CHAPTER-3
THEORETICAL MODELLING

3.0 INTRODUCTION TO ID ANALYSIS

The Ignition Overpressure phenomenon is very complex and has been hard to understand and predict. It has been the cause of structural failures of launch vehicles and the launch pads during lift-off. This chapter presents an introduction to the phenomena of Ignition Overpressure and the various theories that have been developed for theoretical prediction of the same. An analytical model has been presented to predict overpressure values which are then correlated with experimentally measured values. The analytical model is one of many postulated theories and has been presented mainly for understanding of the physics behind the overpressure phenomena.

During the start-up process of a rocket motor, transient pressure waves are generated that propagate to the rocket’s vicinity. The source of the pressure disturbance is the sudden displacement of the ambient air by the combustion products flowing from the rocket nozzle during ignition transients. As air is displaced by the increasing flow rate of exhaust gases, compression and expansion pressure waves are generated. These pressure waves propagate outward from the disturbed region and interact with the launch facilities and the launch vehicle, resulting in a spatial and time-dependant pressure disturbance. The pressure disturbance $\Delta P$ is defined as

$$\Delta P(x, y, z, t) = P(x, y, z, t) - P_\infty$$  \hspace{1cm} (3.1)

Where, $P(x, y, z, t)$ is the transient pressure at the location $(x, y, z)$ at the time ‘t’ and $P_\infty$ is the ambient atmospheric pressure. This pressure disturbance resulting from rocket ignition is called Ignition Overpressure. The phenomena of transient overpressure induced by engine ignition have long been recognized in missiles and launch vehicle design. These emanating pressure waves are found to affect the entire vehicle and therefore, Ignition Overpressure environment is one of the critical design factors to be assessed for lift-off.
3.1 BRIEF DESCRIPTION OF EARLIER THEORIES

3.1.1 BROAD WELL AND TSU THEORY

The Broadwell and Tsu theory was developed for computing the rocket Ignition Overpressure for silo type launchers. A silo is basically a long duct that is usually built underground, which houses the rocket. The rocket when fired is launched along the duct. In their analysis, the SRM is shown to constitute a mass and momentum source in the silo. The mass source generates compression waves into the launch duct (along the direction of motion of rocket) as well as the exhaust duct (along the exhaust direction). The momentum source generates a compression wave into the exhaust duct and an expansion wave into the launch duct. The pressure wave propagating from the source region is given by

\[ \Delta P = \frac{1}{2} \left[ \frac{\gamma m}{a_s \rho_c A_d} K_n \pm \left( \frac{2U_s m}{A_b} \right) K_f \right] \]  

(3.2)

The plus sign is for the exhaust duct and the minus sign is for the launch duct. The subsequent propagation of these waves is treated by the one-dimensional linear theory. Application of this theory results in,

\[ \Delta P \propto \dot{P}_c \]  

(3.3)

3.1.2 RIED’S SCALING RELATIONSHIP

The engineering application of Ried’s formulation helps in bridging the gap between sub-scale test data and flight data. Ried considers the SRM Ignition Overpressure phenomenon as analogous to the blast wave phenomenon whereby the energy is released instantaneously. Therefore, the overpressure (\( \Delta P \)) is related to the energy released during start-up (\( E_0 \)) by

\[ \Delta P \approx \frac{E_0}{\dot{P}_c A_s B} \]  

(3.4)

Where, \( VA+B \) are the affected volume or volume enclosed by the blast wave. For a rocket motor, the total energy released during start-up is determined by
\[ E_0 = A^* \sqrt{h_c \int P_c \, dt} \] (3.5)

Where ‘\( \tau \)’ is the start-up time for the rocket motor, \( h_c \) is the enthalpy of the gases in the combustion chamber, and \( A^* \) is the nozzle throat area. The Ignition Overpressure for one-, two-, and three-dimensional fields can be expressed respectively as

\[ \Delta P \approx \sqrt{\frac{A^* h_c P_c}{\alpha_{\infty}}} \] (3.6)

\[ \Delta P \approx \sqrt{\frac{A^* h_c \dot{P}_c}{\alpha_{\infty}}} \] (3.7)

\[ \Delta P \approx \frac{A^* \sqrt{h_c \dot{P}_c^2}}{\alpha_{\infty}^3 P_c} \] (3.8)

Broadwell and Tsu’s theory implies that \( \Delta P \propto \dot{P}_c \) whereas Ried’s theory indicates that \( \Delta P \propto \frac{\dot{P}_c^2}{P_c} \) for a three-dimensional flow field. Thus one can express the magnitude of overpressure and the \( \dot{P}_c \) in a general relationship

\[ \Delta P \propto \left( \frac{\dot{P}_c}{P_c} \right)^m \] (3.9)

Where ‘\( m \)’ can be determined from a log-log plot of \( \Delta P \) and \( \dot{P}_c \).

