Chapter 4

Second Order Neutral Delay And Advanced Difference Equations

4. Second Order Neutral Delay And Advanced Difference Equations

4.1 Introduction

In this chapter, we study the oscillatory behavior of solutions of the second order neutral delay difference equation of the form

$$\Delta(r_n\Delta(x_n + p_n x_{n-k})) + q_n x_{n-A} + v_n x_{n-m}^{\alpha} = 0, \quad n \in \mathbb{N}_0 \qquad (4.1.1)$$

and the advanced difference equation of the form

$$\Delta(r_n\Delta(x_n + p_n x_{n+k})) + q_n x_{n+A} + v_n x_{n+m}^{\alpha} = 0, \quad n \in \mathbb{N}_0 \qquad (4.1.2)$$

subject to the following conditions:

(C_1) $\{r_n\}$ is a positive real sequence with $\sum_{n=n_0}^{\infty} \frac{1}{r_n} = \infty$;

(C_2) $\{p_n\}$ is a nonnegative real sequence with $0 \le p_n \le p < \infty$;

(C_3) $\{q_n\}$ and $\{v_n\}$ are positive real sequences;

(C_4) k, A and m are positive integers and α is a ratio of odd positive integers.

Let $\theta = \max\{k, A, m\}$. By a solution of equation (4.1.1) ((4.1.2)) we mean a nontrivial real sequence $\{x_n\}$ defined for all $n \ge n_0 - \theta$ and satisfying equation (4.1.1)((4.1.2)) for all $n \ge n_0$. We assume that such solutions exist for the equations (4.1.1) and (4.1.2).

Most of the results established in the literature for neutral type difference equations involve either delay or advanced type arguments, see for example [1, 2, 7, 8, 11, 12, 16, 17, 18, 23, 29, 30, 31, 32, 34, 36, 37, 39, 41, 43, 44, 46, 50, 52, 53, 55, 67, 72, 76, 77, 78, 80, 81, 84, 86, 87, 88, 95, 96, 97]. In [2, 7, 8, 31, 87], the authors studied the oscillatory behavior of solutions of equation (4.1.1) when $v_n \equiv 0$ and in [32, 43, 44, 78, 88, 96], the authors studied the oscillation of the equation (4.1.1) when $q_n \equiv 0$. Therefore in this chapter, we discuss the oscillatory behavior of equation (4.1.1) which unify the results obtained for linear and nonlinear cases. Further the results obtained for the equation (4.1.2) seems to be new even for the linear or nonlinear cases.

In Section 4.2, we discuss the oscillatory behavior of all solutions of equations (4.1.1) and (4.1.2), and in Section 4.3, we present some examples to illustrate the main results.

4.2 Oscillation Theorems

In this section, we obtain some sufficient conditions for the oscillation of all solution of equation (4.1.1) and (4.1.2). We use the following notation throughout this chapter without further mention:

$$z_n = x_n + p_n x_{n-k}, \tag{4.2.1}$$

$$Q_n = \min\{q_n, q_{n-k}\}, \quad V_n = \min\{v_n, v_{n-k}\} \text{ for all } n \in \mathbb{N}_0, \tag{4.2.2}$$

$$Q_n^* = Q_n \sum_{s=n_1}^{n-\ell-1} \frac{1}{r_s}, \tag{4.2.3}$$

$$V_n^* = V_n \left(\sum_{s=n_1}^{n-m-1} \frac{1}{r_s}\right)^{1-\alpha}, \tag{4.2.4}$$

$$U_n = 2^{1-\alpha} V_n \left(\sum_{s=n_1}^{n-m-1} \frac{1}{r_s}\right)^{1-\alpha}, \tag{4.2.5}$$

$$u_n = x_n + p_n x_{n+k}, \tag{4.2.6}$$

$$R_n = \min\{q_n, q_{n+k}\}, \quad S_n = \min\{v_n, v_{n+k}\} \text{ for all } n \in \mathbb{N}_0, \tag{4.2.7}$$

$$R_n^* = \frac{1}{r_n} \sum_{s=n}^{\infty} R_s, \tag{4.2.8}$$

$$S_n^* = \frac{1}{r_n}\sum_{s=n}^{\infty} S_s, \tag{4.2.9}$$

and

$$T_n = 2^{1-\alpha}\frac{1}{r_n}\sum_{s=n}^{\infty} S_s. \tag{4.2.10}$$

Lemma 4.2.1. *If* $0 \le p \le 1$ *and* $0 < \alpha \le 1$, *then* $p^\alpha \ge p$.

