CHAPTER-III

NEW INFORMATION THEORETIC MEASURES OF ENTROPY, R-DIVERGENCE AND IMPROVEMENT FOR FUZZY DISTRIBUTIONS AND THEIR PROPERTIES

3.1. Introduction:

A feature of imperfect information known as fuzziness results from the lack of crisp distinction between the elements belonging and not belonging to a set, that is, the boundaries of the set under consideration are not sharply defined. A measure of fuzziness often used and cited in the literature of information theory, known as fuzzy entropy, was first introduced by Zadeh (1965). The name entropy was chosen due to an intrinsic similarity of equations to the ones in the Shannon entropy (1948). However, the two functions measure fundamentally different types of uncertainty. Basically, the probabilistic measure of entropy due to Shannon measures the average uncertainty in bits associated with the prediction of outcomes in a random experiment whereas fuzzy entropy is the quantitative description of fuzziness in fuzzy sets.

De Luca and Termini (1972) introduced some requirements which capture our intuitive comprehension of the degree of fuzziness. Kaufmann (1975) proposed to measure the degree of fuzziness of any fuzzy set $A$ by a metric distance between its membership function and the characteristic function of its nearest crisp set. Another way given by Yager (1979) was to view the degree of fuzziness in terms of a lack of distinction between the fuzzy set and its complement.

Fuzzy entropy is one of the important digital features of fuzzy sets and occupies an important place in system model and system design. For example, when generalized fuzzy entropy is used as learning criterion for neural networks, efficient structure parameters are obtained quickly. In other words, generalized fuzzy entropy has better guidance function in neural network system design.
Motivated by the existing probabilistic measure of entropy, De Luca and Termini (1972) took the following expression for fuzzy entropy:

\[
H(A) = -\sum_{i=1}^{n} \left[ \mu_A(x_i) \log \mu_A(x_i) + (1 - \mu_A(x_i)) \log (1 - \mu_A(x_i)) \right]
\]  

(3.1.1)


Distance measure is a term that describes the difference between fuzzy sets and can be considered as a dual concept of similarity measure. Many researchers, such as Yager (1979), Kosko (1991) and Kaufmann (1975) had used distance measure to define fuzzy entropy. Using the axiom definition of distance measure, Fan et al. (2001) developed some new formulas of fuzzy entropy induced by distance measure and studied some new properties of distance measure. Dubois and Prade (1983) defined the distance between two fuzzy subsets on a fuzzy subset of \( \text{R}^+ \). Their definition does not generalize the shortest distance between two crisp sets; rather it generalizes the set of distances between two sets. Rosenfeld (1985) defined the shortest distance between two fuzzy sets as a density function on the non-negative reals, which generalizes the definition of shortest distance for crisp sets in a natural way. Thus, corresponding to the probabilistic measure of divergence due to Kullback and Leibler (1951), Bhandari and Pal (1993) introduced the following measure of fuzzy directed divergence:

\[
I(A : B) = \sum_{i=1}^{n} \left[ \mu_A(x_i) \log \frac{\mu_A(x_i)}{\mu_B(x_i)} + (1 - \mu_A(x_i)) \log \frac{1 - \mu_A(x_i)}{1 - \mu_B(x_i)} \right]
\]  

(3.1.2)

Many measures of fuzzy divergence have been developed by Kapur (1997), Parkash (2000), Parkash and Sharma (2005), Pal and Bezdek (1994) etc.

In section 3.2, we have introduced two new generalized measures of fuzzy entropy and to verify their authenticity, we have studied their essential properties. Section 3.3 deals with the introduction of a new measure of fuzzy \( R \)-divergence.
by applying De Luca and Termini’s (1972) fuzzy entropy. The properties of this new measure have been studied. Further, we have generalized the R-divergence introduced in this section and studied its desirable properties. Using the concept of information improvement, we have developed some measures of fuzzy information improvement and studied their important properties. The findings of these results have been presented in section 3.4 of this chapter.

3.2. New Generalized Measures Of Fuzzy Entropy And Their Properties:

In this section, we have proposed two new generalized information measures for a fuzzy distribution \( \{\mu_A(x_1), \mu_A(x_2), \ldots, \mu_A(x_n): 0 \leq \mu_A(x_i) \leq 1\} \) and studied their essential and desirable properties.

### 3.2.1 Generalized Fuzzy Entropy Involving Three Parameters \( \alpha, \beta \) and \( \gamma \)

We propose the generalized fuzzy entropy depending upon three real parameters \( \alpha, \beta \) and \( \gamma \) as given by the following mathematical expression:

\[
H_{\alpha,\beta,\gamma}(A) = \frac{1}{1-\alpha} \sum_{i=1}^{n} \log \frac{\mu_A^{a+\beta+\gamma-1}(x_i) + (1-\mu_A(x_i))^{\alpha+\beta+\gamma-1}}{\mu_A^{\alpha+\beta+\gamma}(x_i) + (1-\mu_A(x_i))^{\beta+\gamma}},
\]

\[\alpha \neq 1, \beta \geq 0, \gamma \geq 0, \alpha + \beta + \gamma > 0, \beta + \gamma - 1 \geq 0 \quad (3.2.1)\]

If \( \alpha \to 1 \), the measure (3.2.1) becomes

\[
H_{\beta,\gamma}(A) = -\sum_{i=1}^{n} \left[ \frac{\mu_A^{\beta+\gamma}(x_i) \log \mu_A(x_i) + (1-\mu_A(x_i))^{\beta+\gamma} \log \{1-\mu_A(x_i)\}}{\mu_A^{\beta+\gamma}(x_i) + (1-\mu_A(x_i))^{\beta+\gamma}} \right] \quad (3.2.2)
\]

If \( \beta = 0 \) and \( \gamma = 1 \), then the above equation (3.2.2) reduces to

\[
H(A) = -\sum_{i=1}^{n} [\mu_A(x_i) \log \mu_A(x_i) + (1-\mu_A(x_i)) \log (1-\mu_A(x_i))] \]

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which is fuzzy entropy introduced by De Luca and Termini (1972).
Further, if $\beta = 1$ and $\gamma = 0$, then the above equation (3.2.2) again reduces to De Luca and Termini’s (1972) fuzzy entropy.
Moreover, if $\beta + \gamma = 1$, then the equation (3.2.1) reduces to

$$H_{\alpha}(A) = \frac{1}{1 - \alpha} \sum_{i=1}^{n} \log [\mu_{A}^{\alpha}(x_{i}) + (1 - \mu_{A}(x_{i}))^{\alpha}], \quad \alpha \neq 1, \alpha > 0$$

which is a fuzzy entropy introduced by Renyi (1961).
Thus, we see that the measure introduced in equation (3.2.1) is a generalized measure of fuzzy entropy.
Next, we study the essential and desirable properties of the generalized measure of fuzzy entropy (3.2.1):

**I. Essential Properties:**

(i) $H_{\alpha,\beta,\gamma}(A) \geq 0$

(ii) \[
\frac{\partial H_{\alpha,\beta,\gamma}(A)}{\partial \mu_{A}(x_{i})} = \frac{1}{1 - \alpha} \left( \alpha + \beta + \gamma - 1 \right) \frac{\mu_{A}^{\alpha+\beta+\gamma-2}(x_{i}) - (1 - \mu_{A}(x_{i}))^{\alpha+\beta+\gamma-2}}{\mu_{A}^{\alpha+\beta+\gamma-1}(x_{i}) + (1 - \mu_{A}(x_{i}))^{\alpha+\beta+\gamma-1}} - \frac{1}{1 - \alpha} \frac{\beta + \gamma}{\mu_{A}^{\alpha+\beta+\gamma-1}(x_{i}) + (1 - \mu_{A}(x_{i}))^{\alpha+\beta+\gamma}}
\]

Also, we have seen that

$$\frac{\partial^{2} H_{\alpha,\beta,\gamma}(A)}{\partial \mu_{A}^{2}(x_{i})} < 0$$

Thus $H_{\alpha,\beta,\gamma}(A)$ is a concave function of $\mu_{A}(x_{i}) \ \forall \ i$

(iii) $H_{\alpha,\beta,\gamma}(A)$ does not change when $\mu_{A}(x_{i})$ is replaced by $1 - \mu_{A}(x_{i})$

(iv) $H_{\alpha,\beta,\gamma}(A)$ is an increasing function of $\mu_{A}(x_{i})$ for $0 \leq \mu_{A}(x_{i}) \leq \frac{1}{2}$
\[\{ H_{\alpha,\beta,\gamma}(A) / \mu_{A}(x_{i}) = 0 \} = 0 \text{ and } \{ H_{\alpha,\beta,\gamma}(A) / \mu_{A}(x_{i}) = \frac{1}{2} \} = n \log 2 > 0\]

(v) $H_{\alpha,\beta,\gamma}(A)$ is decreasing function of $\mu_{A}(x_{i})$ for $\frac{1}{2} \leq \mu_{A}(x_{i}) \leq 1$
\[\{ H_{\alpha,\beta,\gamma}(A) / \mu_{A}(x_{i}) = \frac{1}{2} \} = n \log 2\]
and

\[ \left\{ \frac{H_{\alpha,\beta,\gamma}(A)}{\mu_A(x_i)} = 1 \right\} = 0 \]

(vi) \( H_{\alpha,\beta,\gamma}(A) = 0 \) for \( \mu_A(x_i) = 0 \) or 1

Under these conditions, the measure \( H_{\alpha,\beta,\gamma}(A) \) is a valid measure of fuzzy entropy.

