2}$$

then $\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = 0.$

Proof. The proof is similar to that of Lemma 5.2.2 and hence the details are omitted. □

Lemma 6.2.3. *Assume that* $y_n > 0$, $\Delta y_n \ge 0$, $\Delta^2 y_n \le 0$ *for all* $n \ge n_0$. *Then for each* $\alpha \in (0, 1)$ *there exists a* $N \in \mathrm{N}_0$ *such that*

$$\frac{y_{n-\sigma}}{n-\sigma} \ge \alpha \frac{y_{n+1}}{n+1} \quad \text{for all} \quad n \ge N. \tag{6.2.3}$$

Proof. From the monotonicity of $\{\Delta y_n\}$, we have

$$y_{n+1} - y_{n-\sigma} = \sum_{s=n-\sigma}^{n} \Delta y_s \le (\sigma + 1)\Delta y_{n-\sigma}$$

or

$$\frac{y_{n+1}}{y_{n-\sigma}} \le 1 + \frac{(\sigma+1)\Delta y_{n-\sigma}}{y_{n-\sigma}}. \tag{6.2.4}$$

Also,

$$y_{n-\sigma} \ge y_{n-\sigma} - y_{n_0} \ge (n - \sigma - n_0)\Delta y_{n-\sigma}.$$

So, for each $\alpha \in (0, 1)$, there is a $N \in \mathrm{N}_0$ such that

$$\frac{y_{n-\sigma}}{\Delta y_{n-\sigma}} \ge \alpha(n-\sigma), \quad n \ge N. \tag{6.2.5}$$

Combining (6.2.4) and (6.2.5), we obtain

$$\frac{y_{n+1}}{y_{n-\sigma}} \le 1 + \frac{(\sigma+1)}{\alpha(n-\sigma)} \le \frac{\alpha n - \alpha\sigma + \sigma + 1}{\alpha(n-\sigma)}$$

or

$$\frac{y_{n+1}}{y_{n-\sigma}} \le \frac{(n+1)}{\alpha(n-\sigma)}.$$

This completes the proof. □

Lemma 6.2.4. *Assume that* $z_n > 0$, $\Delta z_n > 0$, $\Delta^2 z_n > 0$, $\Delta^3 z_n \le 0$ *for all* $n \ge N$. *Then*

$$\frac{z_n}{\Delta z_n} \ge \frac{n-N}{2} \quad \text{for all} \quad n \ge N. \tag{6.2.6}$$

Proof. From the monotonicity of $\{\Delta^2 z_n\}$, we have

$$\Delta z_n = \Delta z_N + \sum_{s=N}^{n-1} \Delta^2 z_s \ge (n - N)\Delta^2 z_n.$$

Summing from N to $n-1$, we obtain

$$\begin{aligned} z_n &\geq z_N + \sum_{s=N}^{n-1} (s-N)\Delta^2 z_s \\ &= z_N + (n-N)\Delta z_n - z_{n+1} + z_N. \end{aligned}$$

Hence $z_n \geq \frac{(n-N)}{2}\Delta z_n, \quad n \geq N$. This completes the proof. □

6.3 Oscillation Theorems

In this section, we obtain some new oscillation criteria for the equation (6.1.1) by using the generalized Riccati transformation and Philos type technique.

Theorem 6.3.1. *Assume that condition* (6.2.2) *holds. If there exists a positive nondecreasing real sequence* $\{\rho_n\}$ *such that*

$$\lim_{n\to\infty} \sum_{s=N}^{n-1} \left[Q_s - \frac{(\Delta\rho_s)^2 r_s}{4\rho_{s+1}} \right] = \infty \tag{6.3.1}$$

where

$$Q_n = \rho_n q_n^* \frac{\alpha(n-\sigma)(n+c-\sigma-N)}{2(n+1)}, \tag{6.3.2}$$

and

$$q_n^* = L(1-P)\sum_{s=c}^{d} q_{n,s}, \tag{6.3.3}$$

then every solution of equation (6.1.1) *is either oscillatory or converging to zero.*

Proof. Assume that $\{x_n\}$ is a nonoscillatory solution of equation (6.1.1). Without loss of generality we may assume that $x_n > 0$, $x_{n+s-\tau} > 0$ for $n \geq n_1 \geq n_0 \in \mathbb{N}_0$. Then $\{z_n\}$ satisfies two cases as mentioned in Lemma 6.2.1.