### 3.2 PROPOSED IGNITION OVERPRESSURE METHODOLOGY

The transient burning characteristics of the full scale and sub scale model of the Solid Rocket Motor (SRM) are not similar. Hence, the true scaling parameter could not be determined directly from geometrical scaling. Therefore, a theoretical model is necessary in order to understand both the IOP phenomenon and the behavior for both full scale and subscale vehicles using different motors.
The IOP physical mechanism is highly complex and therefore the present analytical model represents one of the many postulated hypothesis associated with this complex phenomena. In this methodology, the first order IOP effects are predicted as functions of geometric configuration and of plenum chamber pressure (pc) history. The growth rate of wall jet along the duct trench/jet deflector trench, which is produced by the redirection of impinged rocket exhaust at the duct bottom, is postulated as the source of primary IOP wave. The wave form and frequency of externally propagating IOP response possess an N-wave characteristic with positive and negative pressure peaks and are proportional to the $\tilde{P}_c$. The geometric configuration of the launch complex that is used as the base for this methodology is as shown in the Fig.3.1. The method is divided into two sections.

![Fig. 3.1 Geometric Configuration for IOP Model](image)

The first section analyses the duct flow field below the SRM exhaust nozzle, which determines the required forcing functions for the external IOP propagation analysis. The analysis includes the development of transient characteristics of exhaust plume impingement at the bottom surface of the deflector as a function of SRM $P_C$ history. After the exhaust impingement, the flow is redirected along the jet deflector and flows out as a subsonic wall jet. The growth rate of the wall jet as a function of stagnation pressure ($P_{t2}$) behind normal shock at the bottom of the jet deflector is postulated as the source of primary IOP. Shock tube analogy, with an accelerating-decelerating piston, is applied to compute the compression and expansion wave propagation in the SRM exhaust duct opening. The second section consists of the external IOP propagation analysis on the vehicle and surrounding structures. A generalized three-dimensional acoustic theory was developed for this assessment.
3.2.1 PLENUM CHAMBER PROPERTIES

The properties with the SRM plenum chamber, where ignition transient takes place, are defined first. The subsequent flow phenomena that takes place inside the exhaust duct and the pressure propagation phenomena that follows above the mobile launch pedestal (MLP) are related back to the ignition sequence that proceeds with the plenum chamber.

The mass flow rate through the rocket nozzle is considered as an isentropic process. The \( p_c \) history is the only known property; therefore, time variations of rocket properties will be defined as a function of the \( p_c \) history. Since \( p_e << p_c \), \( p_e/p_c < (2/(\gamma +1))^{\gamma-1} \) condition is rapidly reached and a choked flow condition is established in the nozzle (A condition of non-choked throat can be neglected with minimal impact on the flow analysis). The mass flow rate at the throat is expressed as,

\[
\dot{m}_0(t) = p_c \sqrt{\frac{\gamma}{RT_c}} A^* \left( \frac{2}{\gamma +1} \right)^{\frac{1}{2}} \left( \frac{2}{\gamma +1} \right)^{\frac{1}{2} \left( \frac{\gamma +1}{\gamma -1} \right)}
\]  

(3.10)

Where \( P_c=P_c(t) \) and \( T_c=T_c(t) \), which are properties of the SRM chamber.

3.2.2 EXHAUST JET PROPERTIES (Region 1 of Fig 3.1)

The rocket exhaust within the rocket nozzle is isentropically expanded. Since the external pressure is ambient, the area ratio of the exhaust column must satisfy the pressure ratio \( p_e/p_c \) during the ignition transient; hence, the jet velocity leaving the nozzle is also dependent upon the pressure ratio. The exhaust jet Mach no. \( M_j \) at any given time is directly related to the \( p_c \) value at that time by

\[
M_j^2 = \frac{2}{\gamma -1} \left[ \left( \frac{P_e}{P_c} \right)^{\frac{\gamma-2}{\gamma-1}} - 1 \right]
\]  

(3.11)

The acoustic velocity \( (a^*) \) at the throat is defined by

\[
a^* = \sqrt{\frac{2\gamma RT_c}{\gamma +1}}
\]  

(3.12)
Once $M_j$ and $a^*$ are defined, the jet velocity $u_j$ and area ratio of the jet column $\frac{A}{A_*}$ can be expressed as

$$\frac{u_j}{a^*} = \sqrt{\frac{(\gamma + 1)M_j^2}{2 + (\gamma - 1)M_j^2}}$$  \hspace{1cm} (3.13)

$$\frac{A}{A_*} = \frac{1}{M_j} \left( \frac{2 + (\gamma - 1)M_j^2}{\gamma + 1} \right)^{\frac{\gamma - 1}{(\gamma - 1)}}$$  \hspace{1cm} (3.14)

As shown in eqn.(3.13), $U_j$ is proportional to the $p_c$ history; therefore, during the SRM ignition transient, the velocity of exhaust jet leaving the nozzle at a later time is faster than the one exhausted previously. The flow entrainment caused by flow mixing is assumed to be negligible during the SRM ignition transient, and the exhaust particle is assumed to travel in Region 1 (Fig3.1) with constant velocity.

