Chapter 1

Introduction
1.1 General description and historical background

The study of motion of matter which deals with the branch of science is known as Dynamics. It may be divided into two parts viz. rigid bodies and non-rigid bodies. Fluid mechanics is the branch of science that studies the behaviour of fluids (liquids and gases) when the external forces are acting on it. Fluid mechanics has a large number of applications which includes Civil Engineering, Mechanical Engineering, Chemical Engineering, Astrophysics, Geophysics etc. It subdivides into fluid statics (fluids at rest) and fluid dynamics (study of the effect of forces on the fluid motion).

Fluid dynamics is a branch of continuum mechanics. The defining property of fluids, embracing both liquids and gases, lies in the ease with which they may be deformed. Compressibility is the most significant difference between the mechanical properties of liquids and gases which lies in their bulk elasticity. Gases can be compressed much more readily than liquids, and as a result any motion involving appreciable variations in pressure will be accompanied by much larger changes in specific volume in the case of a gas than in the case of liquid.

Batchelor (1993) contributed his idea on continuum hypothesis. Vacuous regions separate the molecules of a gas with linear dimensions which are larger than the molecules themselves. Again, in a liquid in which molecules are nearly as closely packed as the strong short range repulsive forces will allow, the mass of the material is concentrated in the nuclei of the atoms composing a molecule and is very far from being smeared uniformly over the volume occupied by the liquid. When the fluid is viewed on a small scale as to reveal the individual molecules, properties like composition or velocity of a fluid have a violently non-uniform distribution. Fluid dynamics deals with the behaviour of matter on a macroscopic scale which is large enough when compared to the distance between molecules. Moreover, the molecular structure
of a fluid does not often need to be taken into account explicitly. It can be assumed throughout our study that the macroscopic behaviour of fluids is the same as if they were perfectly continuous in structure and physical quantities such as the mass and momentum associated with the matter contained within a given small volume will be regarded as being spread uniformly over that volume instead of, as in strict reality, being concentrated in a small fraction of it. This assumption is known as Continuum hypothesis. Thus, a fluid element can be subdivided indefinitely on the basis of this hypothesis. As a result, a fluid particle can be defined as the smallest mass of fluid material that has sufficiently large number of molecules to allow statistically of a continuum interpretation. In other words, it is defined as the fluid contained within the physically infinitesimal volume.

The scientific principle of fluid dynamics which is a part of fluid mechanics has become rapidly applicable in day-to-day human life. As the earth is mostly covered with air and water, almost all the activities on this earth are governed by the science of fluid dynamics. It also gave rise to new disciplines of study such as hypersonic flow and magneto-fluid dynamics. The Greek mathematician, Archimedes originates the laws relating to dynamics probably for the first time and later it took the shape of ‘Laws of Buoyancy’. The mathematical form of the conservation of mass as the equation of continuity was determined by Leonardo da Vinci (1452-1519). A Frenchman named Edme Mariotte (1620-1684) built a wind tunnel for the first time which is a good source of application of fluids. The laws of Viscosity which further divides fluid dynamics into Newtonian and Non-Newtonian fluids was discovered by Sir Issac Newton (25th December 1642 - 20th March 1727). The classification of fluid dynamics helps to make the distinction between the real and perfect fluids. The urge to work on the function of fluids was further carried by Mathematician like Bernoulli, Euler, D’Alembert, Joseph-Louis
1.1 General description and historical background

Lagrange and Laplace on finding solutions to many frictionless flow problems.

Daniel Bernoulli (1700-1782) was the first person to investigate a fluid flow mathematically. But the development made by Swiss Mathematician Euler (1707-1783) in the field of “Ideal fluid theory or Non-viscous theory” was remarkable. And this led to the advent of nonlinear field theory. During the first half of the 19th century, around 1822, Jean Augustin Louis Cauchy (1789-1857) introduced the concept of stress as there was a huge demand on industrial and engineering sectors of the viscous effects in flow. Later, after a year, an equation of motion of a real (viscous) fluid was presented by a French engineer and physicist Navier (1785-1836) contributed by Poisson, Saint-Venant and Stokes. However, the equation was finally termed as Navier-Stokes equation of motion as a remarkable contribution which made by Stokes.

Before 1900, the theory of ideal fluid and the theory of viscous flow were considered distinctly. There was an association between the ideal fluid theory and some great papers of Euler (1707-1783) on the fundamental equations, Helmholtz (1821-1894) on vortex motion and Kelvin (1824-1907) on the circulation theorem. Besides, some important features of water and sound waves were also examined by many researchers. Thus, viscid flow theory was undoubtedly worthwhile. Along with ideal fluid theory G.G.Stokes (1819-1903) put forward the viscid flow theory adding a new dimension to the theories of flow. Stokes not only alleviate our hard labour by formulating equations related to motion but also found elementary solutions associated to viscous flow. Reynolds number which is a dimensionless number was invented by Osborne Reynolds (1842-1912). In 1851, Stokes discussed about the low Reynolds number flow in an important paper. H.S.Hele-Shaw (1854-1941) experimented with irrational stream line patterns. Inspite of such efforts, a major problem of accounting for the motion of fluid of small viscosity past a solid body was left unnoticed. In 1904, Ludwing Prandtl (1875-1953) presented
a paper on the motion of fluids of very small viscosity in the 3rd International Congress of Mathematicians in Heidelberg which was an influential contribution in fluid dynamics. He also recommended the concept of boundary layer (precisely the velocity boundary layer). He helped the fluidicist to find solutions to simplify the Navier Stokes equation.

James Clerk Maxwell, a Scottish physicist stated a modern definition of heat in his classic, ‘Theory of Heat’. Following his theory, the physicists identified heat as a form of thermal energy which flows spontaneously from systems of higher temperature to lower temperature till a thermal equilibrium is achieved. The two systems of different temperatures exchange thermal energy when they come into thermal contact, i.e. the hotter body gives more thermal energy to the colder body than it takes from it, until they share of thermal equilibrium at that point. The motion of heat transfer plays an important role in industrial and environmental problems. Industrial problems comprise production and conversion of energy and electrical power generation, maximization of the rates of heat transfer for maintaining the integrity of materials on high temperature environments, working and designing of population systems and cryogenics. Environmental problems comprise the climatology of local and global areas. Heat transfer principles are applicable in the studies of the thermal effects on buildings and other structures. It is also used to study human physiology. Researchers like Grober et al. (1961), Bayley et al. (1972), Ozisik (1977), Gebhart (1971), Thomas (1980), Wolf (1983), Holman (1981), Incropera and Dewitt (1981), Lienhard (1981) had accepted the motion of heat transfer as an important topic for their research work.

Among several situations especially in industries and chemical engineering processes, a system that contains two or more components vary in their concentration from point to point. This system exists a natural tendency for mass transfer which diminishes the concentration
differences within the system. Mass transfer is the transportation of one constituent from a region of higher concentration to that of a lower concentration. Heat and mass transfer are kinetic processes that can be studied separately or jointly. In case of diffusion and connection both the processes are shaped by similar mathematical equations. Thus it is more effective to consider them jointly. The concept of heat transfer, mass transfer and momentum transfer are jointly regarded as a new discipline after the book on ‘Transport Phenomena’ by Bird et al. (1966). A German physiologist Fick (1855) proposed the basic kinetic law for mass diffusion. The Fick’s law was established by imitating Fourier law (1822). Alike heat transfer, mass transfer also has a wide area of application such as transpiration cooling of jet engines, rocket motors and in ablative of space vehicles during reentry to the atmosphere and in many day to day areas like the working system in home humidifier and dispersion of smoke released from a chimney into the atmosphere etc. Mass transfer is also applicable in biological activities like oxygenation of blood, food and drug assimilation, respiration mechanism etc.

As the earth is 75% submerged in water and 100% enveloped in air, the scope of fluid mechanics is very vast and it influences almost every human endeavour. The impact of naturally occurring fluid flow not only encompass the science of meteorology, physical oceanography and hydrology but also medical studies of breathing and blood circulation. The development of fluid dynamics has provided enormous application of it in several sectors. Flood control, irrigation, transportation problems, combustion problems etc are the evidence of implication of fluid dynamics. With the advent of fluid flow measuring techniques and computational methods, fluid dynamics has led to a new avenue in this direction.
1.2 Basic terminologies

(a) Fluid Pressure

When a fluid is contained in a vessel, it exerts a force at each point of the inner side of the vessel, which is known as pressure. The pressure $p$ in a fluid region is defined as

$$p = \lim_{\delta S \to \infty} \frac{\delta F}{\delta S}$$

where $\delta S$ is an elementary area and $\delta F$ is the normal force due to fluid on $\delta S$.