Next, with the help of the data, we have presented the generalized measure (3.2.1) graphically. For this purpose, we have fixed \( \beta = 2 \) and \( \gamma = 3 \). Then, for different values of \( \alpha \), we have computed different values of \( H_{\alpha,\beta,\gamma}(A) \) as shown in the following table- 3.2.1:

**Table-3.2.1: Computations of \( H_{\alpha,\beta,\gamma}(A) \) when \( \beta = 2 \) and \( \gamma = 3 \).**

<table>
<thead>
<tr>
<th>( \mu_A(x_i) )</th>
<th>( \alpha )</th>
<th>( H_{\alpha,\beta,\gamma}(A) )</th>
<th>( \alpha )</th>
<th>( H_{\alpha,\beta,\gamma}(A) )</th>
<th>( \alpha )</th>
<th>( H_{\alpha,\beta,\gamma}(A) )</th>
</tr>
</thead>
<tbody>
<tr>
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<td>0.0000</td>
<td>0.0000</td>
<td>2</td>
<td>0.6931</td>
<td>16</td>
<td>0.6931</td>
</tr>
<tr>
<td>0.1</td>
<td>0.1054</td>
<td>0.1053</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.2</td>
<td>0.2239</td>
<td>0.2236</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.3</td>
<td>0.3648</td>
<td>0.3610</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.4</td>
<td>0.5767</td>
<td>0.5393</td>
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<td></td>
</tr>
<tr>
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<td>0.6931</td>
<td>0.6931</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.6</td>
<td>0.5767</td>
<td>0.5393</td>
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<td></td>
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<tr>
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<td>0.3610</td>
<td></td>
<td></td>
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<tr>
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<td>0.0000</td>
<td>0.0000</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Next, we have presented the values of \( H_{\alpha,\beta,\gamma}(A) \) graphically for \( \alpha = 2, 4 \) and 16 and obtained the following Fig.-3.2.1 which shows that the measure introduced in equation (3.2.1) is a concave function.
II. Desirable Properties:

(i) Maximum Value

Differentiating (3.2.1) with respect to $\mu_A(x_i)$, we get

$$\frac{\partial H_{\alpha, \beta, \gamma}(A)}{\partial \mu_A(x_i)} = \frac{1}{1-\alpha} \left\{ \mu_A^{\alpha+\beta+\gamma-1}(x_i) + (1-\mu_A(x_i))^{\alpha+\beta+\gamma-1} \right\}$$

$$-\frac{1}{1-\alpha} \left\{ \mu_A^{\beta+\gamma-1}(x_i) - (1-\mu_A(x_i))^{\beta+\gamma-1} \right\}$$

Thus, $\frac{\partial H_{\alpha, \beta, \gamma}(A)}{\partial \mu_A(x_i)} = 0$ gives

$$(\alpha + \beta + \gamma - 1) \left\{ \mu_A^{\alpha+\beta+\gamma-2}(x_i) - (1-\mu_A(x_i))^{\alpha+\beta+\gamma-2} \right\} \left\{ \mu_A^{\beta+\gamma}(x_i) + (1-\mu_A(x_i))^{\beta+\gamma} \right\}$$
\[ (\alpha - 1) \left\{ \frac{\mu_A^{\alpha + \beta + \gamma - 2}(x_i)(1 - \mu_A(x_i))^{\alpha + \beta + \gamma - 2} + \mu_A^{\alpha + \beta + \gamma - 2}(x_i)(1 - \mu_A(x_i))^{\beta + \gamma} - (1 - \mu_A(x_i))^{\alpha + 2\beta + 2\gamma - 2}}{1 - \alpha} \right\} \]

or

\[ + (\beta + \gamma) \left\{ \frac{\mu_A^{\alpha + \beta + \gamma - 2}(x_i)(1 - \mu_A(x_i))^{\beta + \gamma - 1} - \mu_A^{\alpha + \beta + \gamma - 1}(x_i)(1 - \mu_A(x_i))^{\alpha + \beta + \gamma - 2}}{1 - \alpha} \right\} = 0 \]

or

\[ (\alpha - 1) \left\{ \frac{\mu_A^{\alpha + \beta + \gamma - 2}(x_i)(1 - \mu_A(x_i))^{\alpha + \beta + \gamma - 2} + \mu_A^{\beta + \gamma - 1}(x_i)(1 - \mu_A(x_i))^{\beta + \gamma} + (1 - \mu_A(x_i))^{\beta + \gamma}}{1 - \alpha} \right\} \]

+ (\beta + \gamma) \left\{ \frac{\mu_A^{\alpha + \beta + \gamma - 1}(x_i)(1 - \mu_A(x_i))^{\beta + \gamma - 1}}{1 - \alpha} \right\} \] = 0

The above equality is the sum of two positive term quantities and each term equated to zero gives \( \mu_A(x_i) = \frac{1}{2} \).

Again, we have

\[ \frac{\partial^2 H_{\alpha, \beta, \gamma}(A)}{\partial \mu_A^2} = \frac{(\alpha + \beta + \gamma - 1)}{1 - \alpha} \]

\[ \left[ (\alpha + \beta + \gamma - 2) \left\{ \mu_A^{\alpha + \beta + \gamma - 1}(x_i) + (1 - \mu_A(x_i))^{\alpha + \beta + \gamma - 1} \right\} \right] \]

\[ - (\alpha + \beta + \gamma - 1) \left\{ \mu_A^{\alpha + \beta + \gamma - 2}(x_i) - (1 - \mu_A(x_i))^{\alpha + \beta + \gamma - 2} \right\} \]

\[ \frac{\mu_A^{\alpha + \beta + \gamma - 1}(x_i)(1 - \mu_A(x_i))^{\alpha + \beta + \gamma - 1}}{1 - \alpha} \]

Taking \( \mu_A(x_i) = \frac{1}{2} \), the above equation gives
Thus, we see that the maximum value of the fuzzy entropy exists at \( \mu_A(x_i) = \frac{1}{2} \).

If, we denote the maximum value by \( f(n) \), then

\[ f(n) = n \log 2 \]

Further, we have

\[ f'(n) = \log 2 > 0 \]

This shows that the maximum value of the generalized fuzzy entropy is an increasing function of \( n \) which is most desirable result.

(ii) Monotonicity

Differentiating equation (3.2.1) with respect to \( \alpha \), we get

\[
\begin{aligned}
(1-\alpha)^2 \frac{dH_{a,\beta,\gamma}(A)}{d\alpha} & = \sum_{i=1}^{N} \left\{ \mu_A^{a+\beta+\gamma-1}(x_i) \log \mu_A^{1-\alpha}(x_i) + (1-\mu_A(x_i))^{a+\beta+\gamma-1} \log(1-\mu_A(x_i))^{1-\alpha} \right\} \\
& \quad \bigg/ \left\{ \mu_A^{a+\beta+\gamma-1}(x_i) + (1-\mu_A(x_i))^{a+\beta+\gamma-1} \right\} \\
& + \sum_{i=1}^{N} \log \left[ \frac{\mu_A^{a+\beta+\gamma-1}(x_i) + (1-\mu_A(x_i))^{a+\beta+\gamma-1}}{\mu_A^{a+\beta+\gamma}(x_i) + (1-\mu_A(x_i))^{a+\beta+\gamma-1}} \right] (3.2.3)
\end{aligned}
\]

Now, we discuss the following two cases:

Case-I. When \( \alpha > 1 \)

In this case, we have

\[
(1-\alpha)^2 \frac{d}{d\alpha} H_{a,\beta,\gamma}(A) =
\]
(1 − α) \sum_{i=1}^{n} \left[ \frac{\mu_A^{\alpha + \beta + \gamma - 1}(x_i) \log \mu_A(x_i) + (1 - \mu_A(x_i))^{\alpha + \beta + \gamma - 1} \log (1 - \mu_A(x_i))}{\mu_A^{\alpha + \beta + \gamma - 1}(x_i) + (1 - \mu_A(x_i))^{\alpha + \beta + \gamma - 1}} \right] \\
+ \sum_{i=1}^{n} \log \left[ \frac{\mu_A^{\alpha + \beta + \gamma - 1}(x_i) + (1 - \mu_A(x_i))^{\alpha + \beta + \gamma - 1}}{\mu_A^{\beta + \gamma}(x_i) + (1 - \mu_A(x_i))^{\beta + \gamma}} \right] \\
= X + Y 
(3.2.4)

where 
Y = \sum_{i=1}^{n} \log \left[ \frac{\mu_A^{\alpha + \beta + \gamma - 1}(x_i) + (1 - \mu_A(x_i))^{\alpha + \beta + \gamma - 1}}{\mu_A^{\beta + \gamma}(x_i) + (1 - \mu_A(x_i))^{\beta + \gamma}} \right] < 0, \\
for 0 < \mu_A(x_i) < 1, \; \alpha > 1, \; \beta + \gamma \geq 0 

and 
X = (1 - \alpha) \sum_{i=1}^{n} \left[ \frac{\mu_A^{\alpha + \beta + \gamma - 1}(x_i) \log \mu_A(x_i) + (1 - \mu_A(x_i))^{\alpha + \beta + \gamma - 1} \log (1 - \mu_A(x_i))}{\mu_A^{\alpha + \beta + \gamma - 1}(x_i) + (1 - \mu_A(x_i))^{\alpha + \beta + \gamma - 1}} \right] > 0, \\
\alpha > 1, 0 < \mu_A(x_i) < 1, \; \beta + \gamma \geq 0 

Next, we claim that \( X - Y \leq 0 \) with equality if \( \alpha = 1 \).