The important parameters to be considered for computing the process of IOP phenomena are the rate of change of flow properties. The rate of change must be defined at the point of interest and is related to the $p_c$ in the plenum chamber. At a fixed point ($x$) in Region 1, the arrival time of the signal is given as

$$\tau_i = t_i + \left( \frac{x}{u_j} \right)_i$$  \hspace{1cm} (3.15)

where the subscript ‘$i$’ corresponds to the $i_{th}$ event. The rate of change of any flow property at a distance ‘$x$’ below the nozzle exit plane is expressed as

$$\frac{df}{d\varepsilon} = \frac{df}{dt} \left(1 - \frac{x}{u_j} \frac{du_j}{dt}\right)^{-1}$$  \hspace{1cm} (3.16)

Where, $df/dt$ is defined at the plenum chamber. The local rate of change is corrected as shown in eqn. (3.15) and exists when the term inside the curly braces is
positive. In the region where that is not satisfied, the exhaust front of the previous event is overrun by the exhaust front of the later event. In this region, the analogy for shock wave formation caused by the coalescence of weak compression waves is applied; therefore, the flame front is assumed to travel with the faster jet velocity. The rate of change of jet velocity is given by

$$\frac{du_j}{dt} = u_j \left[ \frac{d\ln T_c}{dt} + \frac{2}{2 - (\gamma - 1)M_j^2} \frac{d\ln M_j^2}{dt} \right]$$

(3.17)

The plenum chamber temperature transient is much shorter than the pressure transient and so the rate of change of $T_c$ with time can be neglected. The rate of change of jet Mach number $M_j$ is expressed as

$$\frac{dM_j^2}{dt} = \frac{2}{\gamma} \left( \frac{p_j}{p_t} \right)^{\gamma-1} \frac{1}{p_t} \frac{dp}{dt}$$

(3.18)

### 3.2.3 SUBSONIC REGION BEHIND NORMAL SHOCK

When the supersonic exhaust jet impinges on the bottom of the duct, the supersonic jet is compressed by a normal shock process. The jump conditions for flow properties are determined by the Rankine-Hugoniot relationship. The stagnation pressure ratio is given as a function of jet Mach number as

$$\frac{p_c}{p_{t2}} = \left( \frac{2 + (\gamma - 1)M_j^2}{\gamma + 1} \right)^{\frac{\gamma}{\gamma-1}} \left( \frac{2\gamma M_j^2 - \gamma + 1}{\gamma + 1} \right)^{\frac{1}{\gamma-1}}$$

(3.19)

Since the exhaust duct has a confined dimension (assume a two-dimensional trench), the flow is redirected along the trench and escapes as a wall jet. The interface between the subsonic region and the wall jet is assumed to be the sonic plane because the static pressure at the interface will be greater than the pressure in Region 4 of figure 3.1 therefore, the wall jet is accelerated to supersonic flow to satisfy the pressure continuity. The outgoing mass flux at the interface dictates the shock stand-off distance.
\[(\rho \mathbf{u} \cdot A)_{in} = (\rho \mathbf{u} \cdot A)_{out}\]  \hspace{1cm} (3.20)

The incoming mass flux is equal to that at the throat and hence the shock stand-off distance can be expressed as

\[\Delta = A^* \left( \frac{p_c}{p_{t2}} \right)\]  \hspace{1cm} (3.21)

Where, \(p_c/p_{t2}\) is defined by equation (3.19).

The rate of change of shock stand-off distance is expressed as

\[\frac{d\Delta}{d\tau} = \frac{\gamma \Delta}{M_j^2} \left( \frac{2(M_j^2 - 1)^2}{(2 + (\gamma - 1)M_j^2)(2\gamma M_j^2 - \gamma + 1)} \right) \frac{dM_j^2}{d\tau}\]  \hspace{1cm} (3.22)

Where \(\frac{dM_j^2}{d\tau} = \frac{1}{1 - \frac{x}{u_j^2}} \frac{dM_j^2}{dt}\) ... (3.23)

### 3.2.4 SUPersonic WALL JET (Region 3 of Fig 3.1)

Since a large gap exists between the bottom of the MLP and the trench, the rocket exhaust mass is assumed to escape through the trench without being recirculated. Two phenomena, the growth rate of the shock stand-off and the expansion of the wall jet free edge, will combine to act as a piston for propagating pressure pulses into Region 4. An acceleration of the wall jet takes place with the Prandtl-Meyer (P-M) expansion process of the free wall jet boundary until pressure continuity across the contact surface between Region 3 and Region 4 is established. For a perfect gas, the P-M angle through with the stream turns in expanding from \(M=1\) to a supersonic Mach number, \(M\) is given by,

\[\nu = \frac{\gamma + 1}{\gamma - 1} \tan^{-1} \left( \frac{\gamma - 1}{\gamma + 1} (M^2 - 1) - \tan^{-1} \sqrt{(M^2 - 1)} \right)\]  \hspace{1cm} (3.24)

The ratio of static to total pressure for Mach No. \(M\) is given by
\[
\left( \frac{p}{p_t} \right)^{\gamma-1} = \frac{1}{\gamma+1} \left\{ 1 + \cos \left[ 2 \tan^{-1} \left( \frac{\gamma - 1}{\gamma+1} \sqrt{M^2 - 1} \right) \right] \right\} \quad (3.25)
\]

Substituting eqn.(3.25) into eqn.(3.24), we get

\[
v = \frac{1}{\gamma+1} \cos^{-1} \left[ (\gamma+1) \left( \frac{p}{p_t} \right)^{\gamma-1} \right] - \tan^{-1} \left[ \frac{\gamma + 1}{\gamma-1} \tan \left( \frac{1}{2} \cos^{-1} \left\{ (\gamma+1) \left( \frac{p}{p_t} \right)^{\gamma-1} \right\} \right) \right] \quad (3.26)
\]