(b) Density

The density of a fluid is defined as the mass per unit volume. Mathematically, it may be expressed as

$$\rho = \lim_{\delta v \to \infty} \frac{\delta m}{\delta v}$$

where $\delta v$ is the volume element and $\delta m$ is the mass of the fluid within $\delta v$.

(c) Viscosity

Viscosity is a measure of fluid’s resistance to flow. It controls its rate of flow of a fluid. The resistance property possesses by a fluid is known as viscous or real fluid. All fluids in nature including water belong to viscous fluids. Viscosity determines the fluid strain rate that is generated by a given applied shear stress. Therefore it requires more force to move than less viscous materials in highly viscous fluid.

In 1687, Sir Isaac Newton proposed a definition of viscosity which is termed as *Newton’s Law of Viscosity* and formulated as
where \( \tau \) is the tangential (shear) stress, \( \mu \) the constant of proportionality which is known as coefficient of viscosity depends on local measure of the internal friction and \( \frac{\partial u}{\partial y} \) is the velocity gradient. Fluids for which this linear relationship between the shear stress and the velocity gradient hold accurately are usually called Newtonian Fluids.

Another physical property of fluid in terms of kinematic viscosity denoted by \( \nu \) is defined the ratio of dynamic viscosity to density of that fluid. It is the measure of fluid’s resistance to shear stress under the weight of gravity. Kinematic viscosity is expressed as

\[
\nu = \frac{\mu}{\rho}
\]

where \( \rho \) is the fluid density.

The units of \( \tau, \frac{\partial u}{\partial y}, \mu \) and \( \nu \) are \( \text{kg/ms}^2 \) or \( \text{N/m}^2 \), \( \text{s}^{-1} \), \( \text{Pa.s} \) and \( \text{m}^2/\text{s} \) respectively, where \( N \) and \( Pa \) represents the unit of force as Newton and Pascal. A fluid that has no resistance to shear stress is known as ideal or inviscid fluid.

(d) Compressible and incompressible flows

The measure of the change in volume of a fluid under external forces is known as compressible fluids. Due to application of pressure, density of a compressible fluid changes significantly. It appears in many natural and technological processes. Flow of natural gas in a pipe system is an example of a compressible fluid.

If the density of the element of the fluid is not affected by the changes in pressure, then the fluid is termed as incompressible. Liquid and gases may be considered as incompressible fluids
in both microscopic and macroscopic calculations. An application of incompressible fluid is within low-speed aerodynamics.

(e) Laminar flow

A flow is said to be laminar, where the paths of the fluid particles are definite curves and no two paths traced out by two distinct fluid particles intersect. It is characterized by the low, moderate velocity and absence of any vortex motion of the particles. Flow through tubes, pipes etc. considered as laminar flow. The kinetic energy is found to be moderately higher than the viscous resistance in laminar flow.

(f) Steady and unsteady flows

A flow in which fluid properties (like pressure, temperature, velocity, density etc.) and the associated conditions related with the flow remains invariant with time is known as steady flow. On the other hand, the flow is said to be unsteady if the flow of fluid and the associated fluid conditions are dependent on time. The steady or unsteadiness represents the local criterion of a flow field.

1.3 Boundary layer theory

The invention of the first practical airplane by Orville and Wilbur Wright on December 17, 1903, it became very difficult to calculate the lift and drag forces on airplanes. To determine these type of forces, aerodynamicists need to calculate both pressure and shear-stress distributions to integrate them over the surface of the airfoil. With the help of various approximations, pressure distributions can be obtained but the shear-stress distribution needs the inclusion of internal friction and the consideration of viscous flow. Navier-Stokes equations become nonlinear due
to the presence of convective term and the analytical solution of this equation has not been obtained till date. In particular, it is true when friction and inertia forces are of comparable order of magnitude so that neither can be neglected. For high viscous fluid flow, the solutions of the Navier-Stokes equations are possible for very small Reynold’s number, where inertial forces can be neglected completely. Thus it is required to retain both inertial and viscous term for solution in Navier-Stokes equations. In 1904, Ludwig Prandtl investigated on the motion of a fluid with very small viscosity to study the behaviour of the fluid at the wall of the solid boundary. He found that the effects of viscosity are significant within a thin transition layer and that layer is known as the boundary layer. The equation of energy and concentration (analogous to Navier-Stokes equations) are considered to determine the heat and mass transfers. The boundary layer theory is a method of analyzing the behaviour of real fluids. There are various types of boundary layers associated with real fluid flow are as:

(a) **Velocity boundary layer**

For our convenience, the case of laminar two-dimensional flow of a viscous incompressible electrically conducting fluid past a vertical semi-infinite flat plate with a very small viscosity (large Reynold’s number) is considered for discussion. The fluid does not slide (slip) over the plate due to viscosity but sticks to it and as a result, the adherent fluid particles will attain zero velocity of the plate. The fluid velocity makes a transition from zero velocity at the plate to free stream velocity $U_\infty$ asymptotically at a distance far away from the plate. This generates a velocity gradient within a very thin fluid layer in contact with the plate and is known as velocity boundary layer or momentum boundary layer. The boundary layer thickness $\delta$ is very small to be compared with distances parallel to the boundary over the flow velocity which changes appreciably. The velocity boundary layer is mentioned as that region where the fluid velocity
parallel to the surface is less than 99% of the free stream velocity.

In 1904, Prandtl who was the first proposed the method of dividing the fluid into two regions. He suggested for the sake of mathematical analysis the flow region may be divided into the following sub-regions:

- A very thin layer in the vicinity of the plate in which the velocity gradient normal to the wall \( \left( i.e. \frac{\partial u}{\partial y} \right) \) is very large. Accordingly the viscous stress \( \mu \left( \frac{\partial u}{\partial y} \right) \) becomes important even though \( \mu \) is small. Thus viscous and inertial forces are of the same order within the boundary layer.

- In the remaining larger region (i.e. outside of the boundary layer) the velocity gradient \( \frac{\partial u}{\partial y} \) is very small and so the viscous forces may be neglected completely and where the ideal fluid theory offers a very good approximation. This region is characterized by the total dominance of the inertial forces.
1.3 Boundary layer theory

Fig. 1.1 Schematic representation of velocity boundary layer

(b) Thermal boundary layer:

The development of thermal boundary layer due to difference in temperatures between fluid free stream and solid wall. If the solid surface is maintained at a temperature $T_w$, which is different from the fluid temperature $T_\infty$ at a distance far away from the surface, then a variation of the temperature of the fluid is observed and it is assumed that $T_w > T_\infty$ (i.e. the plate is hot). As the fluid particles come in contact with the plate to achieve thermal equilibrium, they exchange thermal energy with the adjacent fluid layers, and temperature gradients develop in the vicinity of its surface. The fluid region near the surface of the plate where temperature gradients exist is known as the thermal boundary layer. The temperature gradient that exists on the solid surface extends up to a normal distance for which the ratio $\frac{T_w - T}{T_w - T_\infty} = 0.99$ is known as thickness ($\delta_t$) of the thermal boundary layer.
Thus thermal boundary layer is regarded as consisting of a stationary fluid film through which heat is conducted and transmitted by fluid motion. Within the boundary layer, the rate of heat flux due to conduction process i.e. \( \dot{q}_k = -\kappa \frac{\partial T}{\partial y} \bigg|_{y=0} \) is comparable to that due to convection process i.e. \( q = h' \delta (T) \). It follows that the local heat transfer coefficient within the thermal boundary layer can be formulated as

\[
h' = -\kappa \frac{\left( \frac{\partial T}{\partial y} \right)_{y=0}}{\delta (T)}
\]

where \( \frac{\partial T}{\partial y} \) is temperature gradient which determine the rate of heat transfer across the thermal boundary layer, \( \delta (T) \) the characteristic temperature and \( \kappa \) is the thermal conductivity. The relative thickness of the thermal boundary layer \( \delta_t \) and the velocity boundary layer \( \delta \) depends on the Prandtl number \( Pr \) of the fluid.

