Chapter 3

Singularity of $\theta$-graphs

In this chapter we establish a necessary and sufficient condition for a graph $G$ to be singular. Further, we have characterized the singularity of $\theta$-graphs and have found the nullity of $\theta$-graphs.

3.1 Introduction

In the following we list some fundamental concepts which are useful for our purpose.

**Definition 3.1.1.** A bicyclic graph is a simple connected graph in which number of edges equal the number of vertices plus one.

The cycle and the path on $n$ vertices are denoted by $C_n$ and $P_n$, respectively. Let $C_p$ and $C_q$ be two vertex-disjoint cycles. Suppose that $v_0$ is a vertex of $C_p$ and $v_l$ is a vertex of $C_q$. Joining $v_0$ and $v_l$ by a path $v_0v_1\ldots v_l$ of length $l$, where $l \geq 0$ ($l = 0$ means identifying $v_0$ with $v_l$), the resulting graph is called an $\infty$-graph and is denoted by $\infty(p,l,q)$ [see Figure 3.1]. We denote by $B_n^*$, the class of all bicyclic graphs that have an $\infty$-graph as an induced subgraph.
Let $P_{l+1}, P_{p+1}$ and $P_{q+1}$ be three vertex-disjoint paths, where $\min\{p, l, q\} \geq 1$ and at most one of them is 1. Identifying the initial vertices and the terminal vertices of $P_{l+1}, P_{p+1}$ and $P_{q+1}$, respectively, the resultant graph is called a $\theta$-graph and is denoted by $\theta(p, l, q)$. By $\mathcal{B}^{**}_n$, we denote the class of all bicyclic graphs that have a $\theta$-graph as an induced subgraph.

![Figure 3.1: $\infty$-graph and $\theta$-graph](image)

Thus the class $\mathcal{B}_n$, of bicyclic graphs can be partitioned into two classes: the class of graphs which contain an $\infty$-graph as an induced subgraph and the class of graphs which contain a $\theta$-graph as an induced subgraph i.e., $\mathcal{B}_n = \mathcal{B}^*_n \cup \mathcal{B}^{**}_n$.

**Definition 3.1.2.** A bicyclic graph $G$ which is either a $\theta$-graph or obtained by attaching some pendent vertices to a $\theta$-graph is called an *elementary* $\theta$-graph.

We will use the following well-known results in computing the nullity of a graph.

**Theorem 3.1.3.** [16] Let $v$ be a pendent vertex of a graph $G$ and $u$ be the vertex in $G$ adjacent to $v$. Then, $\eta(G) = \eta(G - u - v)$, where $G - u - v$ is the induced subgraph of $G$ obtained by deleting $u$ and $v$. 

30
Chapter 3

Singularity of $\theta$-graphs

**Theorem 3.1.4.** [15] A path with four vertices of valency 2 in a graph $G$ can be replaced by an edge [see Figure 3.2] without changing the value of $\eta(G)$.

![Figure 3.2:](image)

**Theorem 3.1.5.** [15] Let $G_1$ and $G_2$ be two bipartite graphs. If $\eta(G_1) = 0$, and if the graph $G$ is obtained by joining an arbitrary vertex of $G_1$ by an edge with an arbitrary vertex of $G_2$, then the relation $\eta(G) = \eta(G_2)$ holds.

**Theorem 3.1.6.** [15] Let $G$ be a bipartite graph in which there does not exist any cycle of length $q \equiv 0 \pmod{4}$, then $\eta(G) = n - 2q$, where $q$ is maximum number mutually nonadjacent edges in $G$.

**Definition 3.1.7.** [49] Let $V(G)$ and $E(G)$ denote the vertex set $\{v_1, v_2, \ldots, v_n\}$ and the edge set of a graph $G$, respectively. The neighborhood of a vertex $v \in V$ in $G$ is defined to be $N(v) = \{u \in V(G) \mid uv \in E(G)\}$. A nonzero vector $(\alpha_1, \alpha_2, \ldots, \alpha_n)^t$ is a null-eigenvector of $G$ if and only if for each $v_i \in V(G)$ we have $\sum_{v_j \in N(v_i)} \alpha_j = 0$. Let $A(G) = [C_1, C_2, \ldots, C_n]$, where $C_j$ is the $j$th column.
vector of $A(G)$. If $G$ is singular and $(\alpha_1, \alpha_2, \ldots, \alpha_n)^t$ is a null-eigenvector of $A(G)$, then the relation

$$\alpha_1C_1 + \alpha_2C_2 + \cdots + \alpha_nC_n = 0$$

is called a kernel relation of $G$.