Where

\[
\frac{p}{p_t} = \frac{p_1}{p_{t2}} \quad (3.27)
\]

The perturbation pressure is expressed as

\[
\Delta p = p_4 - p_e \quad (3.28)
\]

The rate of change of P-M angle is obtained to be

\[
\frac{dv}{d\tau} = \frac{1}{2\gamma} \sqrt{(\gamma - 1) \left( \frac{p}{p_t} \right)^{\gamma-1} - 2 - (\gamma + 1) \left( \frac{p}{p_t} \right)^{\gamma-1}} \left[ 1 - \frac{4(M^2 - 1)^{\gamma-1}}{M^2 [2 + (\gamma - 1)M^2] [2\gamma M^2 - \gamma + 1]} \right] \left( \frac{p}{p_t} \right)^{\gamma-1} \left( \frac{dp}{d\tau} \right) \quad (3.29)
\]

Thus the equivalent piston velocity to generate a perturbation pulse in Region 4 is given as,

\[
u_p = \frac{d\Delta}{d\tau} + KL \frac{dv}{d\tau} \quad (3.30)
\]

Where ‘K’ is a multiplying factor, and ‘L’ is the gap distance between the edge of the MLP and the exhaust plume boundary.

3.2.5 PERTURBATION PRESSURE PROPAGATION (Region 4 of Fig 3.1)

During the short ignition transient, it is assumed that the air occupied in region 4 does not interact with the exhaust plume of region 1; therefore, perturbation wave propagation in region 4 is assumed to be isolated, and a one dimensional shock tube analogy with the known piston driver ‘up’ is used to analyze the wave propagation phenomena.
3.2.6 COMPRESSION WAVE PROPAGATION

The shock tube is as shown in Fig. 3.2. Compression waves occur when the piston is accelerating (dups/dt > 0). When the piston is abruptly set into motion with piston speed, up, at t = 0, a compression wave is induced on the piston surface and propagates with speed us into stationary media. The passage of shock front sets the stationary gas into motion with a speed equal to the piston speed. The piston speed is a function of the pressure ratio that exists across the shock front. In this problem, the piston speed is known and the perturbation pressure must be determined; therefore, the explicit expression of pressure ratio as a function of piston speed is given as

\[
\frac{p_2}{p_1} = 1 + \frac{1}{2}u_k + \frac{1}{2} \sqrt{u_k^2 + \frac{8 \gamma_a}{\gamma_a + 1} u_k}
\]  

(3.31)

Where

\[
u_k = \frac{1}{2} \gamma_a \left( \gamma_a + 1 \right) \left( \frac{u_p}{a_1} \right)^2
\]  

(3.32)

\[\gamma_a = \text{ratio of specific heats, for ambient gas.}\]

Fig. 3.2 Shock Tube Analogy For Compression Waves

If the piston is replaced with a contact surface, pressure continuity must exist across the contact surface. To satisfy pressure continuity, the rate of change of P-M angle and eqn.(3.31) must be iterated. The shock speed and temperature ratios are given as
\[ u_s = a_1 \sqrt{\frac{\gamma_a - 1}{2\gamma_a} + \frac{\gamma_a + 1}{2\gamma_a} \frac{p_2}{p_1}} \]  
(3.33)

\[ \frac{T_2}{T_1} = \frac{p_2}{p_1} \frac{\gamma_a - 1}{\gamma_a - 1} \frac{p_1}{p_1} \left( 1 + \frac{\gamma_a + 1}{\gamma_a - 1} \frac{p_2}{p_1} \right) \]  
(3.34)

\[ \frac{a_2}{a_1} = \sqrt{\frac{T_2}{T_1}} \]  
(3.35)

In equations (3.31) to (3.35) the subscript transformations from those of figure 3.2 to those of Figure 3.1 is done as – subscript ‘2’ to ‘4’, subscript ‘1’ to ‘e’.

### 3.2.7 EXPANSION WAVE PROPAGATION

The Shock Tube is as shown in Fig. 3.3. Expansion waves occur when the piston is decelerating (\( \frac{dp}{d\tau} < 0 \)). The deceleration of the contact surface will result for the condition where \( \bar{P}_c \) is negative. To establish the proper relationship for the expansion wave, wave propagation in the moving media caused by piston deceleration must be established. The formulation of this theory is based on the isentropic wave (Riemann wave) problem. The Riemann invariant of P characteristics can be expressed as

\[ \frac{2}{\gamma_a - 1} a - u = \text{const} . \]  
(3.36)

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Fig. 3.3 Shock Tube Analogy for Expansion Waves
Referring to the schematic of Fig. 3.3, the gas velocity \( u_2 \) in region 2 is flowing left to right, i.e. \( u_2 > 0 \). Since the piston velocity decelerated from \( u_p = u_2 \) to the gas velocity in region 3, we have \( 0 < u_3 < u_2 \). The pressure decreases as the expansion wave propagates, so that \( p_3 < p_2 \). The speed of P characteristic wave is

\[
C = u + a
\]  \hspace{1cm} (3.37)

\( C = \) speed of characteristic wave

Riemann invariant in region 2 gives of figure 3.3

\[
a = a_2 - \frac{\gamma - 1}{2} u_2 + \frac{\gamma - 1}{2} u
\]  \hspace{1cm} (3.38)

Substituting (3.38) in (3.37), we get

\[
C_p = a_2 - \frac{\gamma - 1}{2} u_2 + \frac{\gamma + 1}{2} u_p
\]  \hspace{1cm} (3.39)

Where, the subscript ‘P’ denotes P characteristic waves.