For fluids having \( Pr = 1, \delta_t = \delta \); \( Pr << 1, \delta_t \gg \delta \); \( Pr >> 1, \delta_t << \delta \).
(c) **Concentration boundary layer:**

The concentration boundary layer develops whenever the species concentration is different at the surface and the free stream concentration. Consider the flow of a single chemical species of viscous incompressible electrically conducting fluid over a flat plate where the concentration is kept $C_w$. Assuming that the free stream concentration is at $C_\infty$ where $C_w > C_\infty$. Then the chemical species diffuse from the plate surface into the fluid region, as a result concentration gradient forms in a thin layer near the surface which is known as *concentration boundary layer.* The mass diffusion follows analogous the concept of the second law of thermodynamics that takes from a higher concentration region to a lower concentration region.

![Fig. 1.3 Schematic representation of concentration boundary layer](image)

The concentration gradient that exists on the solid surface extends up to a length for which the ratio $\frac{C_w - C}{C_w - C_\infty} = 0.99$ is known as thickness ($\delta_c$) of the concentration boundary layer. Within a concentration boundary layer, both mass diffusion and convective mass transfer processes are of equal order of magnitude. The species flux quantifies the mass diffusion rate coefficient $h_m$ can be defined by expressing mass flux as
\[ h_m \delta (C) = -D_M \left( \frac{\partial C}{\partial y} \right)_{y=0} \]

which gives

\[ h_m = -D_M \left( \frac{\partial C}{\partial y} \right)_{y=0} \]

where \( D_M \) is the mass diffusivity and \( \delta (C) \) represents some characteristic concentration.

### 1.4 Magnetohydrodynamics (MHD)

Magnetohydrodynamics is the science of motion of electrically conducting fluids in presence of magnetic fields. MHD technology is established on the basis of fundamental laws of electromagnetism which state that the repulsive intersection formed due to the intersection of a magnetic field and an electric current drives the liquid perpendicularly to both the field and the current. For the last several decades, extensive research work and experiments have been attracted in this field of study by several engineers and scientists due to its technological importance and applications in engineering that run on MHD principles and also its use in understanding the diverse cosmic phenomena such as sunspots and the formation of stars from interstellar clouds. It also provides a simplified mathematical explanation and justification of the complicated domain of plasma physics. MHD is based on two basic phenomena such as

- An induced magnetic field associated with this current appears perturbing the original magnetic field.
• An electromotive force due to interact of current and field appears perturbing the original motion.

Essentially, the situation is dealing with a mutual interaction between the fluid velocity field and the magnetic field. A magnetic field is said to be uniform if at all the points, it has same strength in magnitude and direction which is represented by a set of parallel lines of force. The motion effects the magnetic field by carrying the magnetic field lines partially (depends on the electrical conductivity of the fluid) with it and affects the motion by producing a body force, called Lorentz force. The subject matter of MHD is based on Navier-Stokes equations of fluid dynamics and Maxwell’s equations of electromagnetism. Maxwell’s equations describe the properties of electric and magnetic fields and connect them to their sources, current density and charge density. Maxwell’s equations in matter are as follows

Gauss’s law of electrostatics:

$$\nabla \cdot \vec{E} = \frac{\rho_e}{\varepsilon_0} \quad (1.1)$$

Gauss’s law of magnetism:

$$\nabla \cdot \vec{B} = 0 \quad (1.2)$$

Faraday’s law of electromagnetic induction:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad (1.3)$$

Ampere-Maxwell equation:

$$\nabla \times \vec{B} = \mu_e \left[ \vec{J} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \right] \quad (1.4)$$
In addition, we have

Ohm’s law:

\[ \bar{J} = \sigma \left( \bar{E} + \bar{q} \times \bar{B} \right) \]  \hspace{1cm} (1.5)

Lorentz force:

\[ \bar{F}_L = e \left( \bar{E} + \bar{q} \times \bar{B} \right) \]  \hspace{1cm} (1.6)

Magnetic induction equation:

\[ \frac{\partial \bar{B}}{\partial t} = \nabla \times \left( \bar{q} \times \bar{B} \right) + \frac{1}{\mu_0 \sigma} \nabla^2 \bar{B} \]  \hspace{1cm} (1.7)

where \( \bar{E}, \bar{B}, \bar{J}, \bar{F}_L, e, \bar{q}, \varepsilon_0, \rho_e, \mu_e, \sigma \) are respectively the electric field intensity, magnetic induction vector, current density vector, Lorentz force, charge, velocity of the moving charge, permittivity of the medium, charge density, magnetic permeability of the medium and electrical conductivity.

In 1942, Alfven was the first, who published a classical paper on MHD in which he explained that if a highly conducting fluid moves in a magnetic field, the induced current will tend, in some sense, to inhibit relative motion of the fluid and the magnetic field so that the field is convicted by the fluid. According to Alfven, the lines of force are ‘frozen’ into the fluid. As a result, the lines of force are convicted by the fluid in motion. The development of MHD in engineering was slower. But after 1950 the study of MHD developed more rapidly and became well established as a scientific field in various contexts such as astrophysics, planetary magnetism, liquid metal technology and nuclear fusion physics. An early contribution made by Elsasser, Chandrasekhar, Bullard and Cowling in the development of this field. Hannes Alfven won the Nobel Prize for Physics in 1970, for his work on Magnetohydrodynamics. In MHD,
the continuum approximation is made, just as in ordinary hydrodynamics. One postulates that the fluid may be treated as continuous and describable in terms of local properties such as pressure and velocity.

MHD differs from ordinary hydrodynamics in the sense that the fluid is electrically conducting. In MHD heat transfer problem, the additional body force term namely the Lorentz force comes into play and the Joule heating appears in the energy equation. The energy equation remains uncoupled from Maxwell’s equations and Navier-Stokes equation in forced convection system. Thus the electromagnetic and velocity fields can be determined independently of the temperature field. On the other hand, the induced magnetic forces do not exist in the free stream due to zero velocity. Therefore, the free convection MHD problems can be formulated in a much simple way than forced convection problems. Thus the influence of the magnetic field on the boundary layer is exerted through the Lorentz force confined to the boundary layer only, with no additional effects arising out of the free stream pressure gradient.

MHD has got various applications in the fields of science and technology such as geophysics, astrophysics, engineering, medical sciences and many more. The formation and stability of the solar system, formation of stars, coronal heating etc. are some of the applications in astrophysics, applications in engineering (like design of MHD pumps, MHD generators, electromagnetic valves, jet printers etc.) and biomedical engineering which includes cardiac MRI, ECG, Magnetohydrodynamics thermo chemotherapy (MHTCT) of malignant tumors etc.
1.5 Porous medium

Porous medium defines a material consisting of a solid matrix (either rigid or it undergoes small deformation) with an interconnected void (the pores) that can allow the flow of fluids. Many natural substances such as soils, manmade materials (like cements, ceramics etc.), biological tissues (e.g. bones) can be considered as porous media. Flows through porous media is a subject used in many areas of engineering and applied sciences viz. mechanics (like geomechanics, soil mechanics etc.), petroleum engineering, construction engineering, biology and biophysics, geosciences (like hydrogeology, geophysics, petroleum geology), material sciences etc. Research in porous media has significantly increased during recent time due to its practical applications.

(a) Porosity: The porosity of a porous medium is defined as the fraction of the total volume of the medium that is occupied by void space. For an isotropic medium, porosity will normally be equal to the surface porosity (i.e. the fraction of void area to total area of a typical cross section). For natural media, the porosity does not normally exceed 0.6, whereas the porosity may approach the value 1 for manmade materials.

(b) Darcy’s law: Henry Philibert Gaspard Darcy (1803-1858), a French Hydraulic Engineer, who was the first scientist to study the interaction between the skeleton (porous solid body) and water. In 1856, Darcy’s investigations into the hydrology of water supply of Dijon and his experiments on steady-state unidirectional flow in a uniform medium revealed a proportionality between flow rate and applied pressure difference and is expressed as

\[ u = -\frac{K \partial p}{\mu \partial x} \]

(1.8)
where $\mu$ is the dynamic viscosity of the fluid, $\frac{\partial p}{\partial x}$ is the pressure gradient in the flow direction and $K$ is known as specific permeability of the medium and it depends on the geometry of the medium but independent of the nature of the fluid.

In three dimensions, the equation (1.8) generalizes to

$$ \vec{q} = -\frac{1}{\mu} K \nabla p $$

where the permeability $K$ is in general a second order tensor. For the case of an isotropic medium the permeability is a scalar and equation (1.9) reduces to

$$ \nabla p = -\frac{1}{K} \vec{q} $$

Several early research workers on convection in porous media following Wooding (1957) used the extension of Darcy’s law of the form:

$$ \rho \left[ \frac{\partial \vec{q}}{\partial t} + (\vec{q} \cdot \nabla) \vec{q} \right] = -\nabla p - \frac{\mu}{K} \vec{q} $$

This equation was obtained by analogy with the Navier-Stokes equation.