To verify the above result, we proceed as follows:
Consider \( X - Y \leq 0 \).
This gives
\[
(1 - \alpha) \sum_{i=1}^{n} \left[ \frac{\mu_A^{\alpha + \beta + \gamma - 1}(x_i) \log \mu_A(x_i) + (1 - \mu_A(x_i))^{\alpha + \beta + \gamma - 1} \log (1 - \mu_A(x_i))}{\mu_A^{\alpha + \beta + \gamma - 1}(x_i) + (1 - \mu_A(x_i))^{\alpha + \beta + \gamma - 1}} \right] \\
- \sum_{i=1}^{n} \log \left[ \frac{\mu_A^{\alpha + \beta + \gamma - 1}(x_i) + (1 - \mu_A(x_i))^{\alpha + \beta + \gamma - 1}}{\mu_A^{\beta + \gamma}(x_i) + (1 - \mu_A(x_i))^{\beta + \gamma}} \right] \leq 0
\]
or
\[
(\alpha - 1) \sum_{i=1}^{n} \left[ \frac{\mu^\alpha A(x_i) \log \mu_A(x_i) + (1 - \mu_A(x_i))^{\alpha - 1} \log (1 - \mu_A(x_i))}{\mu^\alpha A(x_i) + (1 - \mu_A(x_i))^{\alpha - 1}} \right]
\]

\[
+ \sum_{i=1}^{n} \log \left\{ \mu^\alpha A(x_i) + (1 - \mu_A(x_i))^{\alpha - 1} \right\}
\]

\[
\geq \sum_{i=1}^{n} \log \left( \mu^\beta A(x_i) + (1 - \mu_A(x_i))^{\beta + \gamma} \right)
\]

or
\[
\sum_{i=1}^{n} \left[ \mu^\alpha A(x_i) \log \mu_A(x_i) + (1 - \mu_A(x_i))^{\alpha - 1} \log (1 - \mu_A(x_i))^{\alpha - 1} \right]
\]

\[
\geq \sum_{i=1}^{n} \left[ \mu^\beta A(x_i) + (1 - \mu_A(x_i))^{\beta + \gamma} \right] \times \log \left[ \mu^\gamma A(x_i) + (1 - \mu_A(x_i))^{\gamma} \right]
\]

which is always true for \(\alpha > 1\). Thus, equation (3.2.4) is always negative.

Hence, \(\frac{dH_{\alpha,\beta,\gamma}(A)}{d\alpha} \leq 0\) which shows that the generalized measure \(H_{\alpha,\beta,\gamma}(A)\) is monotonically decreasing function of \(\alpha\) for \(\alpha > 1\).

**Note:** It has numerically been verified that \(\frac{dH_{\alpha,\beta,\gamma}(A)}{d\alpha} \leq 0\) as shown in the following table-3.2.2:
Table-3.2.2: Computations of $(1 - \alpha)^2 \frac{dH_{\alpha,\beta,\gamma}(A)}{d\alpha}$

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\mu_A(x_i)$</th>
<th>$(1 - \alpha)^2 \frac{dH_{\alpha,\beta,\gamma}(A)}{d\alpha}$</th>
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</thead>
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<td></td>
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Table-3.2.3: Computations of $H_{\alpha,\beta,\gamma}(A)$ when $\alpha > 1$

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\mu_{A}(x_i)$</th>
<th>$H_{\alpha,\beta,\gamma}(A)$</th>
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<td>0.5</td>
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</tbody>
</table>
Next with the help of the data, we have computed different values of \( H_{\alpha,\beta,\gamma}(A) \) for different values of \( \alpha \) for fixed \( \beta = 2 \) and \( \gamma = 3 \) as shown in table- 3.2.3.

Next, we have presented \( H_{\alpha,\beta,\gamma}(A) \) graphically for different values of \( \alpha \) for fixed \( \beta = 2 \) and \( \gamma = 3 \) and obtained Fig.-3.2.2 which shows that \( H_{\alpha,\beta,\gamma}(A) \) introduced in (3.2.1) is monotonically decreasing function of \( \alpha \), \( \alpha > 1 \).

![Graph](image)

**Fig.-3.2.2: Monotonicity of \( H_{\alpha,\beta,\gamma}(A) \) when \( \alpha > 1 \)**

**Case-II.** When \( \alpha < 1 \)

Using equation (3.2.3), we have

\[
\begin{align*}
f(0) &= \sum_{i=1}^{n} \left\{ \mu_A^{\beta+y-1}(x_i) \log \mu_A(x_i) + (1 - \mu_A(x_i))^{\beta+y-1} \log(1 - \mu_A(x_i)) \right\} < 0
\end{align*}
\]

since \( \beta + \gamma - 1 \geq 0 \) and \( 0 \leq \mu_A(x_i) \leq 1 \)

Also \( f(1) = \sum_{i=1}^{n} \left\{ \mu_A^{\beta+y}(x_i) \log 1 + (1 - \mu_A(x_i))^{\beta+y} \log 1 \right\} = 0 \)

Thus, \( f(0) < 0 \) and \( f(1) = 0 \) which shows that \( f(\alpha) \) is negative for \( 0 < \alpha < 1 \).

Hence \( \frac{dH_{\alpha,\beta,\gamma}(A)}{d\alpha} \leq 0 \), which shows that \( H_{\alpha,\beta,\gamma}(A) \) is monotonically decreasing function of \( \alpha \).
Table-3.2.4: Computations of $H_{\alpha,\beta,\gamma}(A)$ when $0 < \alpha < 1$

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\mu_A(x_i)$</th>
<th>$H_{\alpha,\beta,\gamma}(A)$</th>
</tr>
</thead>
<tbody>
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<td>0.1</td>
<td>0.1</td>
<td>1.9652</td>
</tr>
<tr>
<td></td>
<td>0.2</td>
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<td>0.5</td>
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<tr>
<td>0.2</td>
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<td>0.4</td>
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<td>1.9591</td>
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<td>0.5</td>
<td>0.1</td>
<td>1.9574</td>
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<td>0.5</td>
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</tbody>
</table>
Next, with the help of the data, we have computed different values of $H_{\alpha,\beta,\gamma}(A)$ for different values of $\alpha$ for fixed $\beta = 2$ and $\gamma = 3$ as shown in the table- 3.2.4:

Next, we have presented the values of $H_{\alpha,\beta,\gamma}(A)$ graphically for different values of the parameter $\alpha$ for fixed $\beta = 2$ and $\gamma = 3$ and obtained the following Fig.-3.2.3 which shows that the measure of generalized fuzzy entropy $H_{\alpha,\beta,\gamma}(A)$ introduced in (3.2.1) is monotonically decreasing function of $\alpha$ for $0 < \alpha < 1$.

![Graph showing monotonically decreasing function of $H_{\alpha,\beta,\gamma}(A)$](image)

**Fig.-3.2.3: Monotonicity of $H_{\alpha,\beta,\gamma}(A)$ when $0 < \alpha < 1$**

### 3.2.2. Generalized Fuzzy Entropy Involving Two Parameters $\alpha$ And $\beta$:

Now, we introduce another parametric measure of fuzzy entropy depending upon two real parameters $\alpha$ and $\beta$. This measure is given by

$$H^{\alpha,\beta}(A) = \sum_{i=1}^{n} \left\{ \mu_A(x_i)^{\alpha+\beta} + (1-\mu_A(x_i))^{\alpha+\beta} - 1 \right\} \quad \frac{2^{1-\alpha-\beta}-1}{2^{1-\alpha-\beta}-1} \quad \alpha + \beta \neq 1, \alpha + \beta > 0 \quad (3.2.5)$$

If $\alpha = 0$ and $\beta = 1$, then the above measure (3.2.5) becomes

$$H(A) = \sum_{i=1}^{n} \left[ \mu_A(x_i) \log \mu_A(x_i) + (1-\mu_A(x_i)) \log (1-\mu_A(x_i)) \right]$$

which is fuzzy entropy introduced by De Luca and Termini (1972).

Further, if $\alpha = 1$ and $\beta = 0$, then the above equation (3.2.5) again reduces to De Luca and Termini’s (1972) fuzzy entropy.
Moreover, if $\alpha + \beta \to 1$, then the equation (3.2.5) reduces to

$$H^{\alpha, \beta}(A) = -\sum_{i=1}^{n} \left[ \mu_A(x_i) \log \mu_A(x_i) + (1 - \mu_A(x_i)) \log (1 - \mu_A(x_i)) \right]$$

which is again a fuzzy entropy introduced by De Luca and Termini (1972). Thus, we see that the measure of entropy introduced in equation (3.2.5) is a generalized measure of fuzzy entropy.