To obtain the pressure relationship between region 2 and 3 of Fig. 3.33

\[
dp = \rho adu
\]  \hspace{1cm} (3.40)

for \( du < 0 \) and \( dp < 0 \).

For a perfect gas, density and acoustic velocity are related as

\[
\rho = \text{cont.}(a)^{\frac{2}{\gamma - 1}}
\]  \hspace{1cm} (3.41)

Then eqn. (3.40) becomes,

\[
dp = K_p \left[ a_2 - \frac{1}{2} (\gamma - 1) u_2 + \frac{1}{2} (\gamma - 1) u \right]^{\frac{\gamma + 1}{\gamma - 1}} du
\]  \hspace{1cm} (3.42)
When integrating eqn.(33) and the solution is solved at \( u = u_s \), the constant of integration is found to be zero. Finally, the pressure ratio induced by the decelerating piston is given as,

\[
\frac{p}{p_2} = \left\{ 1 - \frac{1}{2} (\gamma_a - 1) \frac{u_2 - u_p}{a_2} \right\}^{\frac{2\gamma_a}{\gamma_a - 1}}
\]

where \( u_p < u_2 \), \( p_2 \) and \( u_2 \) are defined for the passage of the last compression wave preceding the deceleration of the piston (contact surface).

In eqn.(3.43), ‘p’ corresponds to \( p_4 \) in Fig. 3.1. This equation is iterated along with the P-M angle equation to get a pressure ratio that satisfies pressure continuity. The time delay for the arrival of the perturbation pressure pulse at the duct opening is defined as,

\[
\tau_i = t_i + \left( \frac{x_D}{u_j} \right)_i + \left( \frac{x_U}{u_j} \right)_i \text{ for a compression wave}
\]

\[
\tau_i = t_i + \left( \frac{x_D}{u_j} \right)_i + \left( \frac{x_U}{C_p} \right)_i \text{ for an expansion wave}
\]

where \( C_P = u_2 + a_2 \), \( x_D \) and \( x_U \) correspond to the distances traveled by the signal in Region 1 and Region 4 of Fig. 3.1.

### 3.2.8 EXTERNAL SOUND WAVE PROPOGATION

A method to predict the IOP propagation and the amplitude decaying phenomena outside and above the launch platform is developed. The perturbation pressure waves generated by the rocket motor ignition transient emerge from the SRM duct opening. The external wave propagation of the pressure waves can be described by the acoustic theory. The governing equation of cylindrical wave propagation with a finite length line source is given by

\[
C_\infty^2 \left\{ \phi_{rr} + \frac{1}{r} \phi_r + \phi_{zz} \right\} = \phi_{tt}
\]

(3.46)
The perturbation velocity and pressure induced by the acoustic wave propagation are given by,

\[ u = \phi_t \]

\[ \Delta p = p - p_\infty = -\rho_\infty \phi_t = -\frac{\gamma \rho_\infty}{C_\infty^2} \phi_t \] (3.47)

The solution that satisfies these foregoing equation is approximated by the integration of the acoustic strength rising from the spherical sources distributed uniformly along the Z-axis (Fig. 3.4)

\[
\phi = -\frac{1}{2\pi} \int_{z_R}^{z_I} \frac{q(t - R/C_\infty)}{R} dz
\] (3.48)

Where, \( q(t - R/C) \) is a forcing function proportional to the source strength defined in Eq. (3.31) and (3.43) and is a function of delaying time. For a fixed height, the Z-axis can be represented by the following transformation,

\[ R^2 = r^2 + (z - z_0)^2, \quad z - z_0 = r \tan \theta, \quad \eta = t - (r/C) \sec \theta \]

The transformed solution then is given by,
\[
\phi = -\frac{1}{2\pi} \int_{\eta_{i}}^{\eta_{e}} \frac{q(\eta)}{\sqrt{(t - \eta)^2 - (r / C)^2}} d\eta 
\]

(3.49)

Its time derivative is expressed as,

\[
\phi_t = -\frac{1}{2\pi} \int_{\eta_{i}}^{\eta_{e}} \frac{q'(\eta)}{\sqrt{(t - \eta)^2 - (r / C)^2}} d\eta 
\]

(3.50)

The source function ‘q’, is the instantaneous mass flux escaping from the SRM duct opening, which possesses the delta function behavior. However the forcing function is more conveniently defined with the perturbation pressure history because it is prescribed in the present approach.

\[
q'(\eta) = K_f \left( \frac{\Delta p}{p_\infty} \right)_{(r=0)} 
\]

(3.51)