An alternative to Darcy’s equation was obtained by adding a term in the equation (1.10) in 1947, which is commonly known as Brinkman’s equation and it takes of the form as

$$ \nabla p = -\frac{\mu}{K} \vec{q} + \mu \nabla^2 \vec{q} $$
There are two viscous terms. The first is the usual Darcy term and the second is Brinkman’s term which is analogous to the Laplacian term that appears in the Navier-Stokes equation. The coefficient $\mu$ is an effective viscosity. Brinkman set dynamic viscosity and effective viscosity equal to each other, but in general they are only approximately equal. Liu and Masliyah (2005) suggested that one can think of the difference between dynamic and effective viscosity as being due to momentum dispersion. Thus the Darcy model is valid for slow flow with low permeability, whereas with the increase in fluid velocity, boundary effects become more prominent and Brinkman’s model is more effective. In most cases of flows through porous media, there are two regions: one region, where the fluid is free to flow and the other where, the fluid flows through the porous media. Hence, in studies concerning a porous or fluid composite system, the Brinkman’s model may be used by considering the inertia term as used in the Navier-Stokes equations. So, for viscous incompressible fluid through porous media, Navier-Stokes equation gives

$$\frac{\partial \bar{q}}{\partial t} + (\bar{q} \cdot \nabla) \bar{q} = \bar{F} - \frac{1}{\rho} \nabla p + \nu \nabla^2 \bar{q} - \frac{\nu}{K} \bar{q} \quad (1.13)$$

where $\bar{F}$ is the external force acting on the fluid per unit mass.

### 1.6 Heat Transfer

Heat transfer is a discipline of thermal engineering that concerns with the exchange of thermal energy between physical systems due to temperature difference. It is commonly associated with fluid dynamics and also supplements the law of thermodynamics by providing additional experimental rules to establish energy transfer rates. Heat transfer flows in the direction of
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decreasing temperature (with negative temperature gradient). The science of heat transfer is concerned with the analysis and estimation of the rate of heat transfer, the duration of heating and cooling for certain heat aspect and the surface area required to accomplish that heat aspect. The heat flux is defined as the amount of thermal energy transferred per unit time per unit area of surface, in a direction normal to the surface. The temperature distribution in a medium is controlled by three distinct modes of heat transfer viz. conduction, convection and radiation. It is actually not possible to separate entirely one mode from interactions with the other mode. However for the sake of simplicity in analysis one mode of transfer can be considered separately by assuming the other modes as less prominent.

(a) Conduction:

Heat conduction may occur through two mechanisms:

(i) By the atomic motion/ kinetic motion or direct impact of molecules which results in exchange of thermal energy from relatively higher temperature molecule to adjacent molecules with lower temperature and the complete process persists till the existences of temperature gradient in the medium.

(ii) By the drift of free (valence) electrons, effecting due to difference in concentrations of free electrons as in the case of liquid metals, electrolytes and metallic liquids. The contribution to the heat current by the drifting of electrons increases the thermal conductivities of metallic liquids to values up to 10 to 1000 times the values for non-metallic liquids.

The conduction heat transfer may occur between two material bodies which are at zero relative velocity to one another or within two parts of the same body. As the driving potential for heat transfer is the temperature difference, there exists a linear relationship between heat flow and temperature difference, and it was in 1822, a French mathematician Baron Jean Baptiste
Joseph Fourier (1768-1830), in his monumental work “Theorie Analytique de la chaleur”, formulated an empirical law, called Fourier’s law of heat conduction.

The rate of propagation of heat can be predicted from the macroscopic phenomenological relation as depicted by the Fourier’s law of heat conduction according to which the rate of heat transfer is linearly proportional to the temperature gradient. The mathematical formulation of the stated law is expressed as

$$\vec{q}_k = -\kappa \vec{\nabla} T$$

where $\vec{q}_k$ is the rate of heat flux, a vector quantity with unit $W/m^2$, $\vec{\nabla} T$ is the temperature gradient in the direction of heat flow and $\kappa$ is the constant of proportionality which is the property of material through which heat propagates and is known as thermal conductivity (with unit $W/mK$). The negative sign indicates the direction of heat flow from a high to low temperature. The magnitude of the thermal conductivity for a given substance very much depends on its microscopic structure and also tends to vary somewhat with temperature.

(b) Convection:

The mode of heat transfer that occurs between a solid surface and a fluid system in motion, when the both are at different temperatures is known as convection. When fluid flows over a solid body or inside a channel (where temperatures of the fluid and the solid surface differs), the heat transfer between the fluid and the solid surface takes place as a consequence of the motion of fluid relative to the surface. It is interesting to observe that if no-slip criteria exist, heat is transferred between the fluid particles adherent to the solid surface and the solid (which are in zero relative motion) occurs purely by conduction. The convective heat transfer is dependent on fluid motion in that the faster the fluid movement, the greater is the heat transfer by convection.
When bulk fluid motion is absent, heat transfer between a solid surface and the adjacent fluid is by conduction (random molecular motion of surface molecules) only.

According to the nature of flow, convective heat transfer is classified as Free convection and Forced convection. If the relative motion is artificially induced by some external agencies, called forced convection. It occurs in a variety of flow configurations such as flow of fluids over flat surfaces flow of water through nuclear heating elements, flow in the liquid heating tubes of a solar collector flow of a cryogenic liquid coolant in certain digital computers. The driving force is external to the fluid and the flow velocities are high in forced convection. If the fluid motion is set up by buoyancy effects resulting from density difference, caused by temperature difference in the fluid, the heat transfer is known as free convection (or natural convection). In free convection flow, velocities are generally much smaller than those associated with forced convection and thus the corresponding free convection heat transfer rate is also smaller than those of forced convection. Free convection flows under the influence of gravitational forces have been investigated most extensively because they are encountered frequently in nature as well as in engineering applications. Free convection is used in various applications like electric baseboard heaters, refrigerating coils, transmission lines, hot radiators and several practical situations in everyday life.

Again in many convection problems, both free as well as forced convection effects are of same order of magnitudes so that not a single effect is negligible for study and is termed as mixed convection problem. Cooling of heating pipes in nuclear reactors is a good example of such type of problem.

In general the quantification of convective heat transfer relation suggested by Newton,
known as *Newton's law of cooling* can be expressed as

\[ q = \hbar' \delta (T) \]

where \( q \) is the heat transfer by convection, \( h' \) the heat transfer coefficient and \( \delta (T) \) represents reference temperature.

(c) **Radiation:**

Heat transfer due to radiation takes place in the form of electromagnetic waves (due to Maxwell's electromagnetic theory) or by photons (due to Max Plank's theory). It is called thermal radiation, since the radiation represents a conversion of a body’s thermal energy into electromagnetic energy. Thermal radiation is a spontaneous process of radioactive distribution of entropy. It refers to the radiant energy emitted by bodies by virtue of their own temperatures, resulting from thermal excitation of the molecules. The molecules and atoms of a matter get excited and start vibrating due to heat, which produce charges and from the experiments of Hertz and others, the oscillating charges emit electromagnetic radiation. A distinguishing feature of radiative heat transfer is that no medium required for heat propagation like conduction and convection processes.

Thermal radiation is a function of several components which includes its surface reflectivity, emissivity, surface area, temperature and geometric orientation with respect to other thermally participating objects. Radiative heat flux determines the rate of heat transfer as thermal radiation per unit area. Radiative heat transfer rates are generally proportional to the differences to the fourth power of temperature. Therefore it becomes more important with rising temperature levels, in nuclear reactions (such as in the sun, in a fusion reactor or in
nuclear bombs), during atmospheric re-entry of space vehicles etc. As modern technology strives for higher efficiency devices, thermal radiation even more important some applications that are increasing in importance include bio-mathematics and medical sciences especially for treating cancer and skin disease patients. Some ray therapy such as ultraviolet ray, cameo etc. uses the concept of radiation theory. A black body is an idealized body considered to absorb all incident radiation from all directions at all wavelengths without reflecting, transmitting or scattering. It is a theoretical concept which is used as a standard of perfection against the radiation characteristics of real bodies is compared.

