Chapter 2

Singularity of graphs

In this chapter we derive some results regarding the singularity of graphs.

2.1 Introduction

Let $G$ be a simple graph. The multiplicity of 0 as an eigenvalue of $A(G)$ is the nullity of $G$ and is denoted by $\eta(G)$. For a singular graph $G$, the eigenvectors of $A(G)$ corresponding to the eigenvalue 0 are the null-eigenvectors of $G$ and the null-space of $A(G)$ is the null-space of $G$.

We will use the following well-known results in computing the nullity of a graph.

**Theorem 2.1.1.** [16] Let $v$ be a pendent vertex of a graph $G$ and $u$ be the vertex in $G$ adjacent to $v$. Then $\eta(G) = \eta(G - u - v)$, where $G - u - v$ is the induced subgraph of $G$ obtained by deleting $u$ and $v$.

**Theorem 2.1.2.** [52] Let $G$ be the graph obtained by joining the vertex $x$ of a graph $G_1$ to the vertex $y$ of a graph $G_2$ by an edge. Then

$$\det A(G) = \det A(G_1) \det A(G_2) - \det A(G_1 - x) \det A(G_2 - y).$$
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**Theorem 2.1.3.** [52] Let $P_6[1, 2, 3, 4, 5, 6]$ be an induced subgraph of $G$ with $\text{deg}(2) = \text{deg}(3) = \text{deg}(4) = \text{deg}(5) = 2$. If $H$ is the graph formed from $G - \{2, 3, 4, 5\}$ by joining vertices 1 and 6 with an edge, then $\det A(G) = \det A(H)$.

**Corollary 2.1.4.** [52] Let $C_4[1, 2, 3, 4, 1]$ be a subgraph of $G$, where $\text{deg}(1) = 2$. If $G_0$ is the graph obtained from $G$ by removing the edges $[2, 3]$ and $[3, 4]$, then $\det A(G_0) = \det A(G)$.

### 2.2 Some useful results regarding the singularity of a graph

Let $V(G)$ and $E(G)$ denote the vertex set $\{v_1, v_2, \ldots, v_n\}$ and the edge set of a graph $G$, respectively. The *neighborhood* of a vertex $v \in V$ in $G$ is defined to be $N(v) = \{u \in V(G) | uv \in E(G)\}$. A nonzero vector $(\alpha_1, \alpha_2, \ldots, \alpha_n)^t$ is a null-eigenvector of $G$ if and only if for each $v_i \in V(G)$ we have $\sum_{v_j \in N(v_i)} \alpha_j = 0$.

**Definition 2.2.1.** [49] A pair $V_1, V_2$ of subsets of $V(G)$ is said to satisfy property $(N)$ if

(a) $V_1$ and $V_2$ are nonempty and disjoint, and

(b) $\bigcup_{v \in V_1} N(v) = \bigcup_{v \in V_1} N(v)$.

Further, such a pair is said to be *minimal satisfying the property $(N)$* if for any pair $U_1, U_2$ of subsets of $V(G)$ satisfying the property $(N)$ with $U_1 \subseteq V_1$, $U_2 \subseteq V_2$, $\bigcup_{v \in U_1} N(v) = \bigcup_{v \in U_1} N(v)$. 
we have $U_1 = V_1$, $U_2 = V_2$.

$\begin{array}{c}
\text{Figure 2.1: } C_4 \\
\end{array}$

**Example 2.2.2.** For the cycle $C_4$ in Figure 2.1, $V_1 = \{1, 2\}, V_2 = \{3, 4\}$ is a pair satisfying property (N). But $V_1, V_2$ is not a minimal pair as $U_1 = \{2\}, U_2 = \{4\}$ is a pair with property (N) and $U_1 \subset V_1, U_2 \subset V_2$.

**Theorem 2.2.3.** [49] Let $G$ be a connected graph on $n \geq 2$ vertices. If $G$ is singular, then $V(G)$ has a pair of subsets satisfying the property (N).

**Theorem 2.2.4.** [49] A unicyclic graph $G$ is singular if and only if there is a pair of subsets $V_1, V_2$ of $V(G)$ satisfying the property (N).