Case(I). Let $\{z_n\}$ satisfies Case (I) of Lemma 6.2.1. From (6.2.1), we have

$$\begin{aligned} x_n &\geq z_n - \sum_{s=a}^{b} p_{n,s} z_{n+s-\tau} \\ &\geq \left[1 - \sum_{s=a}^{b} p_{n,s}\right] z_n \\ &\geq (1-P)z_n. \end{aligned} \tag{6.3.4}$$

Using condition (C_3) in equation (6.1.1), we have

$$\Delta(r_n\Delta^2 z_n) \leq -\sum_{s=c}^{d} q_{n,s} L x_{n+s-\sigma}. \tag{6.3.5}$$

Now using (6.3.4) in (6.3.5), we obtain

$$\begin{aligned}\Delta(r_n\Delta^2 z_n) &\leq -L(1-P)\sum_{s=c}^{d} q_{n,s} z_{n+s-\sigma}\\ &\leq -q_n^* z_{n+c-\sigma}.\end{aligned} \tag{6.3.6}$$

Define

$$w_n = \rho_n \frac{r_n\Delta^2 z_n}{\Delta z_n}, \quad n \geq n_1. \tag{6.3.7}$$

Then $w_n > 0$ for all $n \geq n_1$ and from (6.3.6), we have

$$\begin{aligned}\Delta w_n &\leq -\rho_n \frac{q_n^* z_{n+c-\sigma}}{\Delta z_{n+1}} + \frac{\Delta\rho_n}{\rho_{n+1}} w_{n+1} - w_{n+1}\frac{\Delta^2 z_n}{\Delta z_n}\\ &\leq -\rho_n \frac{q_n^* z_{n+c-\sigma}}{\Delta z_{n+1}} + \frac{\Delta\rho_n}{\rho_{n+1}} w_{n+1} - \frac{w_{n+1}^2}{\rho_{n+1} r_n}.\end{aligned} \tag{6.3.8}$$

By Lemma 6.2.3 with $y_n = \Delta z_n$, we have

$$\frac{1}{\Delta z_{n+1}} \geq \frac{\alpha(n-\sigma)}{n+1}\frac{1}{\Delta z_{n-\sigma}} \quad \text{for all} \quad n \geq N. \tag{6.3.9}$$

Using (6.3.9) in (6.3.8), we obtain

$$\Delta w_n \leq -\rho_n q_n^* \frac{\alpha(n-\sigma)}{n+1}\frac{z_{n+c-\sigma}}{\Delta z_{n-\sigma}} + \frac{\Delta\rho_n}{\rho_{n+1}} w_{n+1} - \frac{w_{n+1}^2}{\rho_{n+1} r_n}.$$

Now applying Lemma 6.2.4 in the last inequality, we obtain

$$\begin{aligned}\Delta w_n &\leq -\rho_n q_n^* \frac{\alpha(n-\sigma)}{n+1}\frac{(n+c-\sigma-N)}{2} + \frac{\Delta\rho_n}{\rho_{n+1}} w_{n+1} - \frac{w_{n+1}^2}{\rho_{n+1} r_n}\\ &\leq -Q_n + A_n w_{n+1} - B_n w_{n+1}^2\end{aligned}$$

or

$$Q_n \leq -\Delta w_n + A_n w_{n+1} - B_n w_{n+1}^2 \tag{6.3.10}$$

where

$$A_n = \frac{\Delta\rho_n}{\rho_{n+1}}, \qquad B_n = \frac{1}{\rho_{n+1} r_n}.$$

Now, using completing the square on the right hand side of (6.3.10), we have

$$Q_n - \frac{A_n^2}{4B_n} \le -\Delta w_n.$$

Summing the last inequality from N to $n-1$, we have

$$\sum_{s=N}^{n-1}\left[Q_s - \frac{(\Delta\rho_s)^2 r_s}{4\rho_{s+1}}\right] \le w_N - w_n \le w_N.$$

Letting $n \to \infty$, we obtain a contradiction to (6.3.1).

