Chapter 4

On the spectral radius of a class of polycyclic graphs

4.1 Introduction

In this chapter we will study the spectral radius of a class of polycyclic graphs.

Definition 4.1.1. A sequence \( \{C_n\} \) of cycles in \( G \) is said to be a sequence of consecutive cycles if any two consecutive cycles in \( \{C_n\} \) have a common edge.
For the graph $G_1$ in Figure 4.1, consider the sequences 
\[ S_1 \equiv v_1v_2v_3v_1, v_1v_4v_5v_6v_1, v_1v_3v_4v_1 \text{ and } \]
\[ S_2 \equiv v_1v_2v_3v_1, v_1v_4v_5v_6v_1. \]
Then the sequence $S_1$ is a sequence of consecutive cycles in $G_1$ but $S_2$ is not. There exists a sequence of consecutive cycles in $G_1$, (e.g. $S_1$) where all the cycles have at most one common vertex, but in $G_2$ for any sequence of consecutive cycles there exist two vertices in $G$ which are common to all the cycles in the sequence.

**Definition 4.1.2.** Any two cycles $C_1$ and $C_2$ in a graph $G$ are said to be related if there exists a sequence of consecutive cycles in $G$ in which $C_1$ is first term and $C_2$ is the last term. For any graph $G$ with cycles, this defines an equivalence relation on the set of cycles in $G$. For any cycle $C$ in $G$, the equivalence class of $C$ with respect to the above defined relation is denoted by $\overline{C}$.

Let $\mathcal{G}(n, k)$ be the class of all connected graphs $G$ on $n$ vertices and $k$-pendant vertices, in which for any cycle $C$ in $G$, $\overline{C}$ is such that there exist two vertices in $G$ which are common to all the cycles in $C$. Clearly $G_1$ in Figure 4.1, cannot be a subgraph of any graph in $\mathcal{G}(n, k)$. In this chapter, we determine the graph with the largest spectral radius in $\mathcal{G}(n, k)$.

In order to complete the proof of our main result of this chapter we need the following lemmas.

**Lemma 4.1.3.** [11] Let $G_1$ and $G_2$ be two graphs. If $P(G_1, \lambda) < P(G_2, \lambda)$ for $\lambda > \lambda_1(G_2)$, then 
\[ \lambda_1(G_1) > \lambda_1(G_2). \]
Lemma 4.1.4. [15] If $v$ is a vertex of degree 1 in the graph $G$ and $u$ is the vertex adjacent to $v$, then
\[ P(G, \lambda) = \lambda P(G - v, \lambda) - P(G - u - v, \lambda). \]

Lemma 4.1.5. [11] If $G$ is a connected graph and $G'$ is a proper spanning subgraph of $G$, then
\[ P(G', \lambda) > P(G, \lambda) \quad \text{for all} \quad \lambda \geq \lambda_1(G). \]

Lemma 4.1.6. [61] Let $G$ be a connected graph and $\lambda_1(G)$ be the spectral radius of $A(G)$. Let $u, v$ be two vertices of $G$ and $d(v)$ be the degree of $v$. Suppose $v_1, v_2, \ldots, v_s \in N(v) \setminus \{N(u) \cup \{u\}\}$, where $1 \leq s \leq d(v)$, and $x = (x_1, x_2, \ldots, x_n)$ is the perron vector of $A(G)$, where $x_i$ corresponds to the vertices $v_i$ ($1 \leq i \leq n$). Let $G^*$ be the graph obtained from $G$ by deleting the edges $vv_i$ and adding the edges $uv_i$ ($1 \leq i \leq s$). If $x_u \geq x_v$ then $\lambda_1(G) < \lambda_1(G^*)$.

Figure 4.2: The graphs $G$ and $G_1$

Let $G$ be a connected graph consisting of a connected subgraph $H$ and a tree $T$, such that $T$ is attached to a vertex $r$ of $H$ [see Figure 4.2]. The vertex $r$ is
called root of the tree $T$ or the root vertex of $G$. The distance between the root $r$ and a vertex of $T$ which is furthest from $r$ is defined as the height of the tree $T$. Let $|V(T)|$ be the number of vertices of an attached tree $T$ excluding $r$. If $v$ is a vertex of $T$ furthest from the root $r$, then $v$ is a pendent vertex. Let $u$ be the vertex adjacent to $v$. We carry out a transformation on $G$ in the following way, delete the edge $uv$ and join $v$ with $r$. This procedure results a graph $G_1$ displayed in Figure 4.2. If there exist a vertex in $T - v$ which is at a distance more than one from $r$, we can repeat the above process on $G_1$, and finally we get a graph $\hat{G}$ shown in Figure 4.3.