Next, we study some essential and desirable properties of the generalized measure of fuzzy entropy (3.2.5):

I. Essential Properties:

(i) $H^{\alpha, \beta}(A) \geq 0$

(ii) \[
\frac{\partial H^{\alpha, \beta}(A)}{\partial \mu_A(x_i)} = \frac{\alpha + \beta}{2^{\alpha - \beta} - 1} \left[ \mu_A^{\alpha \beta - 1}(x_i) - (1 - \mu_A(x_i))^{\alpha \beta - 1} \right]
\]

Also

\[
\frac{\partial^2 H^{\alpha, \beta}(A)}{\partial \mu_A^2(x_i)} = -\frac{2^{\alpha \beta} \left\{ (\alpha + \beta)^2 - (\alpha + \beta) \right\}}{2^{\alpha \beta} - 2} \left[ \mu_A^{\alpha \beta - 2}(x_i) + (1 - \mu_A(x_i))^{\alpha \beta - 2} \right]
\]

Now, two cases arise:

Case-I. If $\alpha + \beta > 1$, then $\frac{\partial^2 H^{\alpha, \beta}(A)}{\partial \mu_A^2(x_i)} < 0$

Case-II. If $\alpha + \beta < 1$, then $(\alpha + \beta)^2 < (\alpha + \beta)$ and $2^{\alpha \beta} < 2$. Thus, in this case, again, we see that

$$\frac{\partial^2 H^{\alpha, \beta}(A)}{\partial \mu_A^2(x_i)} < 0$$

Thus $H^{\alpha, \beta}(A)$ is a concave function of $\mu_A(x_i) \ \forall \ i$

(iii) $H^{\alpha, \beta}(A)$ does not change when $\mu_A(x_i)$ is replaced by $1 - \mu_A(x_i)$

(iv) $H^{\alpha, \beta}(A)$ is an increasing function of $\mu_A(x_i)$ for $0 \leq \mu_A(x_i) \leq 1/2$

\{ $H^{\alpha, \beta}(A) / \mu_A(x_i) = 0$\} = 0
\{ H^{\alpha,\beta}(A) / \mu_A(x_i) = \frac{1}{2} \} = \sum_{i=1}^{n} \frac{2^{1-\alpha-\beta} - 1}{2^{1-\alpha-\beta} - 1} = n > 0

(v) $H^{\alpha,\beta}(A)$ is decreasing function of $\mu_A(x_i)$ for $\frac{1}{2} \leq \mu_A(x_i) \leq 1$

$\{ H^{\alpha,\beta}(A) / \mu_A(x_i) = \frac{1}{2} \} = n > 0$ and $\{ H^{\alpha,\beta}(A) / \mu_A(x_i) = 1 \} = 0$

(vi) $H^{\alpha,\beta}(A) = 0$ for $\mu_A(x_i) = 0$ or 1

Under these conditions, $H^{\alpha,\beta}(A)$ is a valid measure of fuzzy entropy.

Next, with the help of the data, we have presented the measure (3.2.5) graphically. For this purpose, we have fixed $\beta = 2$. Then, for different values of $\alpha$, we have computed different values of $H^{\alpha,\beta}(A)$ as shown in the following table- 3.2.5:

**Table-3.2.5: Computations of $H^{\alpha,\beta}(A)$ when $\alpha > 1$**

<table>
<thead>
<tr>
<th>$\mu_A(x_i)$</th>
<th>$\alpha$</th>
<th>$H^{\alpha,\beta}(A)$</th>
<th>$\alpha$</th>
<th>$H^{\alpha,\beta}(A)$</th>
<th>$\alpha$</th>
<th>$H^{\alpha,\beta}(A)$</th>
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<th>$H^{\alpha,\beta}(A)$</th>
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<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td></td>
<td>0.0000</td>
</tr>
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<td>0.1</td>
<td></td>
<td>0.3929</td>
<td>0.4836</td>
<td>0.6500</td>
<td>0.8400</td>
<td>0.0000</td>
<td></td>
<td>0.0000</td>
</tr>
<tr>
<td>0.2</td>
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<td>0.7620</td>
<td>0.7616</td>
<td>0.8900</td>
<td>0.9820</td>
<td>0.9999</td>
<td></td>
<td>0.9999</td>
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<tr>
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<td>0.9100</td>
<td>0.9736</td>
<td></td>
<td>0.9984</td>
<td></td>
<td>0.9984</td>
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<td>0.9999</td>
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<td>0.9100</td>
<td>0.9736</td>
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<td>0.7620</td>
<td>0.7616</td>
<td>0.8900</td>
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<td>0.9820</td>
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<td>0.9820</td>
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<td>0.9</td>
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<td>0.3929</td>
<td>0.4836</td>
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<td>0.8400</td>
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<td>0.0000</td>
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</table>

Next, we have presented the values of $H^{\alpha,\beta}(A)$ graphically for $\alpha = 4$ and obtained the following Fig.-3.2.4 which shows that the measure introduced in equation (3.2.2) is a concave function. Similarly, for other values of $\alpha$, we get different concave curves.
II. Desirable Properties

(i) Maximum value

Differentiating (3.2.5) with respect to $\mu_A(x_i)$, we get

$$\frac{\partial H^{\alpha, \beta}(A)}{\partial \mu_A(x_i)} = \frac{\alpha + \beta}{2^{1-\alpha-\beta} - 1} \left\{ \mu_A^{\alpha+\beta-1}(x_i) - (1 - \mu_A(x_i))^{\alpha+\beta-1} \right\}$$

Thus,

$$\frac{\partial H^{\alpha, \beta}(A)}{\partial \mu_A(x_i)} = 0 \text{ gives } \mu_A(x_i) = \frac{1}{2} \forall i.$$

Again, we have

$$\frac{\partial^2 H^{\alpha, \beta}(A)}{\partial \mu_A^2(x_i)} = \frac{(\alpha + \beta)(\alpha + \beta - 1)}{2^{1-\alpha-\beta} - 1} \left\{ \mu_A^{\alpha+\beta-2}(x_i) + (1 - \mu_A(x_i))^{\alpha+\beta-2} \right\}$$

Taking $\mu_A(x_i) = \frac{1}{2}$, the above equation gives
Thus, we see that the maximum value of the fuzzy entropy exists at $\mu_A(x) = \frac{1}{2}$.

Further, if we denote the maximum value by $f(n)$, then

$$f(n) = n$$

which implies that $f'(n) = 1 > 0$

This shows that the maximum value of the generalized fuzzy entropy is an increasing function of $n$ which is most desirable result.

(ii) Monotonicity

Differentiating equation (3.2.5) w.r.t. $\alpha$, we get

$$\frac{d^2 H^{\alpha, \beta}(A)}{d\mu_A^2(x)} = \frac{(\alpha + \beta)(\alpha + \beta - 1)}{2^{\alpha - \beta} - 1} 2^{\alpha - \beta + 2}$$

$$= -8 \left[ \frac{(\alpha + \beta)^2 - (\alpha + \beta)}{2^{\alpha + \beta} - 2} \right] < 0$$

Next, we discuss following two cases:

Case-I. When $\alpha + \beta > 1$

For $0 < \mu_A(x) < 1$, we have $X > 0$, $Y > 0$

Thus, from equation (3.2.6), we have $\left\{2^{\alpha - \beta} - 1\right\}^2 \frac{dH^{\alpha, \beta}(A)}{d\alpha} \geq 0$

which shows that $H^{\alpha, \beta}(A)$ is monotonically increasing function of $\alpha$.

Next, with the help of the data, we have computed different values of $H^{\alpha, \beta}(A)$ for different values of $\alpha$ for fixed $\beta = 2$ as shown in the following table- 3.2.6:
Table-3.2.6: Computations of $H^{\alpha,\beta}(A)$ when $\alpha + \beta > 1$

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\mu_A(x_i)$</th>
<th>$H^{\alpha,\beta}(A)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.1&lt;br&gt;0.2&lt;br&gt;0.3&lt;br&gt;0.4&lt;br&gt;0.5</td>
<td>3.8905</td>
</tr>
<tr>
<td>3</td>
<td>0.1&lt;br&gt;0.2&lt;br&gt;0.3&lt;br&gt;0.4&lt;br&gt;0.5</td>
<td>4.0112</td>
</tr>
<tr>
<td>4</td>
<td>0.1&lt;br&gt;0.2&lt;br&gt;0.3&lt;br&gt;0.4&lt;br&gt;0.5</td>
<td>4.1351</td>
</tr>
<tr>
<td>5</td>
<td>0.1&lt;br&gt;0.2&lt;br&gt;0.3&lt;br&gt;0.4&lt;br&gt;0.5</td>
<td>4.2000</td>
</tr>
<tr>
<td>6</td>
<td>0.1&lt;br&gt;0.2&lt;br&gt;0.3&lt;br&gt;0.4&lt;br&gt;0.5</td>
<td>4.3500</td>
</tr>
</tbody>
</table>
Next, we have presented the values of $H^{\alpha,\beta}(A)$ graphically for different values of the parameter $\alpha$ for fixed $\beta = 2$ and obtained the following Fig.-3.2.5 which shows that the measure of generalized fuzzy entropy $H^{\alpha,\beta}(A)$ introduced in (3.2.5) is monotonically increasing function of $\alpha$ when $\alpha + \beta > 1$.