Lemma 4.2.2. *If* $p \ge 1$ *and* $\alpha \ge 1$, *then* $p^\alpha \ge p$.

The proof of the Lemmas 4.2.1 and 4.2.2 are elementary and hence the details are omitted.

Lemma 4.2.3. *Let* $\{q_n\}$ *be a nonnegative sequence of real numbers and* A *be a positive integer. Suppose that*

$$\liminf_{n\to\infty}\sum_{s=n}^{n+A-1} q_s > \left(\frac{A}{A+1}\right)^{A+1} \tag{4.2.11}$$

holds, then the difference inequality

$$\Delta x_n - q_n x_{n+A} \ge 0, \quad n \in \mathrm{N}_0, \tag{4.2.12}$$

cannot have eventually positive solutions.

Proof. The proof can be found in [26, 40, 51]. □

Lemma 4.2.4. *Let* $\{q_n\}$ *be a nonnegative sequence of real numbers,* α *be a ratio of odd positive integers and* $A \in \{2, 3, \ldots\}$ *be such that* $\sum_{s=n-A+1}^{n-1} q_s > 0$ *for all large* n. *Then the difference inequality* (4.2.12) *has an eventually positive solution if and only if the difference equation*

$$\Delta x_n - q_n x_{n+A}^{\alpha} = 0, \quad n \in \mathrm{N}_0, \tag{4.2.13}$$

has an eventually positive solution.

Proof. The proof can be found in [26]. □

First we study the oscillation of the equation (4.1.1).

Lemma 4.2.5. *If $\{x_n\}$ is a positive solution of equation* (4.1.1)*, then the corresponding $\{z_n\}$ satisfies*

$$z_n > 0, \quad r_n \Delta z_n > 0, \quad \Delta(r_n \Delta z_n) < 0 \tag{4.2.14}$$

eventually.

Proof. Assume that $\{x_n\}$ is a positive solution of equation (4.1.1). Then $z_n > 0$ for all $n \geq n_1 \geq n_0$. From the equation (4.1.1), we have

$$\Delta(r_n \Delta z_n) = -q_n x_{n-A} - v_n x_{n-m}^{\alpha} < 0.$$

Consequently, $r_n \Delta z_n$ is nonincreasing and therefore either $r_n \Delta z_n > 0$ or $r_n \Delta z_n \leq 0$ eventually. If $r_n \Delta z_n \leq 0$, then we have

$$r_n \Delta z_n \leq r_{n_1} \Delta z_{n_1} < 0 \quad \text{for} \quad n \geq n_1.$$

Dividing the last inequality by r_n and then summing the resulting inequality from n_1 to $n-1$, we obtain

$$z_n < z_{n_1} + r_{n_1} \Delta z_{n_1} \sum_{s=n_1}^{n-1} \frac{1}{r_s}.$$

Letting $n \to \infty$, we see that $z_n < 0$ for $n \geq n_1$, which is a contradiction for the positivity of $\{z_n\}$. This completes the proof. □

Theorem 4.2.1. *Let $0 \leq p \leq 1$ and $0 < \alpha \leq 1$. If the first order neutral difference inequality*

$$\Delta w_n + \frac{1}{1+p^{\alpha}} Q_n^{*} w_{n-A+k} + \frac{1}{(1+p^{\alpha})^{\alpha}} V_n^{*} w_{n-m+k}^{\alpha} \leq 0 \tag{4.2.15}$$

has no positive solution, then every solution of equation (4.1.1) *is oscillatory.*