Where, \( K_f \) is an empirical multiplying factor to match test data. From equation (3.31) and (3.43), it can be seen that \( \Delta p \) is proportional to \( \dot{P}_c \) and so \( q'(\eta) \) is proportional to \( \dot{P}_c \) from equations (3.47) and (3.50) we get,

\[
\frac{\Delta p(t)}{p_\infty} = K_f \int_{\eta_{i}}^{\eta_{e}} \frac{(\Delta p / p_\infty)_{(r=0)}}{\sqrt{(t - \eta)^2 - (r / C)^2}} d\eta 
\]

(3.52)

which simplifies to give

\[
\frac{\Delta p(t)}{p_\infty} = K_f \left( \frac{\Delta p}{p_\infty} \right)_{(r=0)} \int_{\theta_{i}}^{\theta_{e}} \sec \theta d\theta 
\]

(3.53)

3.2.9 Riemann Waves

The one dimensional continuity equation is
\[
\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + \rho \frac{\partial u}{\partial x} = 0
\]  
(3.54)

The one dimensional momentum conservation equation is

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{1}{\rho} \frac{\partial p}{\partial x} = 0
\]  
(3.55)

Expansion is isentropic therefore \( ds = 0 \)

\[
dp = \left( \frac{\partial p}{\partial \rho} \right) d\rho + \left( \frac{\partial p}{\partial s} \right) ds = \left( \frac{\partial p}{\partial \rho} \right) d\rho
\]

(3.56)

\[
\left( \frac{\partial p}{\partial \rho} \right) = a^2
\]  
(3.57)

Substituting in Eqn. (3.54) and multiplying Eqn. (3.55) by \( \pm \frac{a}{\rho} \) and adding

\[
\left[ \frac{\partial u}{\partial t} + (u \pm a) \frac{\partial u}{\partial x} \right] \pm \frac{a}{\rho} \left[ \frac{\partial \rho}{\partial t} + (u \pm a) \frac{\partial \rho}{\partial x} \right] = 0
\]

(3.58)

\[
\frac{d^{\pm}}{dt} = \frac{\partial}{\partial t} + (u \pm a) \frac{\partial}{\partial x}
\]

(3.59)

For isentropic case,

\[
\rho = cont.(a)^2
\]

(3.60)

Therefore

\[
\frac{d^{\pm}u}{dt} \pm \frac{2}{\gamma - 1} \frac{a^{\pm}a}{dt} = 0
\]

(3.61)

Hence,

\[
\frac{2a}{\gamma - 1} + u = cont. \quad \text{along the C}^+ \text{ characteristic}
\]

(3.62)
\[
\frac{2a}{\gamma - 1} - u = \text{const.}
\]
along the \(c\) characteristic

(expansion wave characteristics)

### 3.3 NUMERICAL SCHEME ADOPTED

The numerical scheme adopted for the Ignition Overpressure simulation during the start-up of solid rocket motors is described in detail. Governing equations, boundary conditions, turbulence model and convergence criteria are also highlighted in this chapter.

Numerical simulations have been carried out for a typical solid rocket motor and a relative smaller sized motor without nozzle shutter using the CFD software tool FLUENT. The Spalart – Allmaras turbulence model has been adopted in the turbulent compressible flow simulation. The steady state mass flow rate of the full size motor is 1.95 tonnes/s. The exhaust flow from the motor has been simulated for three different stand-off distances 3De, 4.5De and 6De, where De is the nozzle exit diameter and three ignition times of 50 ms, 100 ms and 150 ms (milli seconds). The computational domain with boundary conditions for the full size motor is shown in Figure 3.5. The entire domain was discretized into 2 lakhs structured cells of uniform size. Numerical solution has been obtained by advancing with respect to time, in steps of 0.5 millisecond time increment.

Fig 3.5 Computational domain of full size motor with boundary conditions
The steady state mass flow rate of the small size motor is 13 kg/s. The exhaust flow from the small size motor has been simulated for the ignition time of 5ms. Computational domain for the small size motor is shown in Figure 3.6. The entire domain has been discretized into 1 lakh structured cells of uniform size. Numerical solution was obtained by advancing with respect to time, in steps of 0.5 millisecond time increments

![Computational domain of small size motor with boundary conditions](image)

**Fig 3.6 Computational domain of small size motor with boundary conditions**

3.3.1 **GOVERNING EQUATIONS AND BOUNDARY CONDITIONS**

The unsteady state governing equations in axi-symmetric form for simulating the jet flow are briefed in the following. The flow field has been characterized by the viscous, compressible Navier-Stokes equations.

(I) **Mass Conservation Equation**

For axi-symmetric geometry

\[
\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u_x)}{\partial x} + \frac{\partial (\rho u_r)}{\partial r} + \frac{\rho u_r}{r} = 0
\]  

(3.64)

Where x is the axial coordinate r is the radial coordinate \( u_x \) is the axial velocity and \( u_r \) is the radial velocity.
(II) Momentum Conservation Equations

The axial and radial momentum conservation equations are given by

\[
\frac{\partial}{\partial t} (\rho u_x) + \frac{1}{r} \frac{\partial}{\partial r} (r\rho u_x u_r) + \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \rho u_r u_r) = -\frac{\partial p}{\partial x} + \frac{1}{r} \frac{\partial}{\partial r} \left[ r \mu \left( \frac{\partial u_x}{\partial x} + \frac{\partial u_r}{\partial r} \right) \right] + \frac{1}{r^2} \frac{\partial}{\partial r} \left[ r^2 \mu \left( \frac{\partial u_x}{\partial x} + \frac{\partial u_r}{\partial r} \right) \right]
\]