In 1879, Stefan suggested a relationship based on an experimental data between the temperature and emissive power of a black body. Later, it was derived by Boltzmann on the basis of thermodynamical considerations and is known as Stefan-Boltzmann law, expressed as

\[ E_b = \sigma^* T^4 \]

where \( E_b \) is the black body emissive power, \( \sigma^* \) the Stefan-Boltzmann constant = \( 5.669 \times 10^{-8} \) W/m\(^2\)K\(^4\), \( T \) the temperature of the black body.

1.7 Heat Sink

Heat sink is a medium (usually a protective device such as a metal plate or a layer of material or a coolant fluid) that can absorb excess heat (or thermal energy) without a significant change in temperature. It is a useful device that absorbs and dissipates the excess heat generated by a system. Heat sink is usually made out of copper or aluminium. A large number of importances in engineering applications exist. Metals (like magnesium and aluminium) having high thermal
1.8 Ohmic Dissipation or Joule Heating

Joule heating also known as Ohmic heating or resistive heating, is the process that the energy of an electric current converted into heat as it flows through a resistance. It was first studied by James Prescott Joule in 1841. Joule immersed a length of wire in a fixed mass of water and measured the temperature rise due to a known current flowing through the wire for a 30 minute period. He found that the heat produced was proportional to the square of the current multiplied by the electrical resistance of the wire. In particular, when the electric current flows through a solid or liquid with finite conductivity, electric energy is converted to heat through resistive losses in the material. The mechanism of Ohmic heating is that it causes electroporation of cell membranes. The results of chemical analysis indicate that Ohmic heating
technology provides products with chemical properties similar to those of the products obtained by conventional heating treatment. Joule heating is used in multiple devices and industrial process. Evaporation, extraction, dehydration, fermentation, blanching and heating of foods etc. are some of applications exist for Ohmic heating.

1.9 Mass Transfer

Mass transfer is the mass in transit due to difference in species concentration in a mixture. It will take place as long as a difference in species concentration exists. Therefore, the species concentration gradient acts as the driving potential in mass transfer just as the temperature gradient does in heat transfer. The mass transfer rate of gases is usually higher than that of solids and liquids. Mass transfer occurs in a variety of applications in various branches of engineering and sciences like mechanical, chemical, aerospace and bio-engineering. Absorption, evaporation, drying, chemical reaction and solution are some of the concept of mass transfer. Mass transfer includes oxygenation of blood, drug and food assimilation, respiration mechanism etc. are some of the biological applications.

The subject of mass transfer like heat transfer relates with both bulk mass transport and mass diffusion on a molecular scale which resulting from convection process. The mechanism of mass transfer depends greatly on the dynamics of the system in which it occurs. There are two distinct modes of mass transfer:

(a) Mass transfer by diffusion

The process of transportation of mass is transferred by random molecular motion or laminar flow of fluids, known as diffusion mass transfer. Mass diffusion occurs in liquids and solids
as well as in gases. However since mass transfer is strongly influenced by molecular spacing
diffusion, occurs more readily in gases than in liquids and easily in liquids than in solids.
The rate equation for mass diffusion relates with mass flux of the diffused substance and the
concentration gradient for mass transfer was first established by Fick (1855) and is thus known
as Fick’s law of diffusion. Mathematically it is expressed as

\[ \vec{J} = -D_M \vec{\nabla} C \]

where \( J \) represents the molar flux that measures the amount of substance that flows through
a unit area during a unit time interval, \( D_M \) is the diffusion coefficient or mass diffusivity that
represents the mobility characteristic of the component, \( C \) is the species concentration that
signifies the amount of substance per unit volume of the mixture. Experimentally it is seen that
due to increase in temperature, mass diffusivity coefficient increases.

(b) Mass transfer by convection

Mass transfer by convection takes place in cases where the bulk velocity is appreciable or the
constituents in a binary mixture are moving with significant velocities. As in heat convection,
mass convection may also occur under free or forced convections. The buoyancy force causing
circulation in free convection mass transfer results from the differences in density of the vapour-
air (or vapour-gas) mixtures of varying compositions. The evaporation of alcohol is an example
of free convection mass transfer. Due to the presence of some external agencies, the forced
convection mass transfer occur the mass to transport. An instance of forced convection mass
transfer is the evaporation of water from an ocean when air blows over it.

Convective mass transfer occurs abundantly in nature. The three transport process namely
momentum, energy and mass transfer are analogous to each other and their analogies have
been widely used in the study of mass transfer. Momentum transfer takes place due to velocity
gradient (Newton’s law of viscosity), energy transfer takes place due to temperature gradient
(Fourier’s heat conduction equation) and mass transfer takes place due to concentration gradient
(Fick’s law of diffusion). Skin friction, Nusselt number and Sherwood number are respectively
the dimensionless coefficients of the said transport properties.

1.10 Chemical Reaction

A chemical reaction is the chemical change in one or more substances as a result of the input
energy such as heat, light or electricity into new substances. It takes place at a characteristics
rate at given temperature and concentration. The substance involved in a chemical reaction is
termed as reactant or reagent. A chemical reaction mainly depends on the concentration of its
reactants. The change in concentration over the change in time is known as the rate of chemical
reaction. Chemical reaction occurs the rate which refers to the speed. Temperature, pressure,
concentration and catalysts are some factors which can either slow down or speed up the rate
of chemical reaction. Chemical reaction can be classified as *homogeneous or heterogeneous*
based on the physical state of the substances present. A homogeneous reaction is one that
occurs uniformly in a single phase (gaseous, liquid or solid) and is solely dependent on the
interactions of the reacting substances. The reactions between gases (combination of household
gas and oxygen which produce a flame) and reactions between liquids or substances dissolved
in liquids (e.g. the reactions between aqueous solutions of acids and bases) are some examples
of homogeneous chemical reaction. In contrast, a heterogeneous reaction takes place in a
1.11 Thermal Diffusion or Soret effect

restricted region or within the boundary of a phase. It is less important whether the reaction takes place in one, two, or more phases; at an interface; or whether the reactants are distributed among the phases contained within a single phase. The rate of mass transfer becomes important factor for reaction rate during reaction in heterogeneous system since more than one phase is involved in this system. The reaction of metals with acids, the electrochemical changes that occur in batteries and electrolytic cells and the phenomena of corrosion are part of the subject of heterogeneous reactions.

The order of a reaction is a number that relates the rate of a chemical reaction with the concentrations of the reacting substances. The rate of reaction is independent of the concentration of the reactants or occurring at a constant rate in a chemical reaction, then it is known as zeroth-order reaction. The rate equation of a homogeneous zeroth-order reaction can be expressed as

$$Rate = k[A]^0 = k$$

where $k$ is rate constant and $[A]$ represents for concentration of one of the reactants. On the other hand, the rate of production of a chemical product is directly proportional to the first power of the concentration of the reactants is known as first order chemical reaction. The rate equation gives

$$Rate = k[A]^1 = k[A]$$

### 1.11 Thermal Diffusion or Soret effect

The process of mass transfer that occurs by the combine effects of concentration as well as temperature gradients is known as thermal diffusion or Soret effect. The experimental
investigation of the thermal diffusion effect was first performed by Swiss scientist Charles Soret in 1879 on the basis of experiments with solutions of $NaCl$ and $KNO_3$. In general, the Soret effect is of small order in magnitude than the effect described in Fick’s law and is often neglected in mass transfer process. But in the mass transfer involving low concentration levels, the effect of thermal diffusion becomes significant. Soret effect occurs when two parts of a liquid solution are maintained at different temperatures. Soret effect is said to be ‘positive’ when the particles move from a hot to cold region and ‘negative’ when the reverse is true. The mass flux under a large temperature gradient has been the subject of several experimental and theoretical researches. Such type of theory on Soret effect was published in 1887 by Van’t Hoff. In this theory he investigated the analogy between gases and dilute solutions and predicted that the solute would distribute itself so that its osmotic pressure was constant throughout the system.

Eckert and Drake (1972) emphasized the fact that in mixtures between gases with very light molecular weights like ($H_2$ and $He$) and for medium molecular weight like ($N_2$, air), the Soret effect was found to be of considerable magnitude such that it cannot be neglected. Soret effect is found to be significant when more than one species are present under a very large temperature gradient. Therefore, it may be utilized for isotope separation. Soret effect plays an important role in polymer characterization and the structure of flames. The significance of thermal diffusion effect may be seen in various phenomena like planetary dynamics, thermal reactions, biological systems, electrochemical processes and so on.
1.12 Basic Equations

In MHD, fluid flow models corresponding to the flow of a laminar, Newtonian, electrically conducting, viscous incompressible fluid have been considered for the study of heat and mass transfer problems.