Proof. Let $\{x_n\}$ be a nonoscillatory solution of equation (4.1.1). Without loss of generality we may assume that $x_n > 0$ and $x_{n-k} > 0$ for all $n \geq n_1 \geq n_0 - \theta$. Then $z_n > 0$,

$$z_{n-A} = x_{n-A} + p_{n-A}x_{n-k-A} \leq x_{n-A} + px_{n-k-A}, \tag{4.2.16}$$

and

$$z_{n-m} = x_{n-m} + p_{n-m}x_{n-k-m} \leq x_{n-m} + px_{n-k-m}. \tag{4.2.17}$$

From the equation (4.1.1), we have

$$\Delta(r_n\Delta z_n) + q_n x_{n-A} + v_n x^{\alpha}_{n-m} = 0, \tag{4.2.18}$$

and

$$p^{\alpha}\Delta(r_{n-k}\Delta z_{n-k}) + p^{\alpha}q_{n-k}x_{n-k-A} + p^{\alpha}v_{n-k}x^{\alpha}_{n-k-m} = 0. \tag{4.2.19}$$

Combining (4.2.18) and (4.2.19), then using (4.2.2) we get

$$\Delta(r_n\Delta z_n + p^{\alpha}r_{n-k}\Delta z_{n-k}) + Q_n(x_{n-A} + p^{\alpha}x_{n-k-A}) + V_n(x^{\alpha}_{n-m} + p^{\alpha}x^{\alpha}_{n-k-m}) \leq 0. \tag{4.2.20}$$

Applying Lemma 4.2.1 in the second term of the inequality (4.2.20) and using (4.2.16), we have

$$\Delta(r_n\Delta z_n + p^{\alpha}r_{n-k}\Delta z_{n-k}) + Q_n z_{n-A} + V_n(x^{\alpha}_{n-m} + p^{\alpha}x^{\alpha}_{n-k-m}) \leq 0. \tag{4.2.21}$$

Using Lemma 2.2.1 and (4.2.17) in (4.2.21), we have

$$\Delta(r_n\Delta z_n + p^{\alpha}r_{n-k}\Delta z_{n-k}) + Q_n z_{n-A} + V_n z^{\alpha}_{n-m} \leq 0. \tag{4.2.22}$$

Since $y_n = r_n\Delta z_n > 0$ is decreasing, we have

$$z_n \geq y_n \sum_{s=n_1}^{n-1} \frac{1}{r_s}. \tag{4.2.23}$$

Substituting (4.2.23) in (4.2.22), we get

$$\Delta(y_n + p^\alpha y_{n-k}) + Q_n y_{n-\ell} \left(\sum_{s=n_1}^{n-\ell-1} \frac{1}{r_s}\right) + V_n y^\alpha_{n-m} \left(\sum_{s=n_1}^{n-m-1} \frac{1}{r_s}\right)^\alpha \le 0.$$

By using (4.2.3) and (4.2.4), we have

$$\Delta(y_n + p^\alpha y_{n-k}) + Q^*_n y_{n-\ell} + V^*_n y^\alpha_{n-m} \le 0. \tag{4.2.24}$$

Define a function w_n by

$$w_n = y_n + p^\alpha y_{n-k}.$$

Then $w_n > 0$. By using monotonicity of $\{y_n\}$, we have

$$w_n \le (1 + p^\alpha) y_{n-k}. \tag{4.2.25}$$

Combining (4.2.25) and (4.2.24), we see that $\{w_n\}$ is a positive solution of the following inequality

$$\Delta w_n + \frac{1}{1+p^\alpha} Q^*_n w_{n-\ell+k} + \frac{1}{(1+p^\alpha)^\alpha} V^*_n w^\alpha_{n-m+k} \le 0,$$

which is a contradiction to (4.2.15) and the proof is now complete. □

Theorem 4.2.2. *Let $p \ge 1$ and $\alpha \ge 1$. If the first order neutral difference inequality*

$$\Delta w_n + \frac{1}{1+p^\alpha} Q^*_n w_{n-\ell+k} + \frac{1}{(1+p^\alpha)^\alpha} U_n w^\alpha_{n-m+k} \le 0 \tag{4.2.26}$$

has no positive solution, then every solution of equation (4.1.1) is oscillatory.