Case(II). If $\{z_n\}$ satisfies Case (II) of Lemma 6.2.1, then by condition (6.2.2) we have $\lim_{n\to\infty} x_n = 0$. This completes the proof. □

Before moving to next theorem, we define functions h, $H : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{R}$ such that

(i) $H_{n,n} = 0$ for $n \ge n_0 \ge 0$;

(ii) $H_{n,s} > 0$ for $n > s \ge n_0$;

(iii) $\Delta_2 H_{n,s} = H_{n,s+1} - H_{n,s} \le 0$ for $n > s \ge n_0$ and there exists a positive real sequence $\{\rho_n\}$ such that

$$\Delta_2 H_{n,s} + \frac{\Delta\rho_s}{\rho_{s+1}} H_{n,s} = -h_{n,s}\sqrt{H_{n,s}}$$

for $n > s \ge n_0$.

Theorem 6.3.2. *Assume that* (6.2.2) *holds. If there exists a positive real sequence* $\{\rho_n\}$ *such that*

$$\limsup_{n\to\infty} \frac{1}{H_{n,n_0}} \sum_{s=n_0}^{n-1}\left[H_{n,s}Q_s - \frac{1}{4}\rho_{s+1} r_s h_{n,s}^2\right] = \infty, \tag{6.3.11}$$

then every solution of equation (6.1.1) *is either oscillatory or converging to zero.*

Proof. Assume that $\{x_n\}$ is a nonoscillatory solution of equation (6.1.1). Proceeding as in the proof of Case (I) of Theorem 6.3.1, we have (6.3.10). Now multiplying the inequality (6.3.10) by $H_{n,s}$, and summing the resulting inequality from n_2 to $n-1$ for all $n \geq n_2 \geq n_0$, we have

$$\sum_{s=n_2}^{n-1} H_{n,s}Q_s \leq -\sum_{s=n_2}^{n-1} \Delta w_s H_{n,s} + \sum_{s=n_2}^{n-1} \left[A_s w_{s+1} - B_s w_{s+1}^2\right] H_{n,s}.$$

Using summation by parts on the first term of right hand side, we obtain

$$\begin{aligned}
\sum_{s=n_2}^{n-1} H_{n,s}Q_s &\leq H_{n,n_2}w_{n_2} + \sum_{s=n_2}^{n-1} w_{s+1}\Delta_2 H_{n,s} + \sum_{s=n_2}^{n-1} A_s w_{s+1} H_{n,s} \\
&\quad - \sum_{s=n_2}^{n-1} B_s w_{s+1}^2 H_{n,s} \\
&\leq H_{n,n_2}w_{n_2} + \sum_{s=n_2}^{n-1} \left[\Delta_2 H_{n,s} + \frac{\Delta \rho_s}{\rho_{s+1}} H_{n,s}\right] w_{s+1} \\
&\quad - \sum_{s=n_2}^{n-1} B_s w_{s+1}^2 H_{n,s}. \qquad (6.3.12)
\end{aligned}$$

Using completing the square in the last inequality, we obtain

$$\sum_{s=n_2}^{n-1} \left[H_{n,s}Q_s - \frac{1}{4}\rho_{s+1} r_s h_{n,s}^2\right] \leq H_{n,n_2}w_{n_2}$$

or

$$\frac{1}{H_{n,n_2}} \sum_{s=n_2}^{n-1} \left[H_{n,s}Q_s - \frac{1}{4}\rho_{s+1} r_s h_{n,s}^2\right] \leq w_{n_2}.$$

Taking limit supremum, we obtain a contradiction to (6.3.11).

If $\{z_n\}$ satisfies Case (II) of Lemma 6.2.1, then by condition (6.2.2) we have $\lim_{n\to\infty} x_n = 0$. This completes the proof. □

Corollary 6.3.1. *If* $H_{n,s} = (n-s)^\beta$ *for all* $n \geq s \geq 0$ *and*

$$\lim_{n\to\infty} \sup \frac{1}{(n-n_0)^\beta} \sum_{s=n_0}^{n-1} \left[(n-s)^\beta Q_s - \frac{1}{4}\beta^2 \rho_{s+1} r_s (n-s)^{\beta-2}\right] = \infty, \qquad (6.3.13)$$

for every $\beta \geq 1$*, then every solution of equation* (6.1.1) *is oscillatory.*

Corollary 6.3.2. *If* $H_{n,s} = \left(\log\frac{n+1}{s+1}\right)^{\beta}$ *for all* $n \geq s \geq 0$ *and*

$$\limsup_{n\to\infty} (\log(n+1))^{-\beta} \sum_{s=n_0}^{n-1} \left[\left(\log\frac{n+1}{s+1}\right)^{\beta} Q_n - \frac{\beta^2}{4(s+1)^2}\rho_{s+1} r_s \left(\log\frac{n+1}{s+1}\right)^{\beta-2} \right] = \infty, \qquad (6.3.14)$$

for every $\beta \geq 1$*, then every solution of equation* (6.1.1) *is oscillatory.*

The proofs of Corollaries 6.3.1 and 6.3.2 follow from Theorem 6.3.2 and hence the details are omitted.