![Graph showing the monotonicity of $H^{\alpha,\beta}(A)$](image)

**Fig.-3.2.5: Monotonicity of $H^{\alpha,\beta}(A)$ when $\alpha + \beta > 1$**

**Case-II.** When $\alpha + \beta < 1$. In this case, we have

$$\left\{2^{1-\alpha-\beta} - 1\right\}^2 \frac{dH^{\alpha,\beta}(A)}{d\alpha} = X + Y$$

where

$$X = \left\{2^{1-\alpha-\beta} - 1\right\} \sum_{i=1}^{n} \left[ \mu_A^{\alpha+\beta}(x_i) \log \mu_A(x_i) + (1 - \mu_A(x_i))^{\alpha+\beta} \log(1 - \mu_A(x_i)) \right]$$

and

$$Y = 2^{1-\alpha-\beta} \sum_{i=1}^{n} \left[ \mu_A^{\alpha+\beta}(x_i) + (1 - \mu_A(x_i))^{\alpha+\beta} - 1 \right]$$

Obviously, $X < 0$ and $Y < 0$. Hence, we see that $\frac{dH^{\alpha,\beta}(A)}{d\alpha} \leq 0$

which shows that $H^{\alpha,\beta}(A)$ is monotonically decreasing function of $\alpha$.

Next, with the help of the data, we have computed different values of $H^{\alpha,\beta}(A)$ for different values of $\alpha$ for fixed $\beta = 0.2$ as shown in the following table- 3.2.7:
Table-3.2.7: Computations of $H^{α,β}(A)$ when $0 < α + β < 1$

<table>
<thead>
<tr>
<th>$α$</th>
<th>$μ_A(x_i)$</th>
<th>$H^{α,β}(A)$</th>
</tr>
</thead>
<tbody>
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<tr>
<td></td>
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<td></td>
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<td></td>
<td>0.3</td>
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<td>0.4</td>
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<td>0.5</td>
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<td>0.3</td>
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<td>0.5</td>
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<tr>
<td>0.4</td>
<td>0.1</td>
<td>4.2929</td>
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<td></td>
<td>0.5</td>
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</tbody>
</table>
Next, we have presented the values of $H^{\alpha,\beta}(A)$ graphically and obtained the following Fig.-3.2.6 which shows that the generalized fuzzy entropy $H^{\alpha,\beta}(A)$ introduced in (3.2.5) is monotonically decreasing function of $\alpha$ when $\alpha + \beta < 1$.

![Fig.-3.2.6: Monotonicity of $H^{\alpha,\beta}(A)$ when $\alpha + \beta < 1$](image)

3.3. A New Measure Of Fuzzy R-Divergence, Its Generalization And Properties:

We know that if A and B are two fuzzy sets and $H(A)$ and $H(B)$ are their fuzzy entropies, then R-divergence $R(A,B)$ is defined as

$$R(A,B) = H\left[\frac{A + B}{2}\right] - \frac{H(A) + H(B)}{2}$$

(3.3.1)

Using (3.1.1), equation (3.3.1) gives the following expression:

$$R(A,B) = -\sum_{i=1}^{n} \left[ \frac{\mu_{A}\mu_B(x_i)}{2} \log \frac{\mu_{A+B}(x_i)}{2} + (1 - \frac{\mu_{A+B}(x_i)}{2}) \log (1 - \frac{\mu_{A+B}(x_i)}{2}) \right]$$
\[
+ \frac{1}{2} \sum_{i=1}^{n} \left[ \mu_A(x_i) \log \mu_A(x_i) + (1 - \mu_A(x_i)) \log(1 - \mu_A(x_i)) \right] \\
+ \frac{1}{2} \sum_{i=1}^{n} \left[ \mu_B(x_i) \log \mu_B(x_i) + (1 - \mu_B(x_i)) \log(1 - \mu_B(x_i)) \right] \\
= -\sum_{i=1}^{n} \left[ \frac{\mu_{A+B}(x_i)}{2} \log \frac{\mu_A(x_i)}{1 - \mu_A(x_i)} + \log \frac{1 - \mu_{A+B}(x_i)}{2} \right] \\
+ \frac{1}{2} \sum_{i=1}^{n} \left[ \mu_A(x_i) \log \frac{\mu_A(x_i)}{1 - \mu_A(x_i)} + \log(1 - \mu_A(x_i)) \right] \\
+ \frac{1}{2} \sum_{i=1}^{n} \left[ \mu_B(x_i) \log \frac{\mu_B(x_i)}{1 - \mu_B(x_i)} + \log(1 - \mu_B(x_i)) \right] \\
= -\frac{1}{2} \sum_{i=1}^{n} \mu_{A+B}(x_i) \log \frac{\mu_{A+B}(x_i)}{2} - \frac{1}{2} \sum_{i=1}^{n} \mu_{A+B}(x_i) \log \frac{1 - \mu_{A+B}(x_i)}{2} \\
- \frac{1}{2} \sum_{i=1}^{n} \log(1 - \mu_{A+B}(x_i)) + \frac{1}{2} \sum_{i=1}^{n} \mu_A(x_i) \log \frac{\mu_A(x_i)}{1 - \mu_A(x_i)} + \frac{1}{2} \sum_{i=1}^{n} \log(1 - \mu_A(x_i)) \\
+ \frac{1}{2} \sum_{i=1}^{n} \mu_B(x_i) \log \frac{\mu_B(x_i)}{1 - \mu_B(x_i)} + \frac{1}{2} \sum_{i=1}^{n} \log(1 - \mu_B(x_i)) \\
= -\frac{1}{2} \sum_{i=1}^{n} \mu_{A+B}(x_i) \log \frac{\mu_{A+B}(x_i)}{2} - \frac{1}{2} \sum_{i=1}^{n} \mu_B(x_i) \log \frac{1 - \mu_B(x_i)}{\mu_B(x_i)} \\
- \frac{1}{2} \sum_{i=1}^{n} \left[ \log(1 - \mu_{A+B}(x_i)) - \log(1 - \mu_B(x_i)) \right] - \frac{1}{2} \sum_{i=1}^{n} \mu_{A+B}(x_i) \log \frac{\mu_{A+B}(x_i)}{2} \\
- \frac{1}{2} \sum_{i=1}^{n} \mu_A(x_i) \log \frac{1 - \mu_A(x_i)}{\mu_A(x_i)} - \frac{1}{2} \sum_{i=1}^{n} \left[ \log(1 - \mu_{A+B}(x_i)) - \log(1 - \mu_A(x_i)) \right]
\]

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Thus, we have
\[ R(A, B) = \frac{1}{2} D \left( \frac{A + B}{2}, A \right) + \frac{1}{2} D \left( \frac{A + B}{2}, B \right) \]  
(3.3.2)

where
\[ D(A, B) = \sum_{i=1}^{n} \mu_A(x_i) \log \frac{1 - \mu_A(x_i)}{\mu_A(x_i)} + \mu_B(x_i) \log \frac{\mu_B(x_i)}{1 - \mu_B(x_i)} + \log \frac{1 - \mu_B(x_i)}{1 - \mu_B(x_i)} \]  
(3.3.3)

is a new measure of fuzzy directed divergence.

We, now discuss the validity of the measure introduced in (3.3.3).

Consider
\[ \frac{\partial}{\partial \mu_B(x_i)} D(A, B) = \sum_{i=1}^{n} \log \frac{\mu_B(x_i)}{1 - \mu_B(x_i)} \]
Further, we have

\[
\frac{\partial^2}{\partial^2 \mu_b(x)} D(A, B) = \frac{\partial}{\partial \mu_b(x)} \left[ \sum_{i=1}^{n} \log \frac{\mu_b(x_i)}{1 - \mu_b(x_i)} \right]
\]

\[
= \sum_{i=1}^{n} \frac{1}{\mu_b(x_i)(1 - \mu_b(x_i))} > 0 \forall i
\]

Hence, \( D(A, B) \) is a convex function of \( \mu_b(x_i) \).

Similarly, we can prove that \( D(A, B) \) is a convex function of \( \mu_A(x_i) \).

Thus, we have the following properties:

(i) \( D(A, B) \geq 0 \)

(ii) \( D(A, B) = 0 \) iff \( \mu_A(x_i) = \mu_b(x_i) \) \( \forall i \)

(iii) \( D(A, B) \) is a convex function of both \( \mu_A(x_i) \) and \( \mu_b(x_i) \) \( \forall i \)

Hence, fuzzy directed divergence measure \( D(A, B) \) introduced in (3.3.3) is a valid measure of divergence.

Also the measure \( R(A, B) \) defined in (3.3.2) satisfies the following properties:

(i) \( R(A, B) \geq 0 \)

(ii) \( R(A, B) = 0 \) \( \iff \mu_A(x_i) = \mu_b(x_i) \) \( \forall i \)

(iii) \( R(A, B) \), being the sum of fuzzy directed divergences is convex.

(iv) \( R(A, B) \) is symmetric in the sense that \( R(A, B) = R(B, A) \).

Thus, we see that the fuzzy R-divergence \( R(A, B) \) defined in (3.3.2) is a valid measure of divergence.

Next, we have generalized \( R(A, B) \) and discussed its properties.