Proof. The proof is similar to that of Theorem 4.2.1 by using Lemmas 4.2.2 and 2.2.2 instead of Lemmas 4.2.1 and 2.2.1 and hence the details are omitted. □

Corollary 4.2.1. *Assume that $\ell > k$ and $\ell > m$. If $\alpha = 1$, and*

$$\liminf_{n\to\infty} \sum_{s=n-m+k}^{n-1} (Q^*_s + V^*_s) > (1+p) \left(\frac{m-k}{m-k+1}\right)^{m-k+1} \tag{4.2.27}$$

then every solution of equation (4.1.1) is oscillatory.

Proof. Assume that $\{w_n\}$ is a positive solution of (4.2.15). Then $\{w_n\}$ is decreasing and if $A > m$, then

$$w_{n-A} \geq w_{n-m}.$$

Using the last inequality in (4.2.15), we get that $\{w_n\}$ is a positive solution of the difference inequality

$$\Delta w_n + \frac{1}{1+p}(Q_n^* + V_n^*)w_{n-m+k} \leq 0. \tag{4.2.28}$$

Inview of condition (4.2.27), Lemmas 2.2.3 and 2.2.5 implies that the inequality (4.2.28) has no positive solution, which is a contradiction. Therefore (4.2.15) has no positive solution and now the result follows from Theorem 4.2.1. □

Corollary 4.2.2. *Assume* $A > k$ *and* $A < m$. *If* $\alpha = 1$, *and*

$$\liminf_{n\to\infty} \sum_{s=n-A+k}^{n-1} (Q_s^* + V_s^*) > (1+p) \left(\frac{A-k}{A-k+1}\right)^{A-k+1} \tag{4.2.29}$$

then every solution of equation (4.1.1) *is oscillatory.*

Proof. The proof is similar to that of Corollary 4.2.1 and hence the details are omitted. □

Corollary 4.2.3. *Assume* $q_n \equiv 0$, $m > k$ *and* $0 < \alpha < 1$ *in equation* (4.1.1). *If*

$$\sum_{s=n_0}^{\infty} V_s^* = \infty \tag{4.2.30}$$

then every solution of equation (4.1.1) *is oscillatory.*

Proof. The proof follows by applying Lemmas 2.2.3 and 2.2.4 in Theorem 4.2.1 and hence the details are omitted. □

Corollary 4.2.4. *Assume* $q_n \equiv 0$, $m > k$ *and* $\alpha > 1$ *in equation* (4.1.1). *If there exists a* $\lambda > 0$ *such that* $\lambda > \frac{1}{m-k} \log \alpha$ *and*

$$\liminf_{n\to\infty} \left[U_n \exp(-e^{\lambda n})\right] > 0 \tag{4.2.31}$$

then every solution of equation (4.1.1) *is oscillatory.*

Proof. The proof follows by applying Lemmas 2.2.3 and 2.2.6 in Theorem 4.2.2 and hence the details are omitted. □

Corollary 4.2.5. *Assume* $V_n \equiv 0$ *and* $A > k$. *If*

$$\liminf_{n\to\infty} \sum_{s=n-A+k}^{n-1} Q_s^* > (1+p)\left(\frac{A-k}{A-k+1}\right)^{A-k+1} \tag{4.2.32}$$

then every solution of equation (4.1.1) *is oscillatory.*

Proof. The proof follows by applying Lemmas 2.2.3 and 2.2.5 in Theorem 4.2.1 and hence the details are omitted. □

Now we study the oscillation solution of the equation (4.1.2).

Lemma 4.2.6. *If* $\{x_n\}$ *is a positive solution of equation* (4.1.2)*, then the corresponding* $\{u_n\}$ *satisfies*

$$u_n > 0, \quad r_n\Delta u_n > 0, \quad \Delta(r_n\Delta u_n) < 0 \tag{4.2.33}$$

eventually.