![Figure 4.3: The graph $\hat{G}$](image)

Let $G$ be a connected graph consisting of a connected subgraph $H$ and a path $P_k$ such that an end vertex of $P_k$ is identified with a vertex $r$ of $H$ [see Figure 4.4]. If $v$ is a vertex of $P_k$ furthest from the root $r$, then $v$ is a pendent vertex. Let $w$ be the vertex adjacent to $v$. Suppose $G_0$ be the graph obtained from $G$ by deleting the edge $wv$ and joining $v$ with $r$. Also, let $H_0$ be the graph obtained from $G_0$ by joining the edge $uv$ [see Figure 4.4].

**Lemma 4.1.7.** If $G$ and $G_0$ are the graphs as shown in Figure 4.4, then

$$P(G, \lambda) > P(G_0, \lambda), \text{ for all } \lambda \geq \lambda_1(G).$$
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\begin{center}
\begin{tikzpicture}
\node[shape=circle,draw=black] (a) at (1,0) {$v$};
\node[shape=circle,draw=black] (b) at (2,0) {$w$};
\node[shape=circle,draw=black] (c) at (3,0) {$u$};
\node[shape=circle,draw=black] (d) at (4,0) {$r$};
\node[shape=circle,draw=black] (e) at (5,0) {$w$};
\node[shape=circle,draw=black] (f) at (6,0) {$v$};
\node[shape=circle,draw=black] (g) at (7,0) {$P_k$};
\node[shape=circle,draw=black] (h) at (8,0) {$H$};
\node[shape=circle,draw=black] (i) at (9,0) {$H_0$};
\node[shape=circle,draw=black] (j) at (10,0) {$P_{k-1}$};
\node[shape=circle,draw=black] (k) at (11,0) {$w$};
\draw[thick] (a) -- (b);
\draw[thick] (b) -- (c);
\draw[thick] (c) -- (d);
\draw[thick] (d) -- (e);
\draw[thick] (e) -- (f);
\draw[thick] (f) -- (g);
\draw[thick] (g) -- (h);
\draw[thick] (h) -- (i);
\draw[thick] (i) -- (j);
\draw[thick] (j) -- (k);
\end{tikzpicture}
\end{center}

Figure 4.4: The graphs $G$, $G_0$ and $H_0$

In particular, we have $\lambda_1(G_0) > \lambda_1(G)$.

**Proof.** By Lemma 4.1.4, we have

$$P(G, \lambda) = \lambda P(G - v, \lambda) - P(G - w - v, \lambda),$$ \hfill (4.1.1)

$$P(G_0, \lambda) = \lambda P(G_0 - v, \lambda) - P(G_0 - r - v, \lambda).$$ \hfill (4.1.2)

We have

$$G - v \cong G_0 - v.$$ 

Also $G_0 - r - v$ is a proper spanning subgraph of $G - w - v$. By Lemma 4.1.5, we have

$$P(G_0 - r - v, \lambda) > P(G - w - v, \lambda), \text{ for all } \lambda \geq \lambda_1(G - w - v)$$
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Since \( G - w - v \) is a proper subgraph of \( G - v \), we have \( \lambda_1(G) > \lambda_1(G - w - v) \).

Thus from (4.1.1) and (4.1.2), we have

\[
P(G, \lambda) > P(G_0, \lambda), \quad \text{for all } \lambda \geq \lambda_1(G - v).
\]

Since \( G - v \) is a proper subgraph of \( G \), therefore \( \lambda_1(G) > \lambda_1(G - v) \). Thus

\[
P(G, \lambda) > P(G_0, \lambda), \quad \text{for all } \lambda \geq \lambda_1(G).
\]

\[\blacksquare\]

**Lemma 4.1.8.** If \( G \) and \( H_0 \) are the graphs as shown in Figure 4.4, then

\[
\lambda_1(G) < \lambda_1(H_0).
\]

**Proof.** By Lemma 4.1.7, \( \lambda_1(G) < \lambda_1(H_0 - uv) \). Since \( H_0 - uv \) is a proper subgraph of \( H_0 \) the result follows.