Theorem 6.3.3. *Assume that all conditions of Theorem 6.3.2 are satisfied except condition* (6.3.11). *Also let*

$$0 < \inf_{s\geq n_0} \left[\liminf_{n\to\infty} \frac{H_{n,s}}{H_{n,n_0}} \right] \leq \infty \qquad (6.3.15)$$

and

$$\limsup_{n\to\infty} \frac{1}{H_{n,n_0}} \sum_{s=n_0}^{n-1} \rho_{s+1} r_s h_{n,s}^2 < \infty \qquad (6.3.16)$$

hold. If there exists a positive sequence $\{\psi_n\}$ *such that*

$$\limsup_{n\to\infty} \sum_{s=n_0}^{n-1} \frac{\psi_n^2}{\rho_{s+1} r_s} = \infty \qquad (6.3.17)$$

and

$$\limsup_{n\to\infty} \frac{1}{H_{n,N}} \sum_{s=N}^{n-1} \left[H_{n,s} Q_s - \frac{1}{4}\rho_{s+1} r_s h_{n,s}^2 \right] \geq \psi_N, \qquad (6.3.18)$$

then every solution of equation (6.1.1) *is either oscillatory or converging to zero.*

Proof. Proceeding as in the proof of Theorem 6.3.2, we obtain (6.3.12). Using completing the square in (6.3.12) and rearranging terms, we obtain

$$\limsup_{n\to\infty} \frac{1}{H_{n,n_2}} \sum_{s=n_2}^{n-1} \left[H_{n,s} Q_s - \frac{h_{n,s}^2}{4B_s} \right] \leq w_{n_2} - \liminf_{n\to\infty} \frac{1}{H_{n,n_2}} \times \sum_{s=n_2}^{n-1} \left(\sqrt{H_{n,s} B_s w_{s+1}} + \frac{h_{n,s}}{2\sqrt{B_s}} \right)^2$$

for $n \geq n_2$. It follow from (6.3.18) that

$$w_{n_2} \geq \psi_{n_2} + \liminf_{n \to \infty} \frac{1}{H_{n,n_2}} \sum_{s=n_2}^{n-1} \left[\sqrt{H_{n,s}B_s}\, w_{s+1} + \frac{h_{n,s}}{2\sqrt{B_s}} \right]^2,$$

which means that,

$$w_{n_2} \geq \psi_{n_2} \quad \text{for} \quad n \geq N \tag{6.3.19}$$

and

$$\liminf_{n\to\infty} \frac{1}{H_{n,n_2}} \sum_{s=n_2}^{n-1} \left[\sqrt{H_{n,s}B_s}\, w_{s+1} + \frac{h_{n,s}}{2\sqrt{B_s}} \right]^2 < \infty.$$

Therefore

$$\liminf_{n\to\infty} \left[\frac{1}{H_{n,n_2}} \sum_{s=n_2}^{n-1} H_{n,s}B_s w_{s+1}^2 + \frac{1}{H_{n,n_2}} \sum_{s=n_2}^{n-1} h_{n,s} \sqrt{H_{n,s}}\, w_{s+1} + \frac{1}{4}\frac{1}{H_{n,n_2}} \sum_{s=n_2}^{n-1} \frac{h_{n,s}^2}{B_s} \right] < \infty.$$

Then

$$\liminf_{n\to\infty} \left[\frac{1}{H_{n,n_2}} \sum_{s=n_2}^{n-1} H_{n,s}B_s w_{s+1}^2 + \frac{1}{H_{n,n_2}} \sum_{s=n_2}^{n-1} h_{n,s} \sqrt{H_{n,s}}\, w_{s+1} \right] < \infty. \tag{6.3.20}$$