**Definition 3.1.8.** A subset $A$ of a vector space is said to be minimal dependent set if

(a) $A$ is dependent

(b) any proper subset of $A$ is linearly independent.

**Definition 3.1.9.** [49] A pair $V_1, V_2$ of subsets of $V(G)$ is said to satisfy the property (N) if (a) $V_1$ and $V_2$ are nonempty and disjoint, and (b) $\bigcup\{N(v) \mid v \in V_1\} = \bigcup\{N(v) \mid v \in V_2\}$. Further, such a pair is said to be minimal satisfying the property (N) if for any pair $U_1, U_2$ of $V(G)$ satisfying the property (N) with $U_1 \subseteq V_1, U_2 \subseteq V_2$, we have $U_1 = V_1, U_2 = V_2$.

**Theorem 3.1.10.** [49] Let $G$ be a connected graph on $n \geq 2$ vertices. If $G$ is singular, then $V(G)$ has a pair of subsets satisfying the property (N).

**Definition 3.1.11.** [49] A pair $V_1, V_2$ of subsets of $V(G)$ is said to satisfy the property (S) if it satisfies the property (N) and for all pairs $u, v$ in $V_i, i = 1, 2$, we have $N(u) \cap N(v) = \emptyset$. 

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32
Theorem 3.1.12. [49] If $V(G)$ has a pair of subsets $V_1$ and $V_2$ satisfying the property (S), then $G$ is singular.

Theorem 3.1.13. [49] Let $T$ be a nontrivial tree. Then, the following statements are equivalent.

(a) $T$ is singular.

(b) There exist subsets $V_1$ and $V_2$ of $V(T)$ satisfying the property (N).

(c) There exist subsets $V_1$ and $V_2$ of $V(T)$ satisfying the property (S).

Theorem 3.1.14. [49] A unicyclic graph $G$ is singular if and only if there is a pair of subsets $V_1$ and $V_2$ of $V(G)$ satisfying the property (N).

Definition 3.1.15. [49] An elementary unicyclic graph is a graph $G$ which is either a cycle or is obtained by attaching some pendent vertices to a cycle. An outer matching of a unicyclic graph $G$ which is not elementary is a matching $M_0$ in $G$ such that $G - V(M_0)$ is the disjoint union of an elementary unicyclic graph and a set of isolated vertices (possibly empty).

Proposition 3.1.16. [50] Let $G$ be an elementary unicyclic graph on $n$ vertices having a pendant. Then $\eta(G) = n - 2q$, where $q$ is the maximum number of mutually nonadjacent edges in $G$.

Theorem 3.1.17. [50] A unicyclic graph $G$ is singular if and only if one of the following holds:
(a) \(G\) is singular elementary.

(b) \(G\) is obtained from a singular elementary unicyclic graph \(G_0\) by attaching trees at vertices of \(G_0\) such that the graph \(G - V(G_0)\) has a perfect matching.

(c) There exists a tree \(T_v\) attached at a vertex \(u\) of the cycle with \(uv\) as the attaching edge such that none of \(T_v\) and \(T_v - v\) has a perfect matching.

Theorem 3.1.10 gives a necessary condition for \(G\) to be singular. Theorem 3.1.13 and Theorem 3.1.14 shows that this necessary condition is also sufficient for unicyclic and acyclic graphs. In general, this condition is not sufficient. For example, consider the graph \(\infty(3, 3, 3)\) [see Figure 3.3] on the vertex set \(\{1, 2, 3, 4, 5, 6, 7, 8\}\). Then \(V_1 = \{1, 2, 5, 6\}\), \(V_2 = \{3, 4, 7, 8\}\) is a minimal pair in \(\infty(3, 3, 3)\) satisfying the property (N), though \(\infty(3, 3, 3)\) is nonsingular.

![Figure 3.3: \(\infty(3, 3, 3)\)](image)

In section 2 of this chapter, we derive a necessary and sufficient condition for a graph to be singular. We also prove two results which will be useful to find the nullity of a graph. In section 3, we show how this characterization can be used to find the nullity of a graph in \(B_n^{**}\).  