The equations of motions of a viscous incompressible and electrically conducting fluid in the presence of magnetic field and radiation:

Equation of continuity:

\[ \vec{\nabla}.\vec{q} = 0 \quad (1.14) \]

Momentum equation:

\[
\frac{\partial \vec{q}}{\partial t} + (\vec{q}.\vec{\nabla}) \vec{q} = \vec{F} - \frac{1}{\rho} \vec{\nabla}p + \nu \nabla^2 \vec{q} + \frac{\vec{J} \times \vec{B}}{\rho} - \frac{\nu}{K} \vec{q} \quad (1.15)
\]

Ampere’s circuital law:

\[ \vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon \frac{\partial \vec{E}}{\partial t} \quad (1.16) \]

Maxwell-Faraday equation (Faraday’s law of electromagnetic induction):

\[ \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad (1.17) \]

Gauss’s law of electrostatics:

\[ \vec{\nabla}.\vec{E} = \frac{\rho_0}{\varepsilon} \quad (1.18) \]

Gauss’s law of magnetism:

\[ \vec{\nabla}.\vec{B} = 0 \quad (1.19) \]
Kirchhoff’s first law:

\[ \vec{\nabla} \cdot \vec{J} = 0 \]  

(1.20)

Ohm’s law:

\[ \vec{J} = \sigma \left[ \vec{E} + \vec{q} \times \vec{B} \right] \]  

(1.21)

Generalized Ohm’s law:

\[ \vec{J} + \frac{m}{B_0} \left( \vec{J} \times \vec{B} \right) = \sigma \left[ \vec{E} + \vec{q} \times \vec{B} + \frac{1}{e\eta_e} \vec{\nabla} p_e \right] \]  

(1.22)

Energy equation:

\[ \rho C_p \left[ \frac{\partial T}{\partial t} + \left( \vec{q} \cdot \vec{V} \right) T \right] = \kappa \nabla^2 T + \frac{J^2}{\sigma} + \phi - \vec{\nabla} \cdot \vec{q} r + \dot{Q} \]  

(1.23)

Species continuity equation:

\[ \frac{\partial C}{\partial t} + \left( \vec{q} \cdot \vec{V} \right) C = D_M \nabla^2 C + \dot{R} \]  

(1.24)

The effect of thermal diffusion (Soret) is to be included in equation with proper conditions. In case of high viscous and creeping motion, the pressure gradient in equation gets transformed by using Bernoulli’s pressure equation and Boussinesq approximation.

Equation of state on the basis of classical Boussinesq approximation:

\[ \rho_\infty = \rho \left[ 1 + \beta \left( T - T_\infty \right) + \beta^* \left( C - C_\infty \right) \right] \]  

(1.25)
where

\[ m = \omega_e \tau_e \quad \text{Hall parameter} \]

\[ \vec{F} \quad \text{External body force per unit mass acting on the fluid} \]

\[ T \quad \text{Fluid temperature} \]

\[ C \quad \text{Fluid concentration} \]

\[ t \quad \text{Time} \]

\[ \mu_e \quad \text{Magnetic permeability of the medium} \]

\[ \rho_0 \quad \text{Charge density} \]

\[ \varepsilon \quad \text{Permittivity of the medium} \]

\[ \eta_e \quad \text{Density of electron} \]

\[ \dot{Q} \quad \text{Internal energy function (like heat sink)} \]

\[ \dot{R} \quad \text{Species concentration function (like chemical reaction)} \]

\[ \vec{J} \times \vec{B} \quad \text{Lorentz force per unit volume} \]

\[ \frac{j^2}{\sigma} \quad \text{Joulean heat per unit volume} \]

\[ p_e \quad \text{Electron pressure} \]

\[ \varphi \quad \text{Dissipative energy function} \]
$\vec{q}_r$ Radiative heat flux

$C_p$ Specific heat at constant pressure

$\vec{B}$ Magnetic induction vector

$\vec{J}$ Current density

$\beta$ Coefficient of volume expansion for heat transfer

$\beta^*$ Coefficient of Volume expansion for mass transfer

$\nu$ Kinematic viscosity

$\rho$ Fluid density

$p$ Pressure

$\kappa$ Thermal conductivity

$D_M$ Mass diffusivity

$\rho_\infty$ Density of fluid in free stream

$\vec{E}$ Electric field intensity vector

$\sigma$ Electrical conductivity

$\vec{q}$ Fluid velocity

$B_0$ Strength of the magnetic field
1.13 Non Dimensional Parameters

- $\omega_e$: Electron frequency
- $\tau_e$: Electron collision time
- $e$: Electron charge
- $K$: Permeability of the porous media
- $T_\infty$: Temperature in free stream
- $C_\infty$: Concentration in free stream

1.13 Non Dimensional Parameters

(i) Reynolds number:

Reynolds number is a measure of the relative magnitude of the inertia force to the viscous force occurring in a flow. It is useful for investigating important features of a given flow of a fluid. It was first proposed by Stokes in 1851, although later work by Reynolds in 1883 in a classical experiment on the onset of turbulence in flow, and it termed as Reynolds number. Mathematically it is expressed as

$$Re = \frac{\text{Inertia force}}{\text{Viscous force}} = \frac{VL}{\nu}$$

where $L$ is the characteristic length, $V$ the characteristic velocity and $\nu$ is the kinematic viscosity.

Reynolds number provides an estimate of the relative contribution of the inertia and viscous forces. It makes the viscous force dominant in the flow for smaller value of $Re$ whereas for
greater values of Re, viscous force becomes less effective and inertia force dominates over viscous force.

(ii) Prandtl number:

Prandtl number is the ratio of viscous force to thermal force. It is denoted by $Pr$ and mathematically defined as

$$Pr = \frac{\mu C_p}{\kappa} = \frac{\nu}{\alpha}$$

where $\mu$ is the dynamic viscosity, $\kappa$ the thermal conductivity, $C_p$ the specific heat at constant pressure and $\alpha = \frac{\kappa}{\rho C_p}$ is the thermal diffusivity of the medium. Evidently, Prandtl number depends on the properties of the fluid and controls the relation between the velocity and temperature profiles for a fluid flow.

(iii) Soret number:

Soret number is a measure of relative effectiveness of energy flux over mass flux. Mathematically, it is expressed as

$$Sr = \frac{D_M K_T (\Delta T)}{\nu T (\Delta C)}$$

where $K_T$ is the thermal diffusion ratio, $\Delta T$ is reference temperature difference, $\Delta C$ is some reference concentration difference between species and other symbols are defined in the nomenclature.

(iv) Schmidt number:

Schmidt number is the ratio of momentum diffusivity and mass diffusivity in a fluid medium. It is denoted by $Sc$ and defined as

$$Sc = \frac{\nu}{D_M}$$
where $D_M$ is the coefficient of mass diffusivity and $\nu$ is the kinematic viscosity.

Evidently, Schmidt number governs the relation between the velocity and the molar species concentration distribution for the fluid flow. It is used to characterize fluid flows in mass transfer and plays the same role as the Prandtl number in heat transfer.

**(v) Eckert number:**

The Eckert number $Ec$ is mathematically defined as

$$Ec = \frac{V^2}{(\Delta T)C_p}$$

where $V$ is the reference velocity, $\Delta T$ is the reference temperature difference and $C_p$ is the specific heat at constant pressure.

The Eckert number defines the kinetic energy of the flow relative to some enthalpy difference. It is also a measure of the viscous dissipation effects in the flow. This grows in a proportion to square of the velocity and hence can be neglected for flows associated with very small velocities.

**(vi) Hartmann number:**

Hartmann number is the ratio of electromagnetic force to the viscous force. Mathematically, it is defined as

$$M = B_0L\sqrt{\frac{\sigma}{\mu}}$$

where $B_0$ is the strength of the applied magnetic field, $L$ the characteristic length, $\sigma$ the electrical conductivity and $\mu$ is the dynamic viscosity.