Define the functions

$$U_n = \frac{1}{H_{n,n_2}} \sum_{s=n_2}^{n-1} H_{n,s}B_s w_{s+1}^2$$

and

$$V_n = \frac{1}{H_{n,n_2}} \sum_{s=n_2}^{n-1} h_{n,s} \sqrt{H_{n,s}}\, w_{s+1}.$$

Then, the inequality (6.3.20) becomes

$$\liminf_{n\to\infty} [U_n + V_n] < \infty. \tag{6.3.21}$$

Now, we claim that

$$\sum_{s=n_2}^{\infty} B_s w_{s+1}^2 < \infty. \tag{6.3.22}$$

Suppose to the contrary that

$$\sum_{s=n_2}^{\infty} B_s w_{s+1}^2 = \infty. \tag{6.3.23}$$

By (6.3.15) there exists a positive constant M_1 satisfying

$$\inf_{s\geq n_0}\left(\liminf_{n\to\infty}\frac{H_{n,s}}{H_{n,n_0}}\right) > M_1. \tag{6.3.24}$$

Let M_2 be any arbitrary positive number. Then, it follows from (6.3.23) that there exists a $n_3 > n_2$ such that

$$\sum_{s=n_2}^{n} B_s w_{s+1}^2 \geq \frac{M_2}{M_1} \quad \text{for all} \quad n \geq n_3.$$

Therefore,

$$\begin{aligned}
U_n &= \frac{1}{H_{n,n_2}}\sum_{s=n_2}^{n-1} H_{n,s}\Delta\left(\sum_{t=n_2}^{s-1} B_t w_{t+1}^2\right) + B_{n_2} w_{n_2+1}^2 \\
&= \frac{1}{H_{n,n_2}}\left[-\sum_{s=n_2}^{n-1}\left(\sum_{t=n_2}^{s} B_t w_{t+1}^2\right)\Delta_2 H_{n,s} - H_{n,n_2+1}B_{n_2}w_{n_2+1}^2\right] + B_{n_2}w_{n_2+1}^2 \\
&\geq \frac{1}{H_{n,n_2}}\sum_{s=n_3}^{n-1}\left(\sum_{t=n_2}^{s} B_t w_{t+1}^2\right)(-\Delta_2 H_{n,s}) \\
&\geq \frac{1}{H_{n,n_2}}\frac{M_2}{M_1}\sum_{s=n_3}^{n-1}(-\Delta_2 H_{n,s}) \\
&\geq \frac{M_2}{M_1 H_{n,n_2}} H_{n,n_3} \\
&\geq \frac{M_2}{M_1}\frac{H_{n,n_3}}{H_{n,n_0}} \quad \text{for all} \quad n \geq n_3.
\end{aligned}$$

By (6.3.24), there is a $n_4 \geq n_3$ such that

$$\frac{H_{n,n_3}}{H_{n,n_0}} \geq M_1 \quad \text{for all} \quad n \geq n_4,$$

this implies

$$U_n \geq M_2 \quad \text{for all} \quad n \geq n_4.$$

Since M_2 is arbitrary,

$$\lim_{n\to\infty} U_n = \infty. \tag{6.3.25}$$

Next, consider a sequence $\{n_k\}$ with $\lim_{k\to\infty} n_k = \infty$ satisfying,

$$\lim_{k\to\infty}(U_{n_k} + V_{n_k}) = \liminf_{n\to\infty}(U_n + V_n).$$

It follows from (6.3.21) that there exists a number M such that

$$U_{n_k} + V_{n_k} \le M \quad \text{for} \quad k = 0,\ 1,\ 2,\ \ldots . \tag{6.3.26}$$