34
Chapter 3 Singularity of $\theta$-graphs

3.2 Necessary and sufficient condition for a graph to be singular

By $A[n]$ we denote the multiset obtained by taking $n$ copies of each element of the set $A$. By $A[n] \cup B[m]$ we mean the multiset obtained by taking $n$ copies of each element of the set $A$ and $m$ copies of each element of the set $B$. Clearly $A[1] \cup B[1] = A \cup B$, if and only if $A$ and $B$ are disjoint.

**Definition 3.2.1.** A pair of subsets $V_1 = \{v_i \mid i = 1, 2, \ldots, l\}$ and $V_2 = \{v_i \mid i = l+1, l+2, \ldots, k\}$ of $V(G)$ is said to satisfy the property (NS) if (a) $V_1$ and $V_2$ are nonempty and disjoint, (b) there exist positive integers $\alpha_1, \alpha_2, \ldots, \alpha_l, \beta_{l+1}, \beta_{l+2}, \ldots, \beta_k$ such that $\cup\{N(v_i)[\alpha_i] \mid v_i \in V_1\} = \cup\{N(v_i)[\beta_i] \mid v_i \in V_2\}$. Further, such a pair is said to be minimal satisfying the property (NS) if for any pair $U_1, U_2$ of $V(G)$ satisfying the property (NS) with $U_1 \subseteq V_1$, $U_2 \subseteq V_2$, we have $U_1 = V_1$, $U_2 = V_2$.

Note that a pair $V_1$ and $V_2$ of $V(G)$ satisfying the property (NS) satisfy the property (N). Also a pair $V_1$ and $V_2$ of $V(G)$ satisfying the property (S) satisfy the property (NS).

**Theorem 3.2.2.** A graph $G$ is singular if and only if there exist a minimal pair satisfying the property (NS).

**Proof.** (Proof of the necessary part) Let $G$ be singular, therefore columns of $A(G)$ are linearly dependent. Let $\{C_1, C_2, \cdots, C_l\}$ be minimal dependent set of columns of $A(G)$. There exist non-zero integers $\alpha_1, \alpha_2, \ldots, \alpha_l$ with g.c.d. equal
Section 3.2 Necessary and sufficient condition for a graph to be singular

to 1 such that
\[ \alpha_1 C_1 + \alpha_2 C_2 + \cdots + \alpha_l C_l = 0 \]

Let \( V_1 = \{ v_j \mid \alpha_j > 0 \} \) and \( V_2 = \{ v_j \mid \alpha_j < 0 \} \). Since \( A(G) \) is nonnegative and has no zero columns, \( V_1 \) and \( V_2 \) are nonempty. Clearly, \( V_1 \cap V_2 = \emptyset \), and we have
\[
\sum_{v_j \in V_1} \alpha_j C_j = \sum_{v_j \in V_2} \beta_j C_j, \tag{3.2.1}
\]
where \( \alpha_j = -\beta_j \).

Let
\[ X = \cup \{ N(v_j)[\alpha_j] \mid v_j \in V_1 \} \]

and
\[ Y = \cup \{ N(v_j)[\beta_j] \mid v_j \in V_2 \}. \]

Let \( v_i \in X \) and it appears \( \gamma \) times in \( X \). Therefore there exist
\[ \alpha_i, \alpha_{i+1}, \ldots, \alpha_s \in \{ \alpha_j \mid v_j \in V_1 \} \]
such that \( v_i \in N(v_p) \), where \( p = i, i + 1, \ldots, s; v_i \notin N(v_r) \) for \( r \notin \{ i, i + 1, \ldots, s \} \) and \( \alpha_i + \alpha_{i+1} + \cdots + \alpha_s = \gamma \) since \( G \) is without loops. Therefore \( a_{ip} = 1 \), where \( p = i, i + 1, \ldots, s \); and \( a_{ir} = 0 \) for \( r \notin \{ i, i + 1, \ldots, s \} \). This implies that the \( i \)th entry of the vector \( \sum_{v_j \in V_1} \alpha_j C_j \) is \( \gamma \). In view of (3.2.1), the \( i \)th entry of the vector \( \sum_{v_j \in V_2} \beta_j C_j \) must be \( \gamma \). Consequently, there exist
\[ \beta_j, \beta_{j+1}, \ldots, \beta_t \in \{ \beta_j \mid v_j \in V_2 \} \]
such that \( a_{ip} = 1 \), where \( p = j, j + 1, \ldots, t; a_{ir} = 0 \) for \( r \notin \{ j, j + 1, \ldots, t \} \) and \( \beta_j + \beta_{j+1} + \cdots + \beta_t = \gamma \). Therefore \( v_i \in N(v_p) \), where \( p = j, j + 1, \ldots, t; v_i \notin N(v_r) \) for \( r \notin \{ j, j + 1, \ldots, t \} \), i.e., \( v_i \) appears \( \gamma \) times in \( Y = \cup \{ N(v_j)[\beta_j] \mid v_j \in V_2 \}. \)
Interchanging the role of $X$ and $Y$, we can show that if $v_i$ appears $m$ times in $Y$, then it also appears $m$ times in $X$. Therefore $X = Y$.