\[\blacksquare\]

**4.2  The graph with maximal spectral radius**

![Figure 4.5: The graphs \( G^1(n, k) \) and \( G^1(n) \)](figure)

**Theorem 4.2.1.** If \( G \) is a graph in \( G(n, k) \) and \( G^1(n, k) \) is the graph as shown in Figure 4.5, then \( \lambda_1(G) \leq \lambda_1(G^1(n, k)) \); equality holds if and only if \( G \cong G^1(n, k) \).
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Proof. Let $G \in G(n, k)$ such that the spectral radius of $G$ is as large as possible. Denote the vertex set of $G$ by $V(G) = \{v_1, v_2, \ldots, v_n\}$ and the Perron vector of $G$ by $x = (x_{v_1}, x_{v_2}, x_{v_3}, \ldots, x_{v_n})$, where $x_{v_i}$ corresponds to the vertex $v_i, (1 \leq i \leq n)$. Then $G$ have to satisfy the following facts.

Fact 1. Every path joining two cycles in $G$ must have at least one edge from a cycle in $G$.

Proof. Otherwise there exist cycles $C_1$ and $C_2$ joined by a path $P = u_1u_2\ldots u_l$ of length $l - 1 (l \geq 2)$, where the common vertex of $C_1$ and $P$ is $u_1$ and that of $C_2$ and $P$ is $u_l$. Note that we can further choose the cycles $C_1$ and $C_2$ such that the path has no common edge with any cycle in $G$. Without loss of generality, assume that $x_{u_1} \geq x_{u_l}$. Let $C_1$ and $C_2$ be the equivalence classes of $C_1$ and $C_2$ in $G$, respectively.

If $C_2 = \{C_2\}$, then let $G^* = G - u_ly + u_1y$, where $y$ is a neighbour of $u_l$ on $C_2$. By Lemma 4.1.6, we have $\lambda_1(G^*) > \lambda_1(G)$. Also $G^* \in G(n, k)$, a contradiction.

If $|C_2| \geq 2$, and $u_l$ is none of the two vertices which are common to all cycles in $C_2$, then let $G^* = G - u_ly + u_1y$, where $y$ is a vertex on $C_2$ adjacent to $u_l$. By Lemma 4.1.6, we have $\lambda_1(G^*) > \lambda_1(G)$. Also $G^* \in G(n, k)$, a contradiction.

If $|C_2| \geq 2$, and $u_l$ is one of the two vertices which are common to all cycles in $C_2$, then let $C_2' = \{C_1', C_2', \ldots, C_k'\}$ and $u'$ be the other vertex which is common to all the cycles in $C_2$. Let $U = N(u_l) \cap (\cup_{i=1}^k V(C_i'))$ and $G^* = G - \sum_{y \in U} u_ly + \sum_{y \in U} u_1y$. Then by Lemma 4.1.6, we have $\lambda_1(G^*) > \lambda_1(G)$. If $u_l$ is not a pendant vertex in $G^*$, then $G^* \notin G(n, k)$, a contradiction. Again, if $u_l$ is a pendant vertex in $G^*$, then let $G^{**} = G^* + u_1u'$. But $\lambda_1(G^{**}) > \lambda_1(G^*) > \lambda_1(G)$, and $G^{**} \in G(n, k)$, a contradiction.
Thus the claim.

**Fact 2.** Any two cycles in $G$ have a common vertex.

**Proof.** Suppose $C_1$ and $C_2$ are two cycles in $G$ which have no common vertex. Then $C_1$ and $C_2$ are joined by a path $P = u_1 u_2 \ldots u_l$ of length $l - 1 (l \geq 2)$ which has some edge from a cycle in $G$. Note that we can further choose the cycles $C_1$ and $C_2$ such that all the edges in the path are in a single cycle $C_3$ in $G$. Suppose $C_3$ is related to neither $C_1$ nor $C_2$. Let $u_1$ be the vertex common to $C_1$, $C_3$ and $u_l$ be the vertex common to $C_2$, $C_3$. Without loss of generality assume that $x_{u_1} \geq x_{u_l}$. Let $V_1 = N(u_l) \cap \{ \cup_{C \in C_2} V(C) \}$ and $G^* = G - \sum_{x \in V_1} u_l x + \sum_{x \in V_1} u_1 x$. By Lemma 4.1.6, $\lambda_1(G^*) > \lambda_1(G)$, and $G^* \in \mathcal{G}(n, k)$, a contradiction.