**Definition 2.2.5.** [49] A pair $V_1, V_2$ of subsets of $V(G)$ is said to satisfy property (S) if it satisfies the property (N) and for all pairs $u, v$ in $V_i$, $i = 1, 2$, we have $N(u) \cap N(v) = \emptyset$.

**Example 2.2.6.** In Figure 2.1, the cycle $C_4$ has $U_1 = \{2\}, U_2 = \{4\}$ as a pair satisfying property (S).
Theorem 2.2.7. [49] Let $G$ be a graph and suppose that $V(G)$ has a pair of subsets $V_1, V_2$ satisfying the property (S). Then $G$ is singular.

Corollary 2.2.8. [49] Let $G$ be a graph with the pair $V_1, V_2$ of subsets of $V(G)$ satisfying the property (S). Let $\alpha_j$ be defined by

$$
\alpha_j = \begin{cases} 
1 & \text{if } v_j \in V_1, \\
-1 & \text{if } v_j \in V_2, \\
0 & \text{otherwise.}
\end{cases}
$$

Then $(\alpha_1, \alpha_2, \ldots, \alpha_n)^t$ is a null-eigenvector of $G$.

### 2.3 Singularity and determinant of a graph

The following Theorem is one of the main results of this chapter, which gives a sufficient condition for $G$ to be singular.

Theorem 2.3.1. Let $G$ be a graph with a nonempty subset $V_1$ of $V(G)$, such that

$$\left| \bigcup_{v \in V_1} N(v) \right| \leq |V_1| - 1.$$ 

Then $G$ is singular.

Proof. Let $G$ be a graph with a nonempty subset $V_1$ of $V(G)$, such that

$$\left| \bigcup_{v \in V_1} N(v) \right| \leq |V_1| - 1.$$
Let \( V_1 = \{v_1, v_2, \ldots, v_p\} \). Consider the equation \( \sum_{i=1}^{p} \alpha_i R_i = 0 \), which is equivalent to the system of \( n \) equations
\[
\sum_{i=1}^{p} \alpha_i R_{ij} = 0, \quad j = 1, 2, \ldots, n.
\]
Since at least \( n - p + 1 \) vertices are absent in \( \bigcup_{v \in V_1} N(v) \), so at least that many equations in 2.3.1 take the form
\[
\sum_{i=1}^{p} \alpha_i 0 = 0,
\]
which can be omitted. Thus we are left with at most \( p - 1 \) homogeneous equations in \( p \) variables, which have a nonzero solution. As a consequence, the rows of \( A(G) \) are linearly dependent, implying that \( G \) is singular.

\[\Box\]

**Example 2.3.2.** The graph \( G_1(V_1, E_1) \) in the Figure 2.2, is singular. Since the subset \( U = \{7, 8, 9, 1, 2\} \) of \( V_1 \) is such that \( |U| = 5 \) and
\[
\left| \bigcup \{N(i) \mid i \in \{7, 8, 9, 1, 2\}\} \right| = 4.
\]
We can see that \( G' \) is singular, where vertex of set \( G' \) is \( V' = V_1 \cup V_2 \) and edge set of \( G' \), \( E' = E_1 \cup E_2 \cup E_3 \) where \( G_2(V_2, E_2) \) is any graph and \( E_3 \subseteq \{uv \mid u \in \{3, 4, 5, 6\}, v \in V_2\} \).

**Corollary 2.3.3.** Let \( G \) be a graph of order \( n \). If there exists a subset \( U \) of \( V \), the vertex set of \( G \), such that \( U \) is a vertex independent set and \( |U| > \frac{n}{2} \), then \( G \) is singular.
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Figure 2.2: A singular graph which satisfies the condition of Theorem 2.3.1

**Proof.** Since $U$ is vertex independent set, therefore

$$|\bigcup \{N(v) \mid v \in U\}| \leq n - |U|$$

$$< n - \frac{n}{2}$$

$$= \frac{n}{2}$$

$$= |U|.$$ 

Hence $G$ is singular.

**Corollary 2.3.4.** Let $G$ be a bipartite graph with bipartition $V_1, V_2$, such that $|V_1| \neq |V_2|$. Then $G$ is singular.