\textbf{(Proof of the sufficient part)} Suppose $V(G)$ has a minimal pair $V_1, V_2$ satisfying the property (NS). Let $V_1 = \{v_1, v_2, \ldots, v_l\}$ and $V_2 = \{v_{l+1}, v_{l+2}, \ldots, v_k\}$. Therefore there exist positive integers $\alpha_1, \alpha_2, \ldots, \alpha_l, \beta_{l+1}, \beta_{l+2}, \ldots, \beta_k$ such that

$$\bigcup \{N(v_i)[\alpha_i] \mid v_i \in V_1\} = \bigcup \{N(v_i)[\beta_i] \mid v_i \in V_2\}.$$ 

Now $v_i$ appears $\gamma$ times in $\bigcup \{N(v_i)[\alpha_i] \mid v_i \in V_1\}$ if and only if it appears $\gamma$ times in $\bigcup \{N(v_i)[\beta_i] \mid v_i \in V_2\}$. Therefore,

$$\sum_{v_j \in V_1} \alpha_j C_j = \sum_{v_j \in V_2} \beta_j C_j,$$

which shows that the columns of $A(G)$ are linearly dependent.

\textbf{Corollary 3.2.3.} Let $V_1, V_2$ be a pair in $V(G)$ satisfying the property (NS). Let $x_j$ be defined by

$$x_j = \begin{cases} 
\alpha_j & \text{if } v_j \in V_1, \\
-\beta_j & \text{if } v_j \in V_2, \\
0 & \text{otherwise.}
\end{cases}$$

Then $(x_1, x_2, \ldots, x_n)^t$ is a null-eigenvector of $G$.

\textbf{Example 3.2.4.} For the graph $G$, [see Figure 3.4] $V_1 = \{1, 5, 9, 13\}$ and $V_2 = \{2, 3, 7\}$. 

\textbf{Chapter 3} Singularity of $\theta$-graphs

37
Section 3.2 Necessary and sufficient condition for a graph to be singular

{3, 7, 11} is a pair satisfying property (NS). Since

\[ N(1)[1] \cup N(5)[1] \cup N(9)[1] \cup N(13)[1] = N(3)[1] \cup N(7)[2] \cup N(11)[1], \]

therefore \( G \) is singular. Also, we see that

\[(1, 0, -1, 0, 1, 0, -2, 0, 1, 0, -1, 0, 1)^T\]

is a null eigenvector of \( G \).

Before ending this section, we prove two results which will be useful for the next section.

**Lemma 3.2.5.** Let \( G \) be a singular bipartite graph with bipartition \( V', V'' \). If \( V_1, V_2 \) is a minimal pair satisfying property (NS), then \( V_1 \cup V_2 \subseteq V' \) or \( V_1 \cup V_2 \subseteq V'' \).

**Proof.** The vertices of \( G \) can be labeled so that adjacency matrix takes the form

\[
A = \begin{pmatrix}
0 & B \\
B^T & 0
\end{pmatrix}.
\]
Let \( \begin{pmatrix} x' \\ x'' \end{pmatrix} \) be the kernel eigenvector of \( G \) corresponding to the minimal pair \( V_1, V_2 \). If \( x' \neq 0 \) and \( x'' \neq 0 \), then \( \begin{pmatrix} x' \\ 0 \end{pmatrix} \) and \( \begin{pmatrix} 0 \\ x'' \end{pmatrix} \) are also kernel eigenvectors of \( G \) which are linearly independent of \( \begin{pmatrix} x' \\ x'' \end{pmatrix} \). Therefore \( V_1, V_2 \) is not a minimal pair for \( G \). Thus either \( x' = 0 \) or \( x'' = 0 \). Without loss of generality let \( x'' = 0 \), therefore \( V_1, V_2 \subseteq V' \).