It follows from (6.3.25) that

$$\lim_{k\to\infty} U_{n_k} = \infty. \tag{6.3.27}$$

Combining (6.3.26) and (6.3.27),

$$\lim_{k\to\infty} V_{n_k} = -\infty. \tag{6.3.28}$$

From (6.3.26), we have

$$1 + \frac{V_{n_k}}{U_{n_k}} \le \frac{M}{U_{n_k}} < \frac{1}{2}$$

or

$$\frac{V_{n_k}}{U_{n_k}} < -\frac{1}{2}$$

for k large enough. Using the last inequality in (6.3.28), we obtain

$$\lim_{k\to\infty} \frac{V_{n_k}^2}{U_{n_k}} = \infty. \tag{6.3.29}$$

On the other hand, by the Schwarz inequality, we have

$$\begin{aligned} V_{n_k}^2 &= \left(\frac{1}{H_{n_k,n_2}} \sum_{s=n_2}^{n_k-1} h_{n_k,s} \sqrt{H_{n_k,s}}\, w_{s+1} \right)^2 \\ &\le \left(\frac{1}{H_{n_k,n_2}} \sum_{s=n_2}^{n_k-1} H_{n_k,s} B_s w_{s+1}^2 \right) \left(\frac{1}{H_{n_k,n_2}} \sum_{s=n_2}^{n_k-1} \frac{h_{n_k,s}^2}{B_s} \right) \\ &\le U_{n_k} \frac{1}{H_{n_k,n_2}} \sum_{s=n_2}^{n_k-1} \rho_{s+1} r_s h_{n_k,s}^2. \end{aligned}$$

Consequently,

$$\frac{V_{n_k}^2}{U_{n_k}} \le \frac{1}{H_{n_k,n_2}} \sum_{s=n_2}^{n_k-1} \rho_{s+1} r_s h_{n_k,s}^2$$

for all large k. But (6.3.24) guarantees that

$$\liminf_{n\to\infty} \frac{H_{n,s}}{H_{n,n_0}} > M_1.$$

This means that there exists a $n_5 \geq n_4$ such that

$$\frac{H_{n,n_2}}{H_{n,n_0}} \geq M_1 \quad \text{for all} \quad n \geq n_5.$$

Thus,

$$\frac{H_{n_k,n_2}}{H_{n_k,n_0}} \geq M_1$$

for k large enough and therefore

$$\frac{V_{n_k}^2}{U_{n_k}} \leq \frac{1}{M_1 H_{n_k,n_0}} \sum_{s=n_2}^{n_k-1} \rho_{s+1} r_s h_{n_k,s}^2$$

for all large k. It follows from (6.3.29) that

$$\lim_{k\to\infty} \frac{1}{H_{n_k,n_0}} \sum_{s=n_0}^{n_k-1} \rho_{s+1} r_s h_{n_k,s}^2 = \infty. \tag{6.3.30}$$

This gives

$$\lim_{n\to\infty} \sup \frac{1}{H_{n,n_0}} \sum_{s=n_0}^{n-1} \rho_{s+1} r_s h_{n,s}^2 = \infty,$$

which contradicts (6.3.16). Then, (6.3.22) holds. Hence, by (6.3.19),

$$\sum_{s=n_2}^{\infty} \frac{\psi_n^2}{\rho_{s+1} r_s} \leq \sum_{s=n_2}^{\infty} B_s w_{s+1}^2 < \infty,$$

which contradicts (6.3.17).

If $\{z_n\}$ satisfies Case (II) of Lemma 6.2.1, then by condition (6.2.2) we have $\lim_{n\to\infty} x_n = 0$. This completes the proof. □

Theorem 6.3.4. *Assume that all conditions of Theorem 6.3.3 are satisfied except condition* (6.3.16)*. Also let*

$$\lim_{n\to\infty} \inf \frac{1}{H_{n,n_0}} \sum_{s=n_0}^{n-1} H_{n,s} Q_s < \infty \tag{6.3.31}$$

and

$$\lim_{n\to\infty} \inf \frac{1}{H_{n,N}} \sum_{s=N}^{n-1} \left[H_{n,s} Q_s - \frac{1}{4} \rho_{s+1} r_s h_{n,s}^2 \right] \geq \psi_N. \tag{6.3.32}$$

Then every solution of equation (6.1.1) *is either oscillatory or converging to zero.*

Proof. The proof is similar to that of Theorem 6.3.3 and hence the details are omitted. □

Now, let us define

$$H_{n,s} = (n - s)^{\beta}, \quad n \geq s \geq 0,$$

where $\beta \geq 1$ is a constant. Then $H_{n,n} = 0$ for $n \geq 0$ and $H_{n,s} > 0$ for $n > s \geq 0$. Clearly $\Delta_2 H_{n,s} \leq 0$ for $n > s \geq 0$ and

$$h_{n,s} = \left[(n - s)^{\beta} - (n - s - 1)^{\beta}\right](n - s)^{-(\beta/2)} \leq \beta(n - s)^{(\beta-2)/2}$$

for $n > s \geq 0$. We see that (6.3.15) holds,

$$\lim_{n\to\infty} \frac{H_{n,s}}{H_{n,n_0}} = \lim_{n\to\infty} \frac{(n - s)^{\beta}}{n^{\beta}} = 1.$$

Hence, by Theorems 6.3.3 and 6.3.4, we have the following two corollaries.