**(vii) Thermal Grashof number:**

Thermal Grashof number defines the ratio of thermal buoyancy force to the viscous force acting
on the fluid. It is denoted by $\text{Gr}$ and defined mathematically as

$$Gr = \frac{g\beta L^3(\Delta T)}{V^2}$$

where $g$ is the acceleration due to gravity, $\Delta T$ is a suitable reference temperature difference, $L$ the characteristic length, $\beta$ the volumetric thermal expansion coefficient and other symbols have their usual meanings. Thermal Grashof number is of great importance in free convection heat transfer where the only driving force is the buoyancy force. It may be used as a critical value to indicate a transition from laminar to turbulent flow in free convection. The parameter $\frac{Gr}{Re^2}$ is a measure of relative importance of free convection to forced convection. When $\frac{Gr}{Re^2} \approx 1$, combined effects of free and forced convection must be considered. Free convection is negligible if $\frac{Gr}{Re^2} << 1$ and forced convection is negligible if $\frac{Gr}{Re^2} >> 1$. Here, $\beta$ is a thermodynamic property of the fluid that provides a measure of the amount by which the density changes in response to a change in temperature at constant pressure.

(viii) **Solutal Grashof number**:

The Grashof number for mass transfer is defined mathematically as

$$Gm = \frac{g\beta^* L^3(\Delta C)}{V^2}$$

where $\Delta C$ is some suitable reference concentration difference, $\beta^*$ the volumetric coefficient of expansion with species concentration, $L$ the characteristic length and other symbols have their usual meanings.

Solutal Grashof number represents the ratio of mass buoyancy force and viscous force. Free convection is caused by concentration gradients rather than temperature gradients in mass
transfer cases. Thus \( \beta^* \) is a quantity that measures the change in density in response to a change in molar species concentration of the mixture.

## 1.14 Methods of Solution

**(a) Laplace Transform technique**

Laplace Transform is an integral transform introduced by French mathematician Marquis Pierre-Simon Laplace (1749-1827) during his work on probability theory. But methodically, it was extended by British physicist Oliver Heaviside (1850-1925) in the name of operational calculus in solving linear differential equations arising from electrical networks. Laplace transform is commonly applied to functions that are time-dependent, which are zero for \( t < 0 \) and bounded by an exponentially growing function. It is a technique that converts a system of differential equations to algebraic equations. This integral transform finds application in various fields concerning flow and transport phenomena (like fluid mechanics, thermodynamics etc.), celestial mechanics, geophysics etc.

**Definition 1:** Let \( f(t) \) be a continuous and single valued function of the real variable \( t \) for all \( 0 < t < \infty \), and is of exponential order. Then the Laplace transform of \( f(t) \) denoted by \( \mathcal{F}(s) \) and is defined as

\[
L\{f(t)\} = \mathcal{F}(s) = \int_{0}^{\infty} e^{-st} f(t) \, dt, \quad s > 0
\]

where \( s \) is a parameter which may be real or complex and \( L \) is the Laplace transform operator.
**Definition 2:** A function $f(t)$ is said to be of exponential order as $t \to \infty$ if there exists a positive constant $M$, a number $\alpha$ and a finite number $t_0$ such that

$$e^{-\alpha t} |f(t)| < M \forall t > t_0$$

i.e. $|f(t)| < Me^{\alpha t}$

**Some important properties:**

(i) **Linearity property:**

If $C_1$ and $C_2$ are any constants while $f_1(t)$ and $f_2(t)$ are functions with Laplace transforms $\mathcal{F}_1(s)$ and $\mathcal{F}_2(s)$ respectively then

$$L\{C_1 f_1(t) + C_2 f_2(t)\} = C_1 L\{f_1(t)\} + C_2 L\{f_2(t)\} = C_1 \mathcal{F}_1(s) + C_2 \mathcal{F}_2(s)$$

(ii) **Shifting property:**

(a) If $L\{f(t)\} = \mathcal{F}(s)$ and

$$G(t) = f(t - a), \quad t > a$$

$$0, \quad t < a$$

Then $L\{G(t)\} = e^{-at} \mathcal{F}(s)$

where $a$ is constant.

(b) If $L\{f(t)\} = \mathcal{F}(s)$ then $L\{e^{at} f(t)\} = \mathcal{F}(s - a)$
(iii) Change of scale property:
If \( L \{ f (t) \} = \mathcal{F} (s) \) then \( L \{ f (at) \} = \frac{1}{a} \mathcal{F} \left( \frac{s}{a} \right) \)

**Derivatives of Laplace transform:**

(i) If \( L \{ F (t) \} = \mathcal{F} (s) \) then \( L \{ F' (t) \} = s \mathcal{F} (s) - f (0) \)

(ii) If \( L \{ f'' (t) \} = \mathcal{F} (s) \) then \( L \{ f'' (t) \} = s^2 \mathcal{F} (s) - sf (0) - f' (0) \)

(iii) If \( L \{ f (t) \} = \mathcal{F} (s) \) then \( L \{ t^n f (t) \} = (-1)^n \frac{d^n}{ds^n} \mathcal{F} (s) \)

**Integral of Laplace transform:**

(i) If \( L \{ f (t) \} = \mathcal{F} (s) \) then \( L \left\{ \int_0^t f (x) \, dx \right\} = \frac{\mathcal{F} (s)}{s} \)

(ii) If \( L \{ f (t) \} = \mathcal{F} (s) \) then \( L \left\{ \int_0^\infty f (t) \, dt \right\} = \int_0^\infty \mathcal{F} (s) \, ds \) provided \( \lim_{t \to 0} \frac{f (t)}{t} \) exists.

**Periodic function of Laplace transforms:**

If \( F (t) \) be a periodic function with period \( T > 0 \) then

\[
L \{ F (t) \} = \frac{\int_0^T e^{-st} F (t) \, dt}{1 - e^{-sT}}
\]

**Definition 3:** If the Laplace transform of a function \( f (t) \) is \( \mathcal{F} (s) \) i.e. \( L \{ f (t) \} = \mathcal{F} (s) \). Then

\[
f (t) = L^{-1} \{ \mathcal{F} (s) \}
\]

where \( L^{-1} \) is called the inverse Laplace transformation operator.

**Some properties:**

(i) First translation or shifting property:
If \( L^{-1} \{ \mathcal{F} (s) \} = f (t) \) then \( L^{-1} \{ \mathcal{F} (s - a) \} = e^{at} f (t) \)
1.14 Methods of Solution

where \( a \) is any constant.

(ii) Second translation or shifting property:

If \( L^{-1}\{\mathcal{F}(s)\} = f(t) \) then \( L^{-1}\{e^{-as}\mathcal{F}(s)\} = G(t) \)

where

\[
G(t) = f(t - a), \quad t > a
\]

\[
0, \quad t < a
\]

(iii) Change of scale property:

If \( L^{-1}\{\mathcal{F}(s)\} = f(t) \) then \( L^{-1}\{\mathcal{F}(as)\} = \frac{1}{a}f\left(\frac{t}{a}\right) \)

Derivatives of Inverse Laplace transform:

If \( L^{-1}\{\mathcal{F}(s)\} = f(t) \) then \( L^{-1}\{\mathcal{F}'(s)\} = (-1)^n t^n f(t), \quad n = 1, 2, 3... \)

Integral of Inverse Laplace transform:

(i) If \( L^{-1}\{\mathcal{F}(s)\} = f(t) \) then \( L^{-1}\left\{\int_s^\infty \mathcal{F}(u) \, du\right\} = \frac{f(t)}{t} \)

(ii) If \( L^{-1}\{\mathcal{F}(s)\} = f(t) \) then \( L^{-1}\left\{\frac{\mathcal{F}(s)}{s}\right\} = \int_0^t f(u) \, du \)

Convolution:

Let \( F(t) \) and \( G(t) \) be two functions then the convolution of these two functions is defined by

\[
F \ast G = \int_0^t F(u) G(t - u) \, du
\]

The relation \( F \ast G \) is called resultant of \( F \) and \( G \).
1.14 Methods of Solution

Convolution Theorem:

If $L^{-1}\{f(s)\} = F(t)$ and $L^{-1}\{g(s)\} = G(t)$ then

$$L^{-1}\{f(s)g(s)\} = \int_{0}^{t} F(u)G(t-u)\, du = F * G$$

(b) Perturbation technique

Perturbation technique is one of the pioneering techniques of approaching various types of non-linear problems. Perturbation theory was first devised to solve intractable problems in the calculation of the motions of planets in the solar system. Although in the 18th and 19th century, several notable mathematicians extended and generalized the methods of perturbation theory, but later the application of perturbation theory was used extensively to solve non-linear differential equations. The implementation of perturbation theory is required to make the governing equations to non-dimensionalise. Once the equation is non-dimensionalised, the perturbation theory hinges on the identification of a small parameter $\varepsilon$ in the equation such that $0 < \varepsilon \leq 1$. In general, the differential equation may be classified as linear differential equation and non-linear differential equation. The non-linear equation is considered as a small perturbation through a small parameter, which is known as the perturbation parameter.