**Generalization Of R-Divergence:**

We, now generalize \( R(A, B) \) by introducing a non-negative \( \lambda, 0 \leq \lambda \leq 1 \) as follows:

\[
R_{\lambda}(A, B) = H(\lambda A + (1 - \lambda)B) - \lambda H(A) - (1 - \lambda)H(B)
\]  

(3.3.4)

Using (3.1.1) in (3.3.4), we have
\[ R_{\lambda}(A,B) = -\sum_{i=1}^{n} \left[ \{\lambda \mu_{A}(x_{i}) + (1 - \lambda) \mu_{B}(x_{i})\} \log \{\lambda \mu_{A}(x_{i}) + (1 - \lambda) \mu_{A}(x_{i})\} \right. \\
\left. + \{1 - \lambda \mu_{A}(x_{i}) - (1 - \lambda) \mu_{B}(x_{i})\} \log \{1 - \lambda \mu_{A}(x_{i}) - (1 - \lambda) \mu_{B}(x_{i})\} \right] \\
+ \lambda \sum_{i=1}^{n} \{\mu_{A}(x_{i}) \log \mu_{A}(x_{i}) + (1 - \mu_{A}(x_{i})) \log(1 - \mu_{A}(x_{i}))\} \\
+ (1 - \lambda) \sum_{i=1}^{n} \{\mu_{B}(x_{i}) \log \mu_{B}(x_{i}) + (1 - \mu_{B}(x_{i})) \log(1 - \mu_{B}(x_{i}))\} \]

\[ = -\lambda \sum_{i=1}^{n} \left[ \{\lambda \mu_{A}(x_{i}) + (1 - \lambda) \mu_{B}(x_{i})\} \log \{\lambda \mu_{A}(x_{i}) + (1 - \lambda) \mu_{B}(x_{i})\} \right. \\
\left. + \{1 - \lambda \mu_{A}(x_{i}) - (1 - \lambda) \mu_{B}(x_{i})\} \log \{1 - \lambda \mu_{A}(x_{i}) - (1 - \lambda) \mu_{B}(x_{i})\} \right] \\
- (1 - \lambda) \sum_{i=1}^{n} \left[ \{\lambda \mu_{A}(x_{i}) + (1 - \lambda) \mu_{B}(x_{i})\} \log \{\lambda \mu_{A}(x_{i}) + (1 - \lambda) \mu_{B}(x_{i})\} \right. \\
\left. + \{1 - \lambda \mu_{A}(x_{i}) - (1 - \lambda) \mu_{B}(x_{i})\} \log \{1 - \lambda \mu_{A}(x_{i}) - (1 - \lambda) \mu_{B}(x_{i})\} \right] \\
+ \lambda \sum_{i=1}^{n} \{\mu_{A}(x_{i}) \log \mu_{A}(x_{i}) + (1 - \mu_{A}(x_{i})) \log(1 - \mu_{A}(x_{i}))\} \\
+ (1 - \lambda) \sum_{i=1}^{n} \{\mu_{B}(x_{i}) \log \mu_{B}(x_{i}) + (1 - \mu_{B}(x_{i})) \log(1 - \mu_{B}(x_{i}))\} \]

\[ = -\lambda \sum_{i=1}^{n} \left[ \{\lambda \mu_{A}(x_{i}) + (1 - \lambda) \mu_{B}(x_{i})\} \log \frac{\{\lambda \mu_{A}(x_{i}) + (1 - \lambda) \mu_{B}(x_{i})\}}{1 - \{\lambda \mu_{A}(x_{i}) + (1 - \lambda) \mu_{B}(x_{i})\}} \right. \\
\left. + \mu_{A}(x_{i}) \log \frac{1 - \mu_{A}(x_{i})}{\mu_{A}(x_{i})} + \log \frac{1 - \lambda \mu_{A}(x_{i}) - (1 - \lambda) \mu_{B}(x_{i})}{1 - \mu_{A}(x_{i})} \right] \\
+ (1 - \lambda) \sum_{i=1}^{n} \{\mu_{B}(x_{i}) \log \mu_{B}(x_{i}) + (1 - \mu_{B}(x_{i})) \log(1 - \mu_{B}(x_{i}))\} \]
Thus, we have
\[ R_\lambda(A, B) = \lambda D(\lambda A + (1-\lambda)B, A) + (1-\lambda)D(\lambda A + (1-\lambda)B, B), \]
where \( D(A, B) \) is a fuzzy directed divergence introduced in (3.3.3).

We now discuss some desirable properties of \( R_\lambda(A, B) \):

(i) \( R_\lambda(A, B) \geq 0 \), being the sum of fuzzy directed divergences with +ve constants.

Also \( R_\lambda(A, B) = 0 \Leftrightarrow A = B \)

(ii) \( R_\lambda(A, B) = R_\lambda(B, A) \)

(iii) \( R_\lambda(A, B) = R(A, B) \)

(iv) \( R_{\lambda,A}(A, B) = R_\lambda(B, A) \)

(v) \( R_{\lambda,A}(A, B) = R_{\lambda,A}(B, A) \)

(vi) \( R_\lambda(A, B) \) is a concave function of \( \lambda \).

We prove (vi) where as properties (i) to (v) can easily be verified.

Consider
\[
\frac{d}{d\lambda} R_\lambda(A, B) =
\left[ \begin{array}{c}
\sum_{i=1}^{n} \left\{ \lambda \mu_\lambda(x_i) + (1-\lambda)\mu_\beta(x_i) \right\} \log \left\{ \lambda \mu_\lambda(x_i) + (1-\lambda)\mu_\beta(x_i) \right\}
\end{array} \right]
\]

\[
- \frac{d}{d\lambda} \sum_{i=1}^{n} \left[ \begin{array}{c}
\sum_{i=1}^{n} \left\{ \lambda \mu_\lambda(x_i) + (1-\lambda)\mu_\beta(x_i) \right\} \log \left\{ \lambda \mu_\lambda(x_i) + (1-\lambda)\mu_\beta(x_i) \right\}
\end{array} \right]
\]

\[
+ \sum_{i=1}^{n} \left\{ \mu_\lambda(x_i) \log \mu_\lambda(x_i) + (1-\mu_\lambda(x_i)) \log (1-\mu_\lambda(x_i)) \right\}
\]
Also
\[
\frac{d^2}{d\lambda^2} R_\lambda (A, B) = -\sum_{i=1}^{n} \frac{\left(\lambda A(x_i) - \lambda B(x_i)\right)^2}{\left(\lambda A(x_i) + (1-\lambda)B(x_i)\right)\left(1-\lambda A(x_i) - (1-\lambda)B(x_i)\right)}
\]

< 0 \quad \forall i

which shows that \( R_\lambda (A, B) \) is a concave function of \( \lambda \).

3.4. Measures Of Fuzzy Information Improvement And Their Properties:

In this section, we develop some fuzzy measures which provide improvement in distance and study their desirable properties. For this purpose, we consider the following measures of directed divergence:

3.4.1. Measure Of Fuzzy Information Improvement Based Upon Kullback-Leibler’s (1951) Measure of Divergence

We know that Kullback-Leibler’s (1951) measure of fuzzy directed divergence between two fuzzy sets \( A \) and \( B \) is given by

\[
D(A : B) = \sum_{i=1}^{n} \left[ \mu_A(x_i) \log \frac{\mu_A(x_i)}{\mu_B(x_i)} + (1-\mu_A(x_i)) \log \frac{1-\mu_A(x_i)}{1-\mu_B(x_i)} \right]
\]  (3.4.1)

The directed divergence between two fuzzy sets \( A \) and \( C \) is therefore given by
\[
D(A : C) = \sum_{i=1}^{n} \left[ \mu_A(x_i) \log \frac{\mu_A(x_i)}{\mu_C(x_i)} + (1-\mu_A(x_i)) \log \frac{1-\mu_A(x_i)}{1-\mu_C(x_i)} \right]
\]  \hspace{1cm} (3.4.2)

From equations (3.4.1) and (3.4.2), the reduction in ambiguity is given by

\[
D(A : B : C) = D(A : B) - D(A : C)
\]  \hspace{1cm} (3.4.3)

Thus, the information improvement measure can be expressed in the following mathematical form:

\[
D(A : B : C) = \sum_{i=1}^{n} \left[ \mu_A(x_i) \log \frac{\mu_C(x_i)}{\mu_B(x_i)} + (1-\mu_A(x_i)) \log \frac{1-\mu_C(x_i)}{1-\mu_B(x_i)} \right]
\]  \hspace{1cm} (3.4.4)

To prove that the measure (3.4.4) is a measure of fuzzy information improvement, we study its following properties:

I. Obviously, \( D(A : B : C) = 0 \) iff \( A = B = C \)

II. **Non-negativity**

To prove non-negativity, we proceed as follows:

Let \( \sum_{i=1}^{n} \mu_A(x_i) = \alpha_0 \), \( \sum_{i=1}^{n} \mu_B(x_i) = \beta_0 \) and \( \sum_{i=1}^{n} \mu_C(x_i) = \gamma_0 \)

Now, we know that Theil’s (1967) measure of information improvement in terms of probabilities is given by

\[
D(P : Q : R) = \sum_{i=1}^{n} p_i \ln \frac{r_i}{q_i} \geq 0
\]  \hspace{1cm} (3.4.5)