Suppose $C_3$ is related to $C_2$. Then $C_1$ is not related to both $C_2$ and $C_3$. Let $v_1$ and $v_2$ be the vertices which are common to all the cycles in $C_2$. Clearly $u_1$ is neither $v_1$ nor $v_2$.

If $x_{u_1} \geq x_{v_1}$, then let $V_1 = \{ N(v_1) \cap \{ \cup_{C \in C_2} V(C) \} \} - \{ N(u_1) \cup \{ u_1 \} \}$ and $G^* = G - \sum_{y \in V_1} v_1 y + \sum_{y \in V_1} u_1 y$. By Lemma 4.1.6, $\lambda_1(G^*) > \lambda_1(G)$.

If $G^*$ is connected and $v_1$ is not a pendent vertex in $G^*$, then $G^* \in \mathcal{G}(n, k)$, a contradiction. If $G^*$ is connected and $v_1$ is a pendent vertex in $G^*$, then let $G^{**} = G^* + u_1 v_1$. Now $\lambda_1(G^{**}) > \lambda_1(G)$ and $G^{**} \in \mathcal{G}(n, k)$, a contradiction. If $G^*$ is disconnected, then let $G^{**} = G + u_1 v_1 + v_2 v_1$. But $\lambda_1(G^{**}) > \lambda_1(G)$ and $G^{**} \in \mathcal{G}(n, k)$, a contradiction.

If $x_{u_1} < x_{v_1}$, then let $U_1 = N(u_1) \cap \{ \cup_{C \in C_1} V(C) \}$ and $G^* = G - \sum_{y \in U_1} u_1 y + \sum_{y \in U_1} v_1 y$. Now $\lambda_1(G^*) > \lambda_1(G)$ and $G^* \in \mathcal{G}(n, k)$, a contradiction.

Thus any two cycles in $G$ have a common vertex.

**Fact 3.** Any two cycles in $G$ have two common vertices.
Proof. If possible, let $C_1$ and $C_2$ be two cycles in $G$ having exactly one vertex $u$ in common. Since any two cycles in $G$ have a common vertex, therefore $u$ must be a vertex which is common to all cycles in $\overline{C_1}$ and $\overline{C_2}$, otherwise $C_1$ and $C_2$ will have two vertices in common. Let $v$ be the other vertex which is common to all cycles in $\overline{C_1}$ and $w$ be the other vertex which is common to all cycles in $\overline{C_2}$. Let $V_1 = \{N(v) \cap \cup_{C \in C_1} V(C)\} - \{u\}$. Without loss of generality assume that $x_w \geq x_v$, and consider $G^* = G - \sum_{y \in V_1} vy + \sum_{y \in V_1} wy$. Then $\lambda_1(G^*) > \lambda_1(G)$. If $G^*$ is connected and $v$ is not a pendent vertex in $G^*$, then $G^* \in \mathcal{G}(n, k)$, a contradiction. If $G^*$ is connected and $v$ is a pendent vertex in $G^*$, then let $G^{**} = G^* + vw$. But $\lambda_1(G^{**}) > \lambda_1(G)$ and $G^{**} \in \mathcal{G}(n, k)$, a contradiction. If $G^*$ is disconnected, then let $G^{**} = G^* + uv + vw$. Again $\lambda_1(G^{**}) > \lambda_1(G)$ and $G^{**} \in \mathcal{G}(n, k)$, a contradiction. Therefore any two cycles in $G$ must have two common vertices.

By $u_0$ and $v_0$ we denote the two vertices of $G$ which are common to all cycles in $G$.

Fact 4. There is only one equivalence class in $G$.

Proof. If possible let there exist two equivalence classes $\overline{C_1}$ and $\overline{C_2}$ in $G$. Let $C_1 \in \overline{C_1}$ and $C_2 \in \overline{C_2}$. By Fact 3, there exist two vertices $u_0$ and $v_0$ in $G$ which are common to both $C_1$ and $C_2$. Now the cycle $C_3$ in $G$ formed by the $u_0 - v_0$ paths (one lies on the cycle $C_1$ (but not on $C_2$) and other lies on the cycle $C_2$) is in both the equivalence classes $\overline{C_1}$ and $\overline{C_2}$, a contradiction.