**Corollary 2.3.5.** Let $G$ be a bipartite graph with bipartition $(\Omega, \Phi)$ and $A \subseteq \Omega$, if there exists no matching in $G$ that covers the vertices in $A$ then $G$ is singular.
Proof. By Hall’s theorem, there exists at least one subset \( B \) of \( A \) such that \( |N(B)| < |B| \), where \( N(B) \) is the set of all vertices in \( \Omega \) adjacent to a vertex in \( B \). Therefore \( G \) is singular. ■

**Corollary 2.3.6.** If in \( G \) there exists a subset \( V_1 \) of \( V \) such that

\[
\left| \bigcup_{v \in V_1} N(v) \right| = |V_1| - \lambda,
\]

then \( m(0) \geq \lambda \) where \( m(0) \) denotes the multiplicity of zero as an eigenvalue of \( G \).

**Proof.** Since \( \left| \bigcup_{v \in V_1} N(v) \right| = |V_1| - \lambda \), by Theorem 2.3.1, \( G \) is singular. To get the highest order non vanishing minor of \( A(G) \), we shall have to remove at least \( \lambda \) rows out of those represented by \( v \) in \( V_1 \). Therefore, \( \text{rank}(G) \leq n - \lambda \) and so \( m(0) \geq \lambda \). ■

**Corollary 2.3.7.** If in \( G \) there exists an induced subgraph \( X(m, f) \), such that there are no disjoint subsets \( V_1, V_2 \) of \( V \) with \( \bigcup_{v \in V_1} N(v) = \bigcup_{v \in V_2} N(v) \), then \( m(0) \leq n - m \).

**Proof.** Let \( V(X) = \{v_1, v_2, \ldots, v_m\} \). The given condition and Theorem 2.3.1 together imply that \( X \) is nonsingular. Now out of the rows of \( A(G) \), at least \( m \) rows are linearly independent. Therefore \( m(0) \leq n - m \). ■

**Theorem 2.3.8.** If there exists a subset \( U \) of \( V_1 \) in \( G_1 \) such that

\[
|U| = \left| \bigcup_{v \in U} N(v) \right|,
\]

then \( \det G' = \det G_1 \det G_2 \) where \( G' \) is a graph obtained by joining a vertex \( x \) in

\[
\left( \bigcup_{v \in U} N(v) \right) - U
\]
with any vertex y in any graph G₂.

Proof. By Theorem 2.1.2, we have

\[ \det G' = \det G₁ \det G₂ - \det(G₁ - x) \det(G₂ - y). \]

Since \( x \in \left( \bigcup_{v \in U} N(v) \right) - U \) we will have a nonempty subclass \( U' \) of \( V' \), the vertex set of \( G₁ - x \), such that \( |U'| > |\bigcup_{v \in U} N(v)| \). Therefore \( \det(G₁ - x) = 0 \) and hence \( \det G' = \det G₁ \det G₂ \). □

Theorem 2.3.9. Let \( P₆ \) be an induced subgraph of \( G \) with \( \text{deg}(2) = \text{deg}(3) = \text{deg}(4) = \text{deg}(5) = 2 \). If there exists a subset \( U \) of \( V \) such that \( 2, 3, 4, 5 \notin U, 1 \in U, 6 \in \bigcup_{i \in U} N(i) \) and \( |U| = \left| \bigcup_{i \in U} N(i) \right| \), then \( G \) is singular.

Proof. Let \( H \) be a graph formed from \( G - \{2, 3, 4, 5\} \) by joining 1 and 6 by an edge. By Theorem 2.1.3, \( \det G = \det H \). Suppose \( N(1) \in S, 6 \in \bigcup_{i=1}^p N(i) \). Since \( \text{deg}(2) = \text{deg}(3) = \text{deg}(4) = \text{deg}(5) = 2 \) and \( N(1) \in S \), therefore \( 2 \in \bigcup_{i=1}^p N(i) \) in \( G \).

Again in \( H, 2 \notin \bigcup_{i=1}^p N(i) \) and 2 is replaced by 6. But, since 6 already exists in \( \bigcup_{i=1}^p N(i) \) in \( G \), so for \( H \) we get a nonempty subclass \( S' \) of \( N(H) \) such that \( |S'| > \left| \bigcup_{N(i) \in S'} N(i) \right| \).