Proof. The proof is similar to that of Lemma 4.2.5 and hence the details are omitted. □

Theorem 4.2.3. *Let* $0 \le p \le 1$ *and* $0 < \alpha \le 1$. *If the first order difference inequality*

$$\Delta u_n - \frac{1}{1+p^\alpha}\left(R_n^* u_{n+A} + S_n^* u_{n+m}^\alpha\right) \ge 0, \tag{4.2.34}$$

has no positive solution, then every solution of equation (4.1.2) *is oscillatory.*

Proof. Let $\{x_n\}$ be a nonoscillatory solution of equation (4.1.2). Without loss of generality we may assume that $x_n > 0$ and $x_{n+k} > 0$ for all $n \ge n_1 \ge n_0 - \theta$. Then $u_n > 0$,

$$u_{n+A} = x_{n+A} + p_{n+A}x_{n+A+k} \le x_{n+A} + px_{n+A+k}, \tag{4.2.35}$$

and

$$u_{n+m} = x_{n+m} + p_{n+m}x_{n+m+k} \leq x_{n+m} + px_{n+m+k}. \tag{4.2.36}$$

From the equation (4.1.2), we have

$$\Delta(r_n\Delta u_n) + q_n x_{n+A} + v_n x^{\alpha}_{n+m} = 0, \tag{4.2.37}$$

and

$$p^{\alpha}\Delta(r_{n+k}\Delta u_{n+k}) + p^{\alpha}q_{n+k}x_{n+A+k} + p^{\alpha}v_{n+k}x^{\alpha}_{n+m+k} = 0. \tag{4.2.38}$$

Combining (4.2.37), (4.2.38) and (4.2.7) we get

$$\Delta(r_n\Delta u_n + p^{\alpha}r_{n+k}\Delta u_{n+k}) + R_n(x_{n+A} + p^{\alpha}x_{n+A+k}) + S_n(x^{\alpha}_{n+m} + p^{\alpha}x^{\alpha}_{n+m+k}) \leq 0.$$

Applying Lemma 4.2.1 in the second term of the last inequality, we get

$$\Delta(r_n\Delta u_n + p^{\alpha}r_{n+k}\Delta u_{n+k}) + R_n(x_{n+A} + px_{n+A+k}) + S_n(x^{\alpha}_{n+m} + p^{\alpha}x^{\alpha}_{n+m+k}) \leq 0.$$

Using Lemma 2.2.1, (4.2.35) and (4.2.36) in the last inequality, we have

$$\Delta(r_n\Delta u_n + p^{\alpha}r_{n+k}\Delta u_{n+k}) + R_n u_{n+A} + S_n u^{\alpha}_{n+m} \leq 0. \tag{4.2.39}$$

Summing the inequality (4.2.39) from n to ∞, we have

$$-(r_n\Delta u_n + p^{\alpha}r_{n+k}\Delta u_{n+k}) + \sum_{s=n}^{\infty} R_s u_{s+A} + \sum_{s=n}^{\infty} S_s u^{\alpha}_{s+m} \leq 0$$

or

$$r_n\Delta u_n + p^{\alpha}r_{n+k}\Delta u_{n+k} \geq \sum_{s=n}^{\infty} R_s u_{s+A} + \sum_{s=n}^{\infty} S_s u^{\alpha}_{s+m}.$$

Using the monotonicity of $\{u_n\}$ in the last inequality, we have

$$r_n\Delta u_n(1 + p^{\alpha}) \geq \sum_{s=n}^{\infty} R_s u_{s+A} + \sum_{s=n}^{\infty} S_s u^{\alpha}_{s+m}$$

or

$$\Delta u_n \geq \frac{1}{1 + p^{\alpha}}\left[\frac{1}{r_n}\sum_{s=n}^{\infty} R_s u_{s+A} + \frac{1}{r_n}\sum_{s=n}^{\infty} S_s u^{\alpha}_{s+m}\right].$$