\[ \text{Theorem 3.2.6.} \] Let \( G \) be a singular graph with a minimal pair \( (V_1, V_2) \) satisfying property (NS). If \( v_1 \in V_1 \cup V_2 \) and \( G - v_1 \) is the induced subgraph of \( G \) obtained by deleting \( v_1 \), then \( \eta(G) = \eta(G - v_1) + 1 \).

\[ \text{Proof.} \] Let \( V(G) = \{v_1, v_2, \ldots, v_n\} \). Without loss of generality assume \( V_1 = \{v_1, v_2, \ldots, v_k\} \) and \( V_2 = \{v_{k+1}, v_{k+2}, \ldots, v_m\} \). Therefore there exist non zero real number \( \alpha_i \), where \( i = 1, 2, 3, \ldots, m \) such that

\[ \cup \{N(v_i)[\alpha_i] \mid v_i \in V_1\} = \cup \{N(v_i)[\alpha_j] \mid v_i \in V_2\}. \]

Also \( A(G) \) has the following form

\[
A(G) = \begin{pmatrix}
0 & a_{12} & a_{13} & \cdots & a_{1m} & \cdots & a_{1n} \\
a_{12} & 0 & a_{23} & \cdots & a_{2m} & \cdots & a_{2n} \\
a_{13} & a_{23} & 0 & \cdots & a_{3m} & \cdots & a_{3n} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
a_{1m} & a_{2m} & a_{3m} & \cdots & 0 & \cdots & a_{mn} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & A(G - V_1 \cup V_2) \\
a_{1n} & a_{2n} & a_{3n} & \cdots & a_{mn}
\end{pmatrix}
\]

where \( a_{ij} \) are either 0 or 1. Applying the elementary operations \( R_1 \rightarrow \alpha_1 R_1 + \alpha_2 R_2 + \ldots + \alpha_m R_m \) and \( C_1 \rightarrow \alpha_1 C_1 + \alpha_2 C_2 + \ldots + \alpha_m C_m \) to the matrix \( A(G) \),
Section 3.2 Necessary and sufficient condition for a graph to be singular

we see that

$$A(G) \sim \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 & \ldots & 0 \\ 0 & 0 & a_{23} & \ldots & a_{2m} & \ldots & a_{2n} \\ 0 & a_{23} & 0 & \ldots & a_{3m} & \ldots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & a_{2m} & a_{3m} & \ldots & 0 & \ldots & a_{mn} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & a_{2n} & a_{3n} & \ldots & a_{mn} & \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 \\ A(G - v_1) \end{pmatrix}$$

Thus $\eta(G) = 1 + \eta(G - v_1)$.

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\[G\]

\[G - 8\]

Figure 3.5:
Example 3.2.7. The graph $G$ in Figure 3.5 is singular. Note that $V_1 = \{9, 8\}$ and $V_2 = \{2, 6\}$ is a minimal pair satisfying property(NS). Thus $\eta(G) = 1 + \eta(G - 8)$.

Also $U_1 = \{9, 1\}$ and $U_2 = \{3, 7\}$ is a minimal pair satisfying property(NS).

Therefore $\eta(G - 8) \geq 1$. Since $\eta(G - 8 - 3) = 0$, therefore $\eta(G) = 2$.

3.3 Singularity of a graph in $B_{**}^n$

Proposition 3.3.1. Let $\theta(p, l, q)$ be a $\theta$-graph where $p = l = q \equiv 0 \pmod{2}$, then

$$
\eta(\theta(p, l, q)) = \begin{cases} 
3 & \text{if } p = l = q \equiv 2 \pmod{4} \text{ or } p = l = q \equiv 0 \pmod{4}, \\
1 & \text{if } p = l \equiv 2 \pmod{4}, q \equiv 0 \pmod{4} \\
& \text{or } p = l \equiv 0 \pmod{4}, q \equiv 2 \pmod{4}.
\end{cases}
$$

Proof. By Theorem 3.1.4, we have

$$
\eta(\theta(p, l, q)) = \begin{cases} 
\eta(\theta(2, 2, 2)) & \text{if } p = l = q \equiv 2 \pmod{4}, \\
\eta(\theta(4, 4, 4)) & \text{if } p = l = q \equiv 4 \pmod{4}, \\
\eta(\theta(2, 2, 4)) & \text{if } p = l \equiv 2 \pmod{4}, q \equiv 4 \pmod{4}, \\
\eta(\theta(4, 4, 2)) & \text{if } p = l \equiv 4 \pmod{4}, q \equiv 2 \pmod{4}.
\end{cases}
$$

![Figure 3.6:](image_url)
Now \(\{v_0, v_1\}\) is a minimal pair in \(\theta(2, 2, 2)\) [see Figure 3.6] satisfying property (S). Therefore \(\theta(2, 2, 2)\) is singular. By Theorem 3.2.6, \(\eta(\theta(2, 2, 2)) = 1 + \eta(\theta(2, 2, 2) - v_0) = 1 + \eta(C_4) = 3\). Similarly \(\eta(\theta(4, 4, 4)) = 3\).