Therefore \( \det H = 0 \) and so \( G \) is singular. □

Example 2.3.10. The graph in Figure 2.3 with \( U = \{1, 7, 8, 9\} \), satisfies all the conditions of the Theorem 2.3.9 for any induced subgraph \( G₁, G₂ \) and \( G₃ \). Therefore the graph is singular.
Theorem 2.3.11. Let $C_4 = [1, 2, 3, 4, 1]$ be a subgraph of $G$ where $\deg(1) = 2$. If there exists a subclass $S = \{N(1), N(2), \ldots, N(p)\}$ of $N$ such that $3 \notin \bigcup_{i=1}^{p} N(i), i \neq 2, 4; 2, 4 \notin N(i), \forall i \neq 3$ and $N(1) \notin S$ and either $N(2), N(3) \in S$ or $N(3), N(4) \in S$ and also $|S| = \left| \bigcup_{N(i) \in S} N(i) \right| - 2$, then $\det G' = 0$, where $G'$ is a graph obtained from $G$ by removing the edges $[2, 3]$ and $[3, 4]$.

Proof. Let $G'$ be the graph obtained from $G$ by removing the edges $[2, 3]$ and $[3, 4]$. Therefore, by Corollary 2.1.4 $\det G = \det G'$. Suppose $N(2), N(3) \in S$, since $3 \notin \bigcup_{i=1}^{p} N(i), i \neq 2, 4; 2, 4 \notin N(i), \forall i \neq 3$ and $N(1) \notin S$ therefore $2, 3, 4 \in \bigcup_{N(i) \in S} N(i)$ in case of $G$ but $2, 3, 4 \notin \bigcup_{N(i) \in S} N(i)$ in case of $G'$. Therefore we get a nonempty subclass $S'$ of $N(G')$ in $G'$ such that $|S'| > \left| \bigcup_{N(i) \in S'} N(i) \right|$. Therefore $\det G = 0$, $\det G' = 0$. Similarly considering $N(3), N(4) \in S$ we can show $\det G = \det G' = 0$.

Example 2.3.12. The graph $G$ in Figure 2.4 satisfies the conditions of the Theorem.
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Figure 2.4: A graph which satisfies the conditions of Theorem 2.3.11 for any induced subgraph $G_1$ and $G_2$. Therefore $G'$ is singular.

Theorem 2.3.13. If $\det G_1 = 0$ or $\det G_2 = 0$, then $\det G' = 0$.

Theorem 2.3.14. If in a graph $G$ there exist two disjoint subsets $U, W$ of $V$, the vertex set of $G$ such that $\bigcap_{v \in U} N(v) = \bigcap_{v \in W} N(v)$ and $N(v_1) \cup N(v_2) = V$ $\forall v_1, v_2 \in U$, $v_1 \neq v_2$ and $\forall v_1, v_2 \in W$, $v_1 \neq v_2$, then $\det G = 0$, where $G$ is the complement of $G$.

Proof. Let us denote the neighborhood of any vertex $v$ in $G$ by $N'(v)$. Then
$N'(v) = \overline{N(v)}$, where $N(v)$ is the neighborhood of $v$ in $G$. Now

$$\bigcap_{v \in U} N(v) = \bigcap_{v \in W} N(v)$$

$$\Rightarrow \bigcup_{v \in U} N(v) = \bigcup_{v \in W} \overline{N(v)}$$

$$\Rightarrow \bigcup_{v \in U} N'(v) = \bigcup_{v \in W} N'(v).$$

Again

$$N(v_1) \cup N(v_2) = V$$

Which gives $\overline{N(v)} \cap \overline{N(v)} = \phi$ for all $v_1, v_2 \in U$, $v_1 \neq v_2$ and for all $v_1, v_2 \in W$, $v_1 \neq v_2$. Thus $\det \overline{G} = 0$.

**Theorem 2.3.15.** If in a graph $G$ there exist $p$ vertices $v_1, v_2, \ldots, v_p$ such that

$$\left| \bigcap_{i=1}^{p} N(v_i) \right| \geq n - p + 1,$$

then $\det \overline{G} = 0$.

**Proof.** We have

$$\left| \bigcap_{i=1}^{p} N(v_i) \right| \geq n - p + 1$$

$$\Rightarrow \left| \bigcup_{i=1}^{p} \overline{N(v_i)} \right| \leq n - (n - p + 1)$$

$$\Rightarrow \left| \bigcup_{i=1}^{p} N(v_i) \right| \leq (p - 1)$$

$$\Rightarrow \left| \bigcup_{i=1}^{p} N'(v_i) \right| \leq (p - 1).$$

Therefore $\det \overline{G} = 0$. 