Using (4.2.8) and (4.2.9) the last inequality becomes

$$\Delta u_n \geq \frac{1}{1+p^\alpha}\left[R_n^* u_{n+A} + S_n^* u_{n+m}^\alpha\right],$$

or

$$\Delta u_n - \frac{1}{1+p^\alpha}\left[R_n^* u_{n+A} + S_n^* u_{n+m}^\alpha\right] \geq 0.$$

Thus $\{u_n\}$ is a positive solution of the inequality (4.2.34), which is a contradiction to (4.2.34). The proof is now complete. □

Theorem 4.2.4. *Let $p \geq 1$ and $\alpha \geq 1$. If the first order difference inequality*

$$\Delta u_n - \frac{1}{1+p^\alpha}\left[R_n^* u_{n+A} + T_n u_{n+m}^\alpha\right] \geq 0 \tag{4.2.40}$$

has no positive solution, then every solution of equation (4.1.2) *is oscillatory.*

Proof. The proof is similar to that of Theorem 4.2.3 by using Lemmas 4.2.2 and 2.2.2 instead of Lemmas 4.2.1 and 2.2.1 and hence the details are omitted. □

Corollary 4.2.6. *If $A < m$, $\alpha = 1$ and*

$$\liminf_{n\to\infty} \sum_{s=n}^{n+m-1} (R_s^* + S_s^*) > (1+p)\left(\frac{m}{m+1}\right)^{m+1} \tag{4.2.41}$$

then every solution of equation (4.1.2) *is oscillatory.*

Proof. The proof follows by applying Lemma 4.2.3 in Theorem 4.2.3 and hence the details are omitted. □

Corollary 4.2.7. *If $A > m$, $\alpha = 1$ and*

$$\liminf_{n\to\infty} \sum_{s=n}^{n+A-1} (R_s^* + S_s^*) > (1+p)\left(\frac{A}{A+1}\right)^{A+1} \tag{4.2.42}$$

then every solution of equation (4.1.2) *is oscillatory.*

Proof. The proof follows by applying Lemma 4.2.3 in Theorem 4.2.3 and hence the details are omitted. □

Corollary 4.2.8. *If* $A < m$, $\alpha = 1$ *and*

$$\liminf_{n\to\infty} \sum_{s=n}^{n+m-1} (R_s^* + T_s) > (1+p)\left(\frac{m}{m+1}\right)^{m+1} \tag{4.2.43}$$

then every solution of equation (4.1.2) *is oscillatory.*

Proof. The proof follows by applying Lemma 4.2.3 in Theorem 4.2.4 and hence the details are omitted. □

Corollary 4.2.9. *If* $A > m$, $\alpha = 1$ *and*

$$\liminf_{n\to\infty} \sum_{s=n}^{n+A-1} (R_s^* + T_s) > (1+p)\left(\frac{A}{A+1}\right)^{A+1}, \tag{4.2.44}$$

then every solution of equation (4.1.2) *is oscillatory.*

Proof. The proof follows by applying Lemma 4.2.3 in Theorem 4.2.4 and hence the details are omitted. □

4.3 Examples

In this section, we present some examples to illustrate the main results.

Example 4.3.1. *Consider the neutral difference equation*

$$\Delta\left(\frac{1}{n}\Delta(x_n + 2x_{n-1})\right) + \frac{2}{n}x_{n-4} + \frac{2}{n+1}x_{n-2} = 0, \quad n \geq 1. \tag{4.3.1}$$

Here $r_n = \frac{1}{n}$, $p_n = 2$, $q_n = \frac{2}{n}$, $v_n = \frac{2}{n+1}$, $k = 1$, $A = 4$, $m = 2$, *and* $\alpha = 1$. *It is easy to see that all conditions of Corollary 4.2.1 are satisfied. Hence every solution of equation* (4.3.1) *is oscillatory. In fact* $\{x_n\} = \{(-1)^n\}$ *is one such solution of equation* (4.3.1) *since it satisfies the equation* (4.3.1).