Again \(\{u_0, u_1\}\) is a minimal pair in \(\theta(2, 2, 4)\) [see Figure 3.6] satisfying property (S). Therefore \(\theta(2, 2, 4)\) is singular. By Theorem 3.2.6, \(\eta(\theta(2, 2, 4)) = 1 + \eta(\theta(2, 2, 4) - u_0) = 1 + \eta(C_6) = 1\). Similarly \(\eta(\theta(4, 4, 2)) = 1\). Thus the result follows.

**Proposition 3.3.2.** Let \(\theta(p, l, q)\) be a \(\theta\)-graph where \(p = l = q \equiv 1 \pmod{2}\), then

\[
\eta(\theta(p, l, q)) = 0
\]

\[\begin{align*}
\theta(5, 5, 1) & \quad v_0 \\
\theta(5, 5, 1) & \quad v_2 \quad v_4 \quad v_9 \\
G & \quad v_8 \quad v_6
\end{align*}\]

**Figure 3.7:**

**Proof.** By Theorem 3.1.4, we have

\[
\eta(\theta(p, l, q)) = \begin{cases} 
\eta(\theta(5, 5, 1)) & \text{if } p = l = q \equiv 1 \pmod{4}, \\
\eta(\theta(3, 3, 3)) & \text{if } p = l = q \equiv 3 \pmod{4}, \\
\eta(\theta(5, 1, 3)) & \text{if } p = l \equiv 1 \pmod{4}, q \equiv 3 \pmod{4}, \\
\eta(\theta(3, 3, 1)) & \text{if } p = l \equiv 3 \pmod{4}, q \equiv 1 \pmod{4}.
\end{cases}
\]

Consider the graph \(G\) which is obtained from \(\theta(5, 5, 1)\) by attaching a single pendant vertex [see Figure 3.7]. By Theorem 3.1.3, \(G\) is singular and \(\eta(G) = 1\). Also \(\{v_0, v_4, v_6\}, \{v_2, v_8, v_9\}\) is a minimal pair satisfying property (S). By Theorem 3.2.6,

\[
\eta(G) = 1 + \eta(G - v_9) = 1 + \eta(\theta(5, 5, 1))
\]
and therefore, \( \eta(\theta(5, 5, 1)) = 0 \). Similarly we can show that

\[
\eta(\theta(3, 3, 3)) = 0 = \eta(\theta(5, 1, 3)) = \eta(\theta(3, 3, 1)).
\]

Thus the result follows.

**Proposition 3.3.3.** If \( \theta(p, l, q) \) is a \( \theta \)-graph where \( p, l \) are even and \( q \) is odd, then

\[
\eta(\theta(p, l, q)) = \begin{cases} 
0 & \text{if } p + l \not\equiv 0 \pmod{4}, \\
1 & \text{if } p + l \equiv 0 \pmod{4}.
\end{cases}
\]

![Figure 3.8:](image)

**Proof.** Let \( p = l \equiv 2 \pmod{4} \) and \( q \equiv 1 \pmod{4} \). By Theorem 3.1.4, we have

\[
\eta(\theta(p, l, q)) = \begin{cases} 
\eta(\theta(2, 2, 1)) & \text{if } q \equiv 1 \pmod{4}, \\
\eta(\theta(2, 2, 3)) & \text{if } q \equiv 3 \pmod{4}, \\
\eta(\theta(4, 4, 1)) & \text{if } q \equiv 1 \pmod{4}, \\
\eta(\theta(4, 4, 3)) & \text{if } q \equiv 3 \pmod{4}.
\end{cases}
\]