Corollary 6.3.3. *Let $\beta \geq 1$ be a constant, and suppose that*

$$\limsup_{n\to\infty} \frac{1}{(n - n_0)^{\beta}} \sum_{s=n_0}^{n-1} \beta^2 \rho_{s+1} r_s (n - s)^{\beta-2} < \infty. \tag{6.3.33}$$

If there is a sequence $\{\psi_n\}$ satisfying (6.3.17) *and*

$$\limsup_{n\to\infty} \frac{1}{(n - N)^{\beta}} \sum_{s=N}^{n-1} \left[(n - s)^{\beta} Q_s - \frac{\beta^2}{4} \rho_{s+1} r_s (n - s)^{\beta-2}\right] \geq \psi_N \tag{6.3.34}$$

then every solution of equation (6.1.1) *is oscillatory or converging to zero.*

Proof. The proof follows from Theorem 6.3.3 and hence the details are omitted. □

Corollary 6.3.4. *Let $\beta \geq 1$ be a constant, and suppose that*

$$\liminf_{n\to\infty} \frac{1}{(n - n_0)^{\beta}} \sum_{s=n_0}^{n-1} (n - s)^{\beta} Q_s < \infty. \tag{6.3.35}$$

If there is a sequence $\{\psi_n\}$ satisfying (6.3.17) *and*

$$\liminf_{n\to\infty} \frac{1}{(n - N)^{\beta}} \sum_{s=N}^{n-1} \left[(n - s)^{\beta} Q_s - \frac{\beta^2}{4} \rho_{s+1} r_s (n - s)^{\beta-2}\right] \geq \psi_N \tag{6.3.36}$$

then every solution of equation (6.1.1) *is oscillatory or converging to zero.*

Proof. The proof follows from Theorem 6.3.4 and hence the details are omitted. □

6.4 Examples

In this section, we present some examples to illustrate the main results.

Example 6.4.1. *Consider the difference equation*

$$\Delta\left(n\Delta^2\left(x_n + \sum_{s=1}^{2}\frac{1}{2}x_{n+s-1}\right)\right) + \sum_{s=1}^{2}\left(4n + \frac{4}{3}s\right)x_{n+s-1} = 0. \qquad (6.4.1)$$

Here $r_n = n$, $p_{n,s} = \frac{1}{2}$, $q_{n,s} = 4n + \frac{4}{3}s$, $\sigma = \tau = 1$, $a = 1$, $b = 2$, $c = 1$, $d = 2$, *and* $L = 1$. *It is easy to see that all conditions of Theorem 6.3.1 are satisfied. Hence every solution of equation* (6.4.1) *is oscillatory. In fact* $\{x_n\} = \{(-1)^n\}$ *is one such oscillatory solution of equation* (6.4.1) *since it satisfies the equation* (6.4.1).

Example 6.4.2. *Consider the difference equation*

$$\Delta^3\left(x_n + \sum_{s=1}^{2}\frac{1}{2}x_{n+s-1}\right) + \sum_{s=1}^{2} n(n+1)s\,x_{n+s-1} = 0, \qquad (6.4.2)$$

Here $r_n = 1$, $p_{n,s} = \frac{1}{2}$, $q_{n,s} = n(n+1)s$, $\sigma = \tau = 1$, $a = 1$, $b = 2$, $c = 1$, *and* $d = 2$. *Let* $L = 1$, $\alpha = \frac{1}{2}$, $\beta = 1$, *and* $\rho_n = 1$. *It is easy to see that all conditions of Corollary 6.3.1 are satisfied. Hence every solution of equation* (6.4.2) *is oscillatory.*

We conclude this chapter with the following remark.

Remark 6.4.1. *The results obtained in this chapter generalize and complement to that of in [3, 21, 24, 45, 57, 58]. Further it would be interesting to extend the results of this chapter to the equation* (6.1.1) *when* $\sum_{n=n_0}^{\infty}\frac{1}{r_n} < \infty$.