Perturbation theory is applicable if the problem at hand cannot be solved exactly, but can be formulated by adding a small term to the mathematical expression of the exactly solvable problem. It leads to an expression for the desired solution in terms of a formal power series in some small parameter (perturb parameter) known as perturbation series that quantifies the deviation from the exactly solvable problem. For instance, the approximation to the full solution
A, a series in the small parameter $\varepsilon$ can be expressed as

$$A = A_0 + \varepsilon A_1 + \varepsilon^2 A_2 + \ldots$$

where $A_0$ is the mean part and $A_1$ is the perturbed part. For the small values of $\varepsilon$, the higher order terms in the expression becomes successively smaller. Thus the truncating series of an approximate perturbation solution is obtained as

$$A \approx A_0 + \varepsilon A_1$$

Then the differential equations can be solved by perturbation method using this series in the equation and equating the coefficients of like powers of $\varepsilon$.

### 1.15 Review of Literature

(i) MHD convective flows past flat plate with or without heat sink, chemical reaction and suction:

Magnetohydrodynamics (MHD) has been a subject of research for long time because of its wide range of importance in various fields. The study of natural phenomena is vast importance in astrophysics and geophysics to several engineering applications such as liquid metal, plasma confinement and electromagnetic casting. The extensive work in the field of MHD was carried out by Maxwell (1864) and Lorentz (1952). While significant contribution in MHD was due to Alfvén (1942), for which he was awarded Nobel Prize in Physics in 1970. Further, some other significant contributions were made by several researchers like Shercliff (1965), Lehnert
(1952), Sutton and Sherman (1965), Cowling (1976), Crammer and Pai (1973), Hughes et al. (1966), Pai (1962), Alizadeh et al. (2014), Seth et al. (2016), Mangathai et al. (2016) etc.

The study of MHD convective flow over heated and cooled plates becomes one of the fundamental problems in recent times due to its numerous applications. Such type of work has been carried out by several authors. Some of them are Glauret (1961), Davies (1963), Greenspan et al. (1959), Meksyn (1962), Pop (1967), Tan et al. (1968), Afzal (1972), Das (1970), Gulab et al. (1977), Revankar (1983), Raptis and Perdikis (2006), Soundalgekar (1969, 1970, 1973 and 1975), Devi et al. (1984, 1988), Hossain et al. (1996), Singh (1993), Acharya et al. (2000), Singh et al. (2005), Afify (2009), Alam et al. (2006), Singh et al. (1993) etc.

Presence of suction is of great importance in reduction of friction and prevention of boundary layer separation and thus very much essential especially in the fields of aerodynamics, nuclear reactors and manufacturing industries etc. The development of these phenomena was initiated by Lachmann (1961) for the study of laminar boundary layer flow control. Several researchers considered the case of suction such as Soundalgekar (1969, 1970), Devi and Kandaswamy (2001), Merkin (1972, 1975), Afify (2009), Alam et al. (2006) etc.

(ii) Flows through porous media:
Flows through porous media play a significant role in nature and transport process in its various applications such as petroleum industries, energy conservation, geothermal engineering, thermal insulation, chemical catalytic reactors etc. A detailed account of the applications of the convection flows through porous media has been reported by Niold and Bejan (2006). Investigation of flow problems through porous media are essentially based on Darcy’s experimental law. Modification of Darcy’s law to investigate convective flows through porous media was made by Wooding (1957) and Brinkman (1947a, 1947b). Significant contributions on flows
through porous media were done by Ahmadi et al. (1971), Terrill et al. (1965a, 1965b), Pop et al. (1969a, 1969b, 1969c, 1969d), Muskat (1937), Gulab et al. (1977), Bear (1972), Raptis et al. (1981a, 1981b), Singh et al. (2002), Megahed (1984), Singh et al. (1993), Sattar (1994), Jain et al. (2005), Yamamoto et al. (1976), Acharya et al. (2000) etc.

Analytical solutions to the problems of steady and unsteady MHD heat and mass transfer flows through porous media are presented by several researchers like Ahmed et al. (2005, 2006a), Ahmed and Kalita (2009), Garg et al. (2015), Das (2009), Das and Jana (2010), Barik (2016), Reddy and Rao (2016) etc.

In general, most of the researchers have treated the permeability of porous medium as constant. But a porous material containing the fluid is a non-homogeneous medium. Many authors have analyzed the flow with variable permeability. In this point of view, some of them are Singh and Singh (2000), Singh and Verma (1995), Ahmed et al. (2006b), Sharma et al. (2006), Singh (2011) etc.

(iii) Convective heat and mass transfer flow with or without thermal diffusion and chemical reaction:

The study of heat and mass transfer of an incompressible fluid is a considerable interest in view of their applications in astrophysics (like formation and stability of stars), geophysics (like oil and natural gas exploration) and engineering. Applications in engineering include designing of heat exchangers, rocket science, nuclear reactors, separation of isotopes etc.

The law of heat conduction was formulated by Fourier (1822) and the law of mass diffusion by Fick (1855). Many research works in the field of heat and mass transfer were carried out by Nusselt (1915), Brinkman (1947a, 1947b), Jost (1952), Glasstone et al. (1941), Jakob (1949), Crank (1957), Bird (1956), Grober et al. (1961), Knudsen and Katz (1958), Goldstein (1965),

Forced convection flows with heat transfer past flat plates were made by several authors like Devi et al. (1988), Megahed (1984), Sparrow and Gregg (1957), Sellars et al. (1956), Churchill and Ozoe (1973), Pande (1971), Shah and London (1978), Churchill (1976), Soundalgekar and Bhatt (1984), Singh et al. (1993), Govindaraju (1978), Bhargava and Rani (1984), Gupta and Johari (2001), Singh and Sharma (2001a, 2001b) etc.


Problems of mass convective flows with or without thermal diffusion and chemical reaction past flat plates have been carried out by many researchers due to its applications in geophysics and chemical engineering. The problems of convective flows has been made by numerous authors like Gebhart (1971), Raptis et al. (1982), Sattar and Alam (1994), Rai and Pandey (1974), Acharya et al. (2000), Georgantopoulos et al. (1981), Singh et al. (2003), Raptis and Kafousias (1982), Ahmed and Goswami (2011a, 2011b), Ahmed and Kalita (2009), Alam et al. (2006), Sarada and Shanker (2013), Govindaranjan et al. (2014), Reddy (2016), Rao et al.
(iv) **Unsteady flows in slip regions:**

An unsteady flow problem with time dependent is a subject of interest in research due to its significant technological applications. Unsteady flow, which results due to the impulsive motion of a flat plate in an infinite mass of fluid which is otherwise at rest was first investigated by Stokes in 1851 and is popularly known as Stoke’s first problem. And unsteady flow, which sets up from rest when a plane surface oscillates with a periodic velocity , was first studied by Stokes and later by Rayleigh (1911), called Stoke’s second problem or Rayleigh’s problem. Several authors have been contributed in unsteady flow problems like Tsuji (1953), Rott (1955), Kay (1948), Glauret (1956), Gersten (1965), Hill and Stenning (1960), Watson (1959) etc.

The molecular mean free path in gases is comparable to some characteristic domain, the rarefaction effects are prominent and the continuum assumption is no longer valid. Thus the gas exhibits non-continuum effects such as velocity slip and temperature jump. It is generally established that for Knudsen number $K_n = 0$, means the no-slip condition prevails and the continuum flow assumption is valid for $K_n < 0.001$. The flow is considered as slip flow for the range of $0.001 < K_n < 0.1$. Investigation relates with the slip flow were carried out by Maxwell (1879) considering first order velocity slip boundary conditions. Slip flow in the entrance region of a parallel plate was studied by Sparrow et al. (1962). Subsequent works on the slip flows were also done by Street (1960), Beavers et al. (1974), Schaaf and Chambre (1961), Kennard (1938), Beavers and Joseph (1967), Taylor (1971), Saffman (1971), Jain and Gupta (2005), Richardson (1971), Reddy (1964), Yu et al. (2002), Watanebe et al. (1998), Ebert and Sparrow (1965) etc.
MHD free convective mass transfer flow of a viscous fluid through a porous medium in slip flow regime was investigated by Das et al. (2008), Singh and Gupta (2005), Devi and Raj (2011), Balamurugan et al. (2015), Rajput et al. (2008) etc.