Also \( \{v_0\}, \{v_2\} \) is a minimal pair in \( \theta(2, 2, 1) \) [see Figure 3.8] satisfying property (S). By Theorem 3.2.6, \( \eta(\theta(2, 2, 1)) = 1 + \eta(\theta(2, 2, 1) - v_0) = 1 \). Similarly considering other cases we can show that \( \eta(\theta(p, l, q)) = 1 \).
Again let, \( p + l \not\equiv 0 \pmod{4} \), therefore either \( p \equiv 2 \pmod{4}, l \equiv 0 \pmod{4} \) or \( p \equiv 0 \pmod{4}, l \equiv 2 \pmod{4} \). Suppose \( p \equiv 2 \pmod{4} \) and \( l \equiv 0 \pmod{4} \). By Theorem 3.1.4, we have

\[
\eta(\theta(p, l, q)) = \begin{cases} 
\eta(\theta(2, 4, 3)) & \text{if } q \equiv 3 \pmod{4}, \\
\eta(\theta(2, 4, 1)) & \text{if } q \equiv 1 \pmod{4}.
\end{cases}
\]

Consider the graph \( G \) of Figure 3.8. Then \( \{v_0, v_3\}, \{v_1, v_2\} \) is minimal pair satisfying property (NS). Therefore \( G \) is singular. By Theorem 3.2.6,

\[
\eta(G) = 1 + \eta(G - v_3) = 1 + \eta(\theta(2, 4, 3)).
\]

Also \( \eta(G) = 1 + \eta(G - v_2) = 1 \), therefore, \( \eta(\theta(2, 4, 3)) = 0 \). Similarly we can show that \( \eta(\theta(2, 4, 1)) = 0 \). Thus the result follows.

**Proposition 3.3.4.** If \( \theta(p, l, q) \) is a \( \theta \)-graph where \( p, l \) are odd and \( q \) is even, then

\[
\eta(\theta(p, l, q)) = \begin{cases} 
0 & \text{if } p + l \not\equiv 0 \pmod{4}, \\
1 & \text{if } p + l \equiv 0 \pmod{4}.
\end{cases}
\]
Chapter 3  
Singularity of $\theta$-graphs

**Proof.** Let $p + l \not\equiv 0 \pmod{4}$. Therefore either $p = l \equiv 1 \pmod{4}$ or $p = l \equiv 3 \pmod{4}$. Let $p = l \equiv 1 \pmod{4}$. By Theorem 3.1.4, we have

$$\eta(\theta(p, l, q)) = \begin{cases} 
\eta(\theta(1, 5, 2)) & \text{if } q \equiv 2 \pmod{4}, \\
\eta(\theta(1, 5, 4)) & \text{if } q \equiv 0 \pmod{4}.
\end{cases}$$

Consider the graph $G$ of Figure 3.9. Then $G - v_4 = \theta(1, 5, 2)$. Now $(\{v_0, v_4\}, \{v_2, v_6\})$ is a minimal pair satisfying property (NS). Therefore $G$ is singular. By Theorem 3.2.6,

$$\eta(G) = 1 + \eta(G - v_4) = 1 + \eta(\theta(1, 5, 2)).$$

Also by Theorem 3.2.6, $\eta(G) = 1 + \eta(\infty(3, 0, 3))$. Since $\infty(3, 0, 3)$ is nonsingular, therefore $\eta(\theta(1, 5, 2)) = 0$. Thus $\eta(\theta(p, l, q)) = 0$ if $p = l \equiv 1 \pmod{4}$. Similarly, we can show that, $\eta(\theta(p, l, q)) = 0$ if $p = l \equiv 3 \pmod{4}$.

Let $p + l \equiv 0 \pmod{4}$. So let $p \equiv 1 \pmod{4}, l \equiv 3 \pmod{4}$. By Theorem 3.1.4, we have

$$\eta(\theta(p, l, q)) = \begin{cases} 
\eta(\theta(1, 3, 2)) & \text{if } q \equiv 2 \pmod{4}, \\
\eta(\theta(1, 3, 4)) & \text{if } q \equiv 0 \pmod{4}.
\end{cases}$$

Now $(\{v_0, v_1\}, \{v_2, v_3\})$ is a minimal pair in $\theta(1, 3, 2)$ [see Figure 3.10] satisfying property (NS). Therefore $\theta(1, 3, 2)$ is singular. Also by Theorem 3.2.6, $\eta(\theta(1, 3, 2)) = \eta(\theta(1, 3, 2) - v_0) = 1$. Similarly, we can show that $\eta(\theta(1, 3, 4) = 1$.