Example 4.3.2. *Consider the neutral difference equation*

$$\Delta\left(\frac{1}{n}\Delta(x_n + 2x_{n-1})\right) + \frac{2}{n+1}x_{n-2} + \frac{2}{n}x_{n-4} = 0, \quad n \geq 1. \tag{4.3.2}$$

Here $r_n = \frac{1}{n}$, $p_n = 2$, $q_n = \frac{2}{n+1}$, $v_n = \frac{2}{n}$, $k = 1$, $A = 2$, $m = 4$, *and* $\alpha = 1$. *It is easy to see that all conditions of Corollary 4.2.2 are satisfied. Hence every solution of equation* (4.3.2) *is oscillatory. In fact* $\{x_n\} = \{(-1)^n\}$ *is one such solution of equation* (4.3.2) *since it satisfies the equation* (4.3.2).

Example 4.3.3. *Consider the neutral difference equation*

$$\Delta\left(\frac{1}{n}\Delta(x_n + 2x_{n-2})\right) + \frac{1}{n^{7/5}}x_{n-3}^{1/5} = 0, \quad n \geq 1. \tag{4.3.3}$$

Here $r_n = \frac{1}{n}$, $p_n = 2$, $q_n = 0$, $v_n = \frac{1}{n^{7/5}}$, $k = 2$, $m = 3$, *and* $\alpha = \frac{1}{5}$. *It is easy to see that all conditions of Corollary 4.2.3 are satisfied. Hence every solution of equation* (4.3.3) *is oscillatory.*

Example 4.3.4. *Consider the neutral difference equation*

$$\Delta\left(\frac{1}{n}\Delta(x_n + 3x_{n-2})\right) + \frac{e^{e^n}}{n^{10}}x_{n-4}^5 = 0, \quad n \geq 1. \tag{4.3.4}$$

Here $r_n = \frac{1}{n}$, $p_n = 3$, $q_n = 0$, $v_n = \frac{e^{e^n}}{n^{10}}$, $k = 2$, $m = 4$, *and* $\alpha = 5$. *Choose* $\lambda = 1$. *Then it is easy to see that all conditions of Corollary 4.2.4 are satisfied. Hence every solution of equation* (4.3.4) *is oscillatory.*

Example 4.3.5. *Consider the neutral difference equation*

$$\Delta\left(\frac{1}{n}\Delta(x_n + 2x_{n+1})\right) + \frac{1}{n(n+1)}x_{n+2} + \frac{1}{n(n+1)}x_{n+3} = 0, \quad n \geq 1. \tag{4.3.5}$$

Here $r_n = \frac{1}{n}$, $p_n = 2$, $q_n = \frac{1}{n(n+1)}$, $v_n = \frac{1}{n(n+1)}$, $k = 1$, $A = 2$, $m = 3$, *and* $\alpha = 1$. *It is easy to see that all conditions of Corollary 4.2.6 are satisfied. Hence every solution of equation* (4.3.5) *is oscillatory.*

Example 4.3.6. *Consider the neutral difference equation*

$$\Delta\left(\frac{1}{n}\Delta(x_n + 2x_{n+1})\right) + \frac{1}{n(n+1)}x_{n+3} + \frac{1}{n(n+1)}x_{n+2} = 0, \quad n \geq 1. \tag{4.3.6}$$

Here $r_n = \frac{1}{n}$, $p_n = 2$, $q_n = \frac{1}{n(n+1)}$, $v_n = \frac{1}{n(n+1)}$, $k = 1$, $A = 3$, $m = 2$, *and* $\alpha = 1$. *It is easy to see that all conditions of Corollary 4.2.7 are satisfied. Hence every solution of equation* (4.3.6) *is oscillatory.*

We conclude this chapter with the following remark.

Remark 4.3.1. *The results presented in this chapter extend and generalize some of the known results in [2, 7, 8, 31, 32, 43, 44, 78, 87, 88, 96]. Further it would be interesting to obtain oscillation results for the equations* (4.1.1) *and* (4.1.2) *when* $\sum_{n=n_0}^{\infty} \frac{1}{r_n} < \infty$.