(3.65)

And

\[
\frac{\partial}{\partial t} (\rho u_r) + \frac{1}{r} \frac{\partial}{\partial x} (r\rho u_x u_r) + \frac{1}{r} \frac{\partial}{\partial r} (r\rho u_r u_r) = -\frac{\partial p}{\partial r} + \frac{1}{r} \frac{\partial}{\partial r} \left[ r \mu \left( \frac{\partial u_r}{\partial x} + \frac{\partial u_r}{\partial r} \right) \right] + \frac{1}{r^2} \frac{\partial}{\partial r} \left[ r^2 \mu \left( \frac{\partial u_r}{\partial x} + \frac{\partial u_r}{\partial r} \right) \right]
\]

(3.66)

Where \( p \) is the static pressure and \( \rho \) is the density.

\[
\nabla \cdot \vec{u} = \frac{\partial u_x}{\partial x} + \frac{\partial u_r}{\partial r} + \frac{u_r}{r}
\]

(3.67)

(III) Energy Equation

The total energy equation for the compressible viscous flow is given as

\[
\frac{\partial}{\partial t} (\rho e) + \nabla \cdot (\vec{u}(\rho E + p)) = \nabla \cdot (k_{\text{eff}} \nabla T - \sum_j h_j \vec{J}_j) + (\bar{\rho}_{\text{eff}} \bar{u})
\]

(3.68)

where \( k_{\text{eff}} \) is the effective conductivity \( k_{\text{eff}} = k + k_t \), where \( k_t \) is the turbulent thermal conductivity and \( k \) is the laminar thermal conductivity) and \( \vec{J}_j \) is the diffusion flux of species \( j \). Also the three terms on the right-hand side of above equation represent energy transfer due to conduction, species diffusion, and viscous dissipation, respectively. Also the specific total energy \( e \) is defined as,

\[
\begin{align*}
e &= h - \frac{p}{\rho} + \frac{u_x^2 + u_r^2}{2}
\end{align*}
\]

(3.69)

(IV) Boundary conditions

The boundary conditions employed for jet flow simulations are as listed below.
Mass flow inlet:

Mass flux of the exhaust gas \[ \frac{m^*}{\pi \times D \times l} \times (t + 0.0005) \text{ Kg} / \text{m}^2 \] \hspace{1cm} (3.70)

Where \( m^* \) is the steady state mass flow rate of propellant, Kg/s

\( D \) is the propellant grain minor diameter (meter)

\( l \) is the length of propellant grain, (meter)

\( t \) is time, (sec)

It has been assumed that the mass flow rate of the exhaust gas increases linearly with time during the ignition period.

(V)  **Far field boundary condition**

Atmospheric pressure and temperature boundary condition with zero second gradients are prescribed.

\[ P = P_{\infty}, \frac{\partial^2 u_x}{\partial x^2} \rightarrow 0, \frac{\partial^2 u_r}{\partial r^2} \rightarrow 0, T \rightarrow T_{\infty} \] \hspace{1cm} (3.71)

(VI)  **Axi-symmetric boundary condition**

A zero normal gradient is applied for the flow variables on the axis of symmetry,

\[ u_r = 0, \frac{\partial T}{\partial r} = 0, \frac{\partial u_x}{\partial x} = 0, \frac{\partial u_r}{\partial r} = 0 \] \hspace{1cm} (3.72)

(VII)  **Wall surface**

No slip condition for the velocity components is imposed on the “rigid flat plate” (simulating flat ground) on which the jet is impinging. Due to the short time of contact, this surface is also assumed as adiabatic. Hence

\[ u_x = 0, u_r = 0, \frac{\partial T}{\partial n} = 0 \] \hspace{1cm} (3.73)
3.4  SPALART-ALLMARAS TURBULENCE MODEL

The model proposed by Spalart and Allmaras solves a transport equation for a quantity that is a modified form of the turbulent kinematic viscosity. The transported variable, $\nu$, in the Spalart-Allmaras model is identical to the turbulent kinematic viscosity except in the near wall region.

$$\frac{\partial}{\partial t} (\rho \nu) + \frac{\partial}{\partial x_i} (\rho \nu u_i) = G_i + \frac{1}{\sigma_{tr}} \left[ \frac{\partial}{\partial x_j} (\nu + \rho \nu) \frac{\partial \nu}{\partial x_j} \right] + C_{\mu_2} \rho \left( \frac{\partial \nu}{\partial x_j} \right)^2 - Y_v$$  \hspace{1cm} (3.74)

Where, $G$ is the production of turbulent viscosity and $Y_v$ is the destruction of turbulent viscosity that occurs in the near-wall region due to wall blocking and viscous damping. $\sigma_{tr}$ and $C_{\mu_2}$ are constants and $\nu$ is the molecular kinematic viscosity.