Thus $\eta(\theta(p, l, q)) = 1$, if $p + l \equiv 0 \pmod{4}$ and $q$ is even .
Singularity of elementary $\theta$-graph: Let $G_0$ be an elementary $\theta$-graph with pendent vertices. Let $v$ be a pendent vertex in $G$ attached at $u$ of $G_0$. Then by Theorem 3.1.3, $\eta(G_0) = \eta(G_0 - uv)$. Since $G_0 - uv$ is disjoint union of a tree or a unicyclic graph and a set of isolated vertices (possibly empty), we can find the nullity of $G_0 - uv$.

Definition 3.3.5. A matching $M_0$ in a graph $G$ in $B_n^{**}$ is called an outer matching in $G$ if $G - V(M_0)$ is the disjoint union of an elementary $\theta$-graph and a set of isolated vertices (possibly empty). (Note that $M_0 = \emptyset$, if $G$ is elementary.)

Remark 3.3.6. If $G$ is a graph in $B_n^{**}$ which is not elementary, then we construct an outer matching $M_0$ as follows. Let $u_1$ be a (pendent) vertex which is at a maximum distance from $\theta(p, l, q)$ in $G$ and $v_1$ the vertex adjacent to $u_1$. Then $v_1$ is not on $\theta(p, l, q)$, since $G$ is not elementary. We choose the edge $e_1 = u_1v_1$ as an edge in $M_0$. Clearly, $G - u_1 - v_1$ is a disjoint union of an elementary $\theta$-graph $G_1$ and a set of isolated vertices (possibly empty). If $G_1$ is not elementary, we can choose another edge for $M_0$ by the same process, and then proceed recursively. The process must terminate and an outer matching $M_0$ of $G$ is obtained.

Example 3.3.7. Consider the graph $G$ given in Figure 3.11. Here the set $M_0$ of edges in bold face in the figure is an outer matching of $G$. The corresponding elementary $\theta$-graph is $G_0$ (depicted in the figure) and the set of isolated vertices of $G - M_0$ is $\{17, 21\}$. 

46
We denote the set of isolated vertices and the elementary component of $G - V(M_0)$ by $\Lambda_0$ and $G_0$, respectively.

**Theorem 3.3.8.** A graph $G$ in $B^{**}_n$ is singular if and only if one of the following holds:

(a) $G$ is singular elementary $\theta$-graph.

(b) $G$ is obtained from a singular elementary $\theta$-graph $G_0$ by attaching trees at vertices of $G_0$ such that the graph $G - V(G_0)$ has a perfect matching.

(c) There exists a tree $T_v$ attached at a vertex $u$ of the $\theta$-graph with $uv$ as the attaching edge such that none of $T_v$ and $T_v - v$ has a perfect matching.

**Proof.** Suppose $G$ is not elementary and choose an outer matching $M_0$ of $G$. Let $G - V(M_0)$ be the disjoint union of the elementary $\theta$-graph $G_0$ and a set $\Lambda_0$ of isolated vertices (possibly empty). We note that $G$ is obtained by attaching trees at the vertices of $G_0$. In view of Theorem 3.1.3, we have $\eta(G) = \eta(G_0) + |\Lambda_0|$. Therefore, $G$ is singular if and only if either $\Lambda_0 \neq \emptyset$ or $G_0$ is singular. If $\Lambda_0 = \emptyset$, then $G - V(G_0)$ has a perfect matching, and therefore $G$ is singular if and only
if (b) holds. Suppose $\Lambda_0 \neq \emptyset$ and $w \in \Lambda_0$. Let $T_v$ be a tree in $G$, attached at a vertex $u$ of the $\theta$-graph with $uv$ as the attaching edge, of which $w$ is a vertex. Since $w \in \Lambda_0$, $T_v$ does not have a perfect matching. Moreover, if $T_v - v$ has a perfect matching, then $v$ is a vertex of $G_0$. In that case, $w$ is a vertex of $T_v - v$ and therefore is in $V(M_0)$. Since this is not the case, therefore (c) holds.

**Corollary 3.3.9.** If $G$ is a graph in $\mathcal{B}_n^{**}$ which is not a $\theta$-graph then

$$\eta(G) = \eta(G_0) + |\Lambda_0|$$

**Example 3.3.10.** The graph $G$ in Figure 3.11 has nullity, $\eta(G) = 2 + \eta(G_0) = 4$, since $\eta(G_0) = 2$. 

48