Fact 5. Any tree of the graph $G$ is attached to a common vertex of all cycles.

Proof. If possible let there exist a tree $T$ attached to a vertex $u$ on a cycle $C$ of $G$. Let $y$ be the neighbour of $u$ in $T$. If $x_{vu} \geq x_u$, then let $G^* = G - uy + v_0y$. 

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If \( x_{v_0} < x_u \), then let \( C_1 \) be a cycle in \( G \) containing \( v_0, u, u_0 \); \( V_0 = \{ N(v_0) \cap \{ \cup_{C \in C_1} V(C) \} \} - V(C_1) \) and \( G^* = G - \sum_{y \in V_0} v_0 y + \sum_{y \in V_0} u y \). In either case by Lemma 4.1.6, \( \lambda_1(G^*) > \lambda_1(G) \) and \( G^* \in \mathcal{G}(n,k) \), a contradiction.

Without loss of generality we assume that all the trees are attached at \( v_0 \).

**Fact 6.** All the trees attached at \( v_0 \) are paths of length 1.

**Proof.** First we show that the trees attached at \( v_0 \) are paths.

If possible let \( T \) be a tree attached at \( v_0 \) which is not a path. Let \( u \) be a vertex of degree \( r \geq 3 \) which is furthest from \( v_0 \). If \( x_{v_0} \geq x_u \), then let \( y_1, y_2, \ldots, y_{r-2} \) be \( r-2 \) neighbours of \( u \) in \( T \) and none of \( y_i \)’s be in the \( v_0 - u \) path and let \( G^* = G - \sum_{i=1}^{r-2} u y_i + \sum_{i=1}^{r-2} v_0 y_i \). If \( x_{v_0} < x_u \), then let \( G^* = G - v_0 w + uw \), where \( w \) is a vertex on a cycle and is adjacent to \( v_0 \). In either case, by Lemma 4.1.6, \( \lambda_1(G^*) > \lambda_1(G) \) and \( G^* \in \mathcal{G}(n,k) \), a contradiction.

Thus all the trees attached at \( v_0 \) are paths.

Now by Lemma 4.1.8, all the trees attached at \( v_0 \) are paths of length 1 (i.e. there are \( k \)-pendent vertices attached at \( v_0 \)).

**Fact 7.** Every vertex of \( G \) must be adjacent to \( v_0 \).

**Proof.** Clearly \( u_0 \) is adjacent to \( v_0 \). Otherwise, let \( G^* = G + u_0 v_0 \). Then \( \lambda_1(G^*) > \lambda_1(G) \) and \( G^* \in \mathcal{G}(n,k) \), a contradiction. If possible let \( v_k \) be a vertex in \( G \) which is not adjacent to \( v_0 \). Let \( v_0 v_1 \ldots v_k \) be the path from \( v_0 \) to \( v_k \) not containing the vertex \( u_0 \), where \( k \geq 2 \). If \( x_{v_0} \geq x_{v_1} \), then let \( G^* = G - v_1 v_2 + v_0 v_2 \), and by Lemma 4.1.6, \( \lambda_1(G^*) > \lambda_1(G) \). Consider the graph \( G^{**} = G^* + v_1 u_0 \). Then we have \( \lambda_1(G^{**}) > \lambda_1(G^*) > \lambda_1(G) \), and \( G^{**} \in \mathcal{G}(n,k) \), a contradiction.

And if \( x_{v_0} < x_{v_1} \), then let \( U = N(v_0) - \{ v_1, u_0 \} \) and \( G^* = G - \sum_{v \in U} v_0 v + \sum_{v \in U} v_1 v \). By Lemma 4.1.6, \( \lambda_1(G^*) > \lambda_1(G) \) and \( G^* \in \mathcal{G}(n,k) \), a contradiction.
Thus every vertex (other than $v_0$) of $G$ is adjacent to $v_0$.

Combining all the above facts we have, $G \cong G^1(n, k)$.

Let $G(n)$ be the class of all connected graphs on $n$ vertices in which for each class $C$ in $G$, there exist two vertices in $G$ which are common to all the cycles in $C$. If $G^1(n)$ denote the graph shown in Figure 4.5, then we have the following.

**Corollary 4.2.2.** If $G$ is a graph in $G(n)$, then $\lambda_1(G) \leq \lambda_1(G^1(n))$; equality holds if and only if $G \cong G^1(n)$.