3.4.1  MODEL FOR TURBULENT VISCOSITY

The turbulent viscosity, $\mu_t$, is computed from

$$\mu_t = \rho \nu f_{\nu_1}$$  \hspace{1cm} (3.75)

Where the viscous damping function, $f_{\nu_1}$, is given by

$$f_{\nu_1} = \frac{\chi^3}{\chi^3 + C_{\nu_1}^3}$$  \hspace{1cm} (3.76)

and

$$\chi = \frac{\nu}{\nu}$$  \hspace{1cm} (3.77)

3.4.2  MODEL FOR TURBULENT PRODUCTION

The production term, $G_{\nu}$, is modeled as

$$G_{\nu} = C_{\mu_1} \rho \tilde{S} \nu$$  \hspace{1cm} (3.78)

Where

$$\tilde{S} \equiv S + \frac{\nu}{k^2 d^2} f_{\nu_2}$$  \hspace{1cm} (3.79)
and

\[ f_{v2} = 1 - \frac{\chi}{1 + \chi f_{v1}} \]  \hspace{1cm} (3.80)

In the above expression \( C_{b1} \) and \( k \) are constants, \( d \) is the distance from the wall and \( S \) is the scalar measure of the deformation tensor is given as

\[ S = \sqrt{2\Omega_y \Omega_y} \]  \hspace{1cm} (3.81)

Where \( \Omega_y \) is the mean rate of rotation tensor of the form

\[ \Omega_y = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right) \]  \hspace{1cm} (3.82)

### 3.4.3 MODEL FOR TURBULENT DESTRUCTION

The turbulence destruction term is defined as

\[ Y_r = C_{w1} \rho f_w \left( \frac{v}{d} \right)^2 \]  \hspace{1cm} (3.83)

Where

\[ f_w = g \left[ \frac{1 + C_{w3}^6}{g^6 + C_{w3}^6} \right]^{\frac{1}{6}} \]  \hspace{1cm} (3.84)

\[ g = r + C_{w2} (r^6 - r) \]  \hspace{1cm} (3.85)

\[ r = \frac{v}{S k^2 d^2} \]  \hspace{1cm} (3.86)

and \( C_{w1}, C_{w2} \) and \( C_{w3} \) are constants.

### 3.4.4 MODEL CONSTANTS

The values of model constants for the S-A model are:

\[ C_{b1} = 0.1355, \quad C_{b2} = 0.622, \quad \sigma_r = \frac{2}{3}, \quad C_{v1} = 7.1 \]

\[ C_{w1} = \frac{C_{b1}}{k^2} + \frac{1 + C_{b2}}{\sigma_r} \]
3.5 NUMERICAL SOLUTION PROCEDURE

The governing equations and corresponding boundary conditions, described in section 3.4.4 to single supersonic free jet simulation have been solved using the commercial fluid flow solver, FLUENT. The conservation equations of mass, momentum and energy were incorporated in this solver using a control volume based finite difference method. In addition, the prefect gas equation was also included for describing the thermodynamic state of the fluid. Various turbulence models such as the standard $k-\varepsilon$ (Kinetic Energy), RSM (Reynolds’s Stress Model), ASM (Adoptive Simulation model), RNG (Re Normalization Group theory), modified $k-\varepsilon$ and the realizable $k-\varepsilon$ model can be introduced in the solution procedure.

The user can select the suitable turbulence model and its model constants depending on the problem requirements. In the present work, the Spalart – Allmaras (S-A) turbulence model has been employed because the other models are expensive in terms of both computational time and memory. Also the S-A model gives reasonably accurate predictions for compressible viscous flows.

3.6 SELECTION OF GRID

In CFD simulations, the grid employment plays a crucial and major role. The two basic steps involved in the discretization of the jet flow domain were the grid topology selection and the choice of grid generation scheme employed. In this study, axi-symmetric geometry has been created using the pre-processor GAMBIT, and the geometry was decomposed into mesh-able sections. The mesh has been generated locally for edges and the faces. Global meshing is resorted to wherein the entire volume is meshed. In the present study, the mesh is generated in the shape of a pseudo-rectangular domain. The entire domain has been discretized into 2 lakhs rectangular uniform cells.

3.7 GRID INDEPENDENCE TEST

For any numerical solution, a primary requirement is that the predicted solution should be independent of the grid employed. In the present work, grids of varying sizes were employed to ensure that the numerical solution obtained is grid
independent. The domain for the jet simulation was taken as rectangular with radial dimension of 21.3 m and axial dimension of 46 m.

In the present work Grids with 2 lakh and 1 lakh cells have been considered in the simulations. The results of grid independence study are shown in Figures 3.7(a) and 3.7(b) in terms of chamber pressure rise and overpressure. The predictions do not show marked variations for the grids considered. The final mesh selected for all the subsequent simulations has 2 lakh cells.

![Fig. 3.7(a) Chamber pressure variation](image)
![Fig. 3.7(b) Field Pressure variation](image)

### 3.8 CONVERGENCE CRITERION

The convergence criterion for the present study has been defined as the sum of normalized residue values (for the variables of mass, velocity components and turbulence quantities) being equal to $1 \times 10^{-6}$. The computations were under-relaxed to provide stability to the iterative solution procedure.

**Summary:**

Various postulated theories for understanding be physics Overpressure phenomenon is explained. The geometric configuration and numerical scheme adopted with CFD software tools is explained in this chapter. The next chapter deals with the experimental methodology followed.