Thus, we must have

\[
\sum_{i=1}^{n} \frac{\mu_A(x_i)}{\alpha_0} \log \frac{\mu_C(x_i)}{\mu_B(x_i)} \beta_0 \geq 0
\]  \hspace{1cm} (3.4.6)

where

\[
p_i = \frac{\mu_A(x_i)}{\alpha_0}, \quad q_i = \frac{\mu_B(x_i)}{\beta_0} \quad \text{and} \quad r_i = \frac{\mu_C(x_i)}{\gamma_0}
\]

such that \( \sum_{i=1}^{n} p_i = \sum_{i=1}^{n} q_i = \sum_{i=1}^{n} r_i = 1 \)

Upon simplification, equation (3.4.6) gives

\[
\sum_{i=1}^{n} \mu_A(x_i) \log \frac{\mu_C(x_i)}{\mu_B(x_i)} \geq \alpha_0 \log \frac{\gamma_0}{\beta_0}
\]  \hspace{1cm} (3.4.7)
Again, we have
\[ \sum_{i=1}^{n} \frac{1-\mu_A(x_i)}{n-\alpha_0} \log \frac{(1-\mu_C(x_i))}{(n-\gamma_0)} \geq 0 \]

Proceeding on similar lines, we have the following result:
\[ \sum_{i=1}^{n} \frac{(1-\mu_A(x_i))}{1-\mu_B(x_i)} \geq \frac{(n-\alpha_0)}{(n-\beta_0)} \log \frac{n-\gamma_0}{n-\beta_0} \tag{3.4.8} \]

Adding (3.4.7) and (3.4.8), we get
\[ \sum_{i=1}^{n} \left[ \mu_A(x_i) \log \frac{\mu_C(x_i)}{\mu_B(x_i)} + (1-\mu_A(x_i)) \log \frac{1-\mu_C(x_i)}{1-\mu_B(x_i)} \right] \geq \alpha_0 \log \frac{\gamma_0}{\beta_0} + (n-\alpha_0) \log \frac{n-\gamma_0}{n-\beta_0} \tag{3.4.9} \]

Now, clearly the R.H.S. of equation (3.4.9) is non-negative. Thus, we must have
\[ D(A:B:C) \geq 0 \]

III. Convexity of \( D(A:B:C) \) w.r.t. \( \mu_B(x_i) \)

Let \( f(\alpha_0, \beta_0, \gamma_0) = \alpha_0 \log \frac{\gamma_0}{\beta_0} + (n-\alpha_0) \log \frac{n-\gamma_0}{n-\beta_0} \)

We shall show that \( f(\alpha_0, \beta_0, \gamma_0) \) is a convex function of \( \beta_0 \)

For this, we have the following results:
\[ \frac{\partial f}{\partial \beta_0} = -\frac{\alpha_0}{\beta_0} + \frac{n-\alpha_0}{n-\beta_0} \]
\[ \frac{\partial^2 f}{\partial \beta_0^2} = \frac{\alpha_0(n^2 + \beta_0^2 - 2n\beta_0) + n\beta_0^2 - \alpha_0\beta_0^2}{\beta_0(n-\beta_0)^2} \]
\[ = \frac{n(n\alpha_0 + \beta_0(\beta_0 - 2\alpha_0))}{\beta_0^2(n-\beta_0)^2} > 0 \]

provided \( \beta_0 > 2\alpha_0 \)

Hence \( f(\alpha_0, \beta_0, \gamma_0) \) is a convex function of \( \beta_0 \) and its minimum value occurs if
\[ \frac{\partial f}{\partial \beta_0} = 0 \]
which gives \(-\frac{\alpha_0}{\beta_0} + \frac{n - \alpha_0}{n - \beta_0} = 0\)

or \(\alpha_0 = \beta_0\)

Also, the right hand side of equation (3.4.9) \(\geq 0\). Consequently, \(D(A, B, C)\) is a convex function of \(\beta_0\).

IV. **Concavity of** \(D(A : B : C)\) w.r.t. \(\mu_C(x_i)\). We have

\[
\frac{\partial D(A : B : C)}{\partial \mu_C(x_i)} = \frac{\mu_A(x_i) - 1 - \mu_A(x_i)}{\mu_C(x_i) - 1 - \mu_C(x_i)}
\]

Also

\[
\frac{\partial^2 D(A : B : C)}{\partial \mu_C^2(x_i)} = -\frac{\mu_A(x_i) - 1 - \mu_A(x_i)}{\mu_C^2(x_i) - 1 - \mu_C(x_i)} < 0 \forall i, 0 < \mu_C(x_i) < 1
\]

Hence \(D(A : B : C)\) is a concave function of \(\mu_C(x_i)\).

Thus, we observe that the measure introduced in (3.4.4) is a convex function of \(\mu_A(x_i)\) and concave function of \(\mu_C(x_i)\).

Therefore, we have the following properties:

(i) \(D(A : B : C) \geq 0\)

(ii) \(D(A : B : C) = 0\) iff \(A = B = C\)

(iii) \(D(A : B : C)\) is a convex function of \(\mu_A(x_i)\) and concave function of \(\mu_C(x_i)\).

Under the above properties, the measure introduced in (3.4.4) is a correct measure of fuzzy information improvement.

3.4.2 Measure Of Fuzzy information Improvement Based Upon Parkash and Tuli’s (2005) Measure of Divergence

Parkash and Tuli (2005) have proposed the following expression for the fuzzy directed divergence measure between two fuzzy sets \(A\) and \(B\) which corresponds to Burg’s (1972) probabilistic measure of divergence:
D_a(A : B) = \sum_{i=1}^{n} \left[ \frac{1+a\mu_A(x_i)}{1+a\mu_B(x_i)} - \log \frac{1+a\mu_A(x_i)}{1+a\mu_B(x_i)} + \frac{1+a(1-\mu_A(x_i))}{1+a(1-\mu_B(x_i))} \right. \\
- \left. \log \frac{1+a(1-\mu_A(x_i))}{1+a(1-\mu_B(x_i))} - 2 \right], a > 0

(3.4.10)

Thus, the fuzzy divergence measure between fuzzy sets A and C can be taken as:

D_a(A : C) = \sum_{i=1}^{n} \left[ \frac{1+a\mu_A(x_i)}{1+a\mu_C(x_i)} - \log \frac{1+a\mu_A(x_i)}{1+a\mu_C(x_i)} + \frac{1+a(1-\mu_A(x_i))}{1+a(1-\mu_C(x_i))} \right. \\
- \left. \log \frac{1+a(1-\mu_A(x_i))}{1+a(1-\mu_C(x_i))} - 2 \right], a > 0

(3.4.11)

From equations (3.4.10) and (3.4.11), the reduction in ambiguity is given by

D_a(A : B : C) = D_a(A : B) - D_a(A : C)

(3.4.12)

To prove that the measure (3.4.12) is a measure of fuzzy information improvement, we study its following properties:

I. Obviously, D_a(A : B : C) = 0 iff A = B = C

II. Non-negativity

Obviously, D_a(A : B : C) \geq 0 for D_a(A : B) \geq D_a(A : C)

Convexity of D_a(A : B : C) w.r.t. \mu_B(x_i).

We have

D_a(A : B : C) = \sum_{i=1}^{n} \left[ \frac{1+a\mu_A(x_i)}{1+a\mu_B(x_i)} - \log \frac{1+a\mu_A(x_i)}{1+a\mu_B(x_i)} + \frac{1+a(1-\mu_A(x_i))}{1+a(1-\mu_B(x_i))} \right. \\
- \left. \log \frac{1+a(1-\mu_A(x_i))}{1+a(1-\mu_B(x_i))} - 2 \right] \\
- \sum_{i=1}^{n} \left[ \frac{1+a\mu_A(x_i)}{1+a\mu_C(x_i)} - \log \frac{1+a\mu_A(x_i)}{1+a\mu_C(x_i)} + \frac{1+a(1-\mu_A(x_i))}{1+a(1-\mu_C(x_i))} \right. \\
- \left. \log \frac{1+a(1-\mu_A(x_i))}{1+a(1-\mu_C(x_i))} - 2 \right], a > 0
Further, we have

\[
\frac{\partial}{\partial \mu_b(x_i)} D_a(A : B : C) = -\frac{a[1 + a \mu_a(x_i)]}{[1 + a \mu_b(x_i)]^2} + \frac{a}{1 + a \mu_b(x_i)} + \frac{a[1 + a(1 - \mu_a(x_i))]}{[1 + a(1 - \mu_b(x_i))]^2} - \frac{a}{1 + a(1 - \mu_b(x_i))}
\]

Also

\[
\frac{\partial^2}{\partial \mu_b^2(x_i)} D_a(A : B : C) = 2a^2 \left[ \frac{1 + a \mu_a(x_i)}{[1 + a \mu_b(x_i)]^3} - \frac{1}{[1 + a \mu_b(x_i)]^2} + \frac{1}{1 + a(1 - \mu_b(x_i))]^2} - \frac{2[1 + a(1 - \mu_a(x_i))]}{[1 + a(1 - \mu_b(x_i))]^3} \right] - a^2 \left[ \frac{1}{[1 + a(1 - \mu_b(x_i))]^2} - \frac{2[1 + a(1 - \mu_a(x_i))]}{[1 + a(1 - \mu_b(x_i))]^3} \right]^2
\]

\[
= a^2 (X - Y)
\]

where \( X = \frac{2[1 + a \mu_a(x_i)]}{[1 + a \mu_b(x_i)]^3} - \frac{1}{[1 + a \mu_b(x_i)]^2} \)

and \( Y = \frac{1}{[1 + a(1 - \mu_b(x_i))]^2} - \frac{2[1 + a(1 - \mu_a(x_i))]}{[1 + a(1 - \mu_b(x_i))]^3} \)