$25$
Definition 2.3.16. The tensor product of two graphs $G_1$ and $G_2$, denoted by $G_1 \wedge G_2$, has the vertex set $V = V_1 \times V_2$ and $(u_1, v_1), (u_2, v_2) \in V$ are adjacent in $G_1 \wedge G_2$ if only if $[u_1, u_2] \in E_1$ and $[v_1, v_2] \in E_2$.

Theorem 2.3.17. If $G$ is a graph such that there exists a subclass $S = \{N(v_1), N(v_2), \ldots, N(v_p)\}$ of $N$ such that $|\bigcup_{i=1}^p N(v_i)| < p$, then $\det(G \wedge G_1 \wedge G_2 \wedge \ldots \wedge G_k) = 0$. For any graph $G_i, i = 1, 2, 3, \ldots, k$ the multiplicity of zero as an eigenvalue of $G' = G \wedge G_1 \wedge G_2 \wedge \ldots \wedge G_k$ is at least $\prod_{i=1}^k n_i$ where $n_i$ is the number of vertices in $G_i$.

Proof. Let

$$S = \{N(u, u_1, u_2, \ldots, u_k)|N(u) \in S\}.$$  

Then

$$|S| = p \times n_1 \times n_2 \times \ldots \times n_k.$$  

Also

$$\bigcup_{N(x) \in S} N(x) \subseteq \{(u, u_1, \ldots, u_k), u \in \bigcup_{i=1}^p N(v_i)\}.$$  

But

$$\left|\bigcup_{i=1}^p N(v_i)\right| \leq (p - 1)$$  

$$\Rightarrow \left|\bigcup_{N(x) \in S} N(x)\right| \leq (p - 1) \times n_1 \times n_2 \ldots \times n_k = |S| - \prod_{i=1}^k n_i$$
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Which gives

\[ |S| \geq \left| \bigcup_{N(u) \in S} \right| + \prod_{i=1}^{k} n_i. \]

Therefore \( \det(G \wedge G_1 \wedge G_2 \wedge \ldots \wedge G_k) = 0 \) and \( m(0) \geq \prod_{i=1}^{k} n_i. \)

**Definition 2.3.18.** The total graph \( T(G) \) of a graph \( G \) has the vertex set \( V \cup E \) and two vertices of \( T(G) \) are adjacent if one of the following holds

(a) they are \( v_i, v_j \in V \) and \([v_i, v_j] \in E\),

(b) one is \( v \in V \) and the other \( e \in E \) and \( e \) is incident with vertex \( v \) in \( G \),

(c) they are \( e_i, e_j \in E \) and the edges \( e_i, e_j \) are adjacent in \( G \).

**Theorem 2.3.19.** If \( G \) is singular, then there exist two disjoint subsets \( X \) and \( Y \) of the vertex set of \( T(G) \) such that \( \bigcup_{v \in X} N(v) \) and \( \bigcup_{v \in Y} N(v) \) have some vertices in common and the remaining vertices in each of the unions are adjacent to at least one of the remaining vertices of the other union and are adjacent to exactly one of the common vertices.

**Proof.** Let \( \det G = 0 \). Then there exist two disjoint subsets \( U \) and \( W \) of \( V \), the vertex set of \( G \) such that

\[ \bigcup_{v \in X} N(v) = \bigcup_{v \in Y} N(v) = \Omega \text{ (say)}. \]

Now consider \( X \) and \( Y \) as subsets of vertex set of \( T(G) \) then \( \bigcup_{v \in X} N(v) = \Omega \cup W_1 \), \( \bigcup_{v \in Y} N(v) = \Omega \cup W_2 \) where \( W_1 \) and \( W_2 \) are the sets of edges from \( X \) to \( \Omega \) and from \( Y \) to \( \Omega \) respectively in \( G \), \( \left( \bigcup_{v \in X} N(v) \right) \cap \left( \bigcup_{v \in Y} N(v) \right) = V \) and obviously...
the vertices in \( W_1 \) and \( W_2 \) are adjacent to at least one vertex in each other and exactly one vertex in \( \Omega \).