Consider the case when \( 2 \mu_a(x_i) > \mu_b(x_i) \forall i \). Then, we have

\[
X = \frac{2 + 2a \mu_a(x_i) - 1 - a \mu_b(x_i)}{[1 + a \mu_b(x_i)]^3} = \frac{1 + a[2 \mu_a(x_i) - \mu_b(x_i)]}{[1 + a \mu_b(x_i)]^3} > 0 \forall i
\]

and

\[
Y = \frac{1 + a - a \mu_b(x_i) - 2 - 2a + 2a \mu_a(x_i)}{[1 + a(1 - \mu_b(x_i))]^3} = \frac{-1 - a + a[2 \mu_a(x_i) - \mu_b(x_i)]}{[1 + a(1 - \mu_b(x_i))]^3} < 0 \forall i
\]
Hence, we can say that \( \frac{\partial^2}{\partial \mu_B^2(x_i)} D_a(A : B : C) > 0 \forall i \). Consequently \( D_a(A, B, C) \) is a convex function of \( \mu_B(x_i) \).

**IV. Concavity of** \( D_a(A : B : C) \) **w.r.t.** \( \mu_C(x_i) \).

We have

\[
\frac{\partial}{\partial \mu_C(x_i)} D_a(A : B : C) = \frac{a[1 + a \mu_A(x_i)]}{[1 + a \mu_C(x_i)]^2} - \frac{a}{1 + a \mu_C(x_i)} - \frac{a[1 + a(1 - \mu_A(x_i))]}{[1 + a(1 - \mu_C(x_i))]^2} + \frac{a}{1 + a(1 - \mu_C(x_i))}
\]

Also

\[
\frac{\partial^2}{\partial \mu_C^2(x_i)} D_a(A : B : C) = - \frac{2a^2[1 + a \mu_A(x_i)]}{[1 + a \mu_C(x_i)]^2} + \frac{a^2}{[1 + a \mu_C(x_i)]^2} - \frac{2a^2[1 + a(1 - \mu_A(x_i))]}{[1 + a(1 - \mu_C(x_i))]^2} + \frac{a^2}{[1 + a(1 - \mu_C(x_i))]^2}
\]

\[
= a^2 \left[ \frac{1}{[1 + a \mu_C(x_i)]^2} - \frac{2[1 + a \mu_A(x_i)]}{[1 + a \mu_C(x_i)]^3} \right] - a^2 \left[ \frac{2[1 + a(1 - \mu_A(x_i))]}{[1 + a(1 - \mu_C(x_i))]^3} - \frac{1}{[1 + a(1 - \mu_C(x_i))]^2} \right]
\]

\[
= a^2 (X_i - Y_i)
\]

where

\[
X_i = \frac{1}{[1 + a \mu_C(x_i)]^2} - \frac{2[1 + a \mu_A(x_i)]}{[1 + a \mu_C(x_i)]^3}
\]

and

\[
Y_i = \frac{2[1 + a \mu_A(x_i)]}{[1 + a(1 - \mu_C(x_i))]^3} - \frac{1}{[1 + a(1 - \mu_C(x_i))]^2}
\]

Consider the case when \( 2 \mu_A(x_i) > \mu_C(x_i) \forall i \).

Then, we have
\[ X_i = \frac{1}{[1 + a\mu_c(x_i)]^2} - \frac{2[1 + a\mu_a(x_i)]}{[1 + a\mu_c(x_i)]^3} = \frac{-1 - a[2\mu_a(x_i) - \mu_c(x_i)]}{[1 + a\mu_c(x_i)]^3} < 0 \quad \forall i \]

Also
\[ Y_i = \frac{2[1 + a\mu_a(x_i)]}{[1 + a(1 - \mu_c(x_i))]^3} - \frac{1}{[1 + a(1 - \mu_c(x_i))]^2} = \frac{1 - a - a[2\mu_a(x_i) - \mu_c(x_i)]}{[1 + a(1 - \mu_c(x_i))]^3} > 0 \quad \forall i \]

Hence, we can say that \( \frac{\partial^2}{\partial \mu_c^2(x_i)} D_a(A : B : C) < 0 \forall i \). Consequently, \( D_a(A : B : C) \) is a concave function of \( \mu_c(x_i) \).

Thus, we observe that the measure introduced in (3.4.12) is a convex function of \( \mu_a(x_i) \) and concave function of \( \mu_c(x_i) \). Thus, we have the following properties

(i) \( D_a(A : B : C) \geq 0 \)

(ii) \( D_a(A : B : C) = 0 \iff A = B = C \)

(iii) \( D_a(A : B : C) \) is a convex function of \( \mu_a(x_i) \) and concave function of \( \mu_c(x_i) \).

Under the above properties, the measure introduced in (3.4.12) is a valid measure of fuzzy information improvement.

Next, we study the monotonicity of this improvement measure \( D_a(A : B : C) \):

V. Monotonicity: We have

\[
\frac{d}{da} D_a(A : B : C) = \sum_{i=1}^{\infty} \left\{ \frac{\mu_a(x_i) - \mu_b(x_i)}{[1 + a\mu_b(x_i)]^2} - \frac{\mu_a(x_i) - \mu_c(x_i)}{[1 + a\mu_c(x_i)]^2} \right\} + \left\{ \frac{\mu_b(x_i) - \mu_a(x_i)}{[1 + a(1 - \mu_b(x_i))]^3} - \frac{\mu_b(x_i) - \mu_c(x_i)}{[1 + a(1 - \mu_c(x_i))]^3} \right\}
\]

or \( \frac{d}{da} D_a(A : B : C) = X - Y \) (3.4.13)
where

\[
X = \sum_{i=1}^{n} a \left\{ \mu_a(x_i) - \mu_b(x_i) \right\}^2 \left[ \frac{1}{1 + a\mu_b(x_i)} \right]^2 \left\{ \frac{1}{1 + a\mu_a(x_i)} \right\}^2 + \frac{1}{\left\{ 1 + a(1 - \mu_b(x_i)) \right\}^2 \left\{ 1 + a(1 - \mu_a(x_i)) \right\}^2}
\]

and

\[
Y = \sum_{i=1}^{n} a \left\{ \mu_a(x_i) - \mu_c(x_i) \right\}^2 \left[ \frac{1}{1 + a\mu_c(x_i)} \right]^2 \left\{ \frac{1}{1 + a\mu_a(x_i)} \right\}^2 + \frac{1}{\left\{ 1 + a(1 - \mu_c(x_i)) \right\}^2 \left\{ 1 + a(1 - \mu_a(x_i)) \right\}^2}
\]

But \( Y < X \). Thus equation (3.4.13) implies that

\[
\frac{d}{da} D_a(A : B : C) \geq 0.
\]

Hence \( D_a(A : B : C) \) is monotonically increasing function of \( a \) for \( a > 0 \).

Next, we have presented graphically the measure \( D_a(A : B : C) \) introduced above.

For this purpose, we consider the following fuzzy sets:

\( A = (0.2, 0.4, 0.6, 0.8) \), \( B = (0.8, 0.4, 0.4, 0.2) \) and \( C = (0.3, 0.4, 0.7, 0.9) \)

For different values of the parameter \( a \), we have calculated different values of \( D_a(A : B : C) \), the results of which have been presented as shown in the following table- 3.4.1:

<table>
<thead>
<tr>
<th>( a )</th>
<th>( D_a(A : B) )</th>
<th>( D_a(A : C) )</th>
<th>( D_a(A : B : C) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>8.337</td>
<td>8.000</td>
<td>0.337</td>
</tr>
<tr>
<td>2</td>
<td>8.798</td>
<td>8.005</td>
<td>0.793</td>
</tr>
<tr>
<td>3</td>
<td>9.245</td>
<td>8.040</td>
<td>1.205</td>
</tr>
<tr>
<td>4</td>
<td>9.588</td>
<td>8.000</td>
<td>1.588</td>
</tr>
</tbody>
</table>
Next, we have presented the values of $D_a(A : B : C)$ graphically for different values of the parameter and obtained the following Fig.-3.4.1 which shows that the measure of fuzzy information improvement introduced in (3.4.12) is monotonically increasing function of $a$.

![Monotonicity of $D_a(A : B : C)$](image)

**Concluding Remarks:** It is known fact that there exists many growth models in Biological Sciences, many diffusion models in real life situations, a variety of income models in Economics, many parametric and non-parametric models in Social Sciences and even in Physical Sciences, thus we need a variety of information measures for each field to extend the scope of their applications. Hence, the development of new generalized parametric, non-parametric, probabilistic and fuzzy measures is necessary. It is therefore concluded that taking into consideration the existing fuzzy measures of information, some new generalized measures of entropy, R-divergence and information improvement for discrete fuzzy distributions have been developed. With similar arguments, many new fuzzy measures for continuous fuzzy distributions can be developed.