Chapter 7

Unreliable Server Bulk Queue with Phase Service and Phase Repair

7.1 Introduction

In the field of manufacturing, production, telecommunication and computer systems etc., the stochastic models of queueing problems with different assumptions have been frequently used to analyse the problem mathematically and to predict the performance measures of the system. When the server breakdowns unpredictably at any stage of the service, the arrival rates of the units may be affected. In various congestion situations, no server can always provide the service without any interruption because sometimes it may fail due to some internal fault or any other reasons during the service of the units. In many physical situations, the equipment may break down randomly and can stop the service for a long time. It is also possible that there may be delay in repair due to the unavailability of the repairman.

In the recent past, many researchers developed different models of queueing problems by incorporating the concept of service interruption due to server failure under different assumptions; they assumed that the failed service system immediately gets repaired or may be delayed in repair due to non-availability of the repairman or any other reasons. Many queue theorists have developed various models for optional service using the supplementary variable technique to discuss the behaviour of the queue size distribution.

In industrial scenarios, there may be many real life congestion situations, wherein the failed service system during first phase essential service/ second phase additional service may not repair immediately due to unavailability of the repairmen or due to other technical reasons. In automobile service station, the customers join the service station for essential service of their vehicles, but some of them may demand for other services such as painting, replacement of the old parts with new parts, etc. It can also be observed that the service of the vehicles gets completed in different phases. Choudhury and Tadj (2009) analysed a model with additional second phase of optional service for unreliable server and delayed repair. They generalized the classical models by incorporating the concepts of random breakdowns, repair and optional service. Further,
by using the probability generating function they obtained the steady state queue size distribution, busy period and the waiting time distributions. Choudhury et al. (2009) considered another $M^X/G/1$ model with the second phase of optional service and single phase of repair for the unreliable server under $N-\text{policy}$. They assumed that the arrival rates of the units are uniform and the idle server immediately begins the first phase of regular service to the waiting unit at the head of the line when the queue size becomes at least $N(\geq 1)$.

In the field of flexible manufacturing systems, digital telecommunication systems and call centers, when the customers arrive for service in bulk, the performance of the service system may be affected due to random breakdown of the service system during the first phase of compulsory service/second phase of optional service. The failed server undergoes for repair and repair station has the provision of many optional services along with essential services. The customers may avail the optional services at the repair station. Due to sudden breakdown of the server, the functioning of the system is interrupted and the arrival rate of the customers in the system may not be uniform. Another popular example can be seen in health care centers wherein the patients arrive for the treatment and regular check-up of the patient can be treated as essential service but some of patients are advised for specialized treatment which comes in the secondary optional service. In medical laboratory, a large number of patients wait for their turn but due to sudden breakdown of testing equipments, the smooth functioning of the laboratory may deteriorate.

In chapter 6, we have developed the stochastic model for unreliable non-Markovian queueing system with two phase heterogeneous service and exponential distributed repair. In this chapter, we consider the more realistic situation by incorporating the additional assumptions namely (i) $N$-policy (ii) general distributed second optional service after completion of the first phase of regular service (iii) delayed repair (iv) $m$ phases general distributed repair on the line of Choudhury et al. (2009). Furthermore, we assume that the units arrive in bulk with varying arrival rates of the units. To explore the behaviour of the queueing system, the supplementary variable and probability generating function approaches are employed. To obtain the approximate results of the system probabilities which may be further utilized for the evaluation of other performance measures, the maximum entropy principle is also applied. To examine the effect of system parameters on various performance indices, we
provide the numerical results by taking an illustration. By stating requisite assumptions and notations, the model description is presented in section 7.2. Section 7.3 contains the set of governing equations along with boundary conditions which are constructed by introducing the supplementary variables concerned with elapsed service time, repair time and delay time. In section 7.4, we analyse the model to study the steady state behaviour of the queue length distribution by using the probability generating functions. Some performance measures are presented in section 7.5. The maximum entropy principle is used to find the queue size distribution in section 7.6. Special cases are deduced in section 7.7 by taking appropriate parameter values. In section 7.8, numerical illustration and sensitivity analysis are facilitated to examine the effect of system parameters on various performance indices.

7.2 Model Description

In many industrial scenarios, the main objective of any manufacturing/production process is to produce the quality products with low cost and minimum time. To produce the optimal results in various congestion situations including the field of digital communication systems, manufacturing/production systems, etc. the service may be rendered in two phases. In the present investigation, we assume that there is a provision to go for the immediate repair on unpredictable breakdowns of the server; the repair may be done in different phases. The repair may be delayed due to unavailability of the repairman or any other reasons but it may affect the smooth functioning of the system. The flow of the units during repair may be influenced by the server status. Keeping in view the above mentioned situation, we consider a queueing system under $N$–policy, wherein the units join the system according to the Poisson process in batches of random size $X$ with the state dependent arrival $\lambda_0, \lambda_1, \lambda_2$ and $\lambda_3$, when server is in idle state, busy in providing the service, waiting for repair and under repair, respectively. There is a provision of two stage service available in the system. After availing the regular service the arrived units may join the optional service with probability $p$ or leave the system with probability $q = 1 - p$. It is assumed that the breakdowns of the server may occur in Poisson fashion during first stage regular service (second stage optional service) with the rate $\alpha_1$ ($\alpha_2$). The failed server joins the repair station wherein $m$ phase repair of the failed server is available in which first phase repair is essential and other phases of optional repair. When essential repair of a server is completed, the
server may join the repair system for second phase of repair with probability \( r_1 \).
Similarly after completing the second phase of optional repair, the repairman immediately starts subsequent third phase of repair with probability \( r_2 \); otherwise the server may leave the repair system and start to provide the service. On a similar pattern, the server may require a maximum of \( m \) phases of repair, including first phase essential repair, with probabilities \( r_{l-1} \) \((l = 2, 3, \ldots, m)\) for moving from \((l - 1)^{th}\) phase to \(l^{th}\) phase of repair.

To analyse the non-markovian model, we introduce the supplementary variable corresponding to elapsed service time, the elapsed delay time and elapsed repair time. Suppose \( N_q(t) \) be the number of units in the system including the unit being in service and \( B^0_i(t) \) as elapsed service time of the unit for \( i^{th} (i = 1, 2) \) phase of service at time \( t \).

It is also assumed that \( \psi_i(t) (\varphi_i(t)) l = 1, 2, \ldots, m \) as the elapsed delay (repair) time of the server when the broken down occurs in \( i^{th} (i = 1, 2) \) phase of service at time \( t \).

To define the server's state, we introduce the random variables \( \zeta(t) \) defined as

\[
\zeta(t) = \begin{cases} 
0, & \text{if the server is idle at time } t, \\
1, & \text{if the server is busy with first phase service at time } t, \\
2, & \text{if the server is busy with second optional phases service at time } t, \\
3, & \text{if the server is waiting for repair when failed during first phase service at time } t, \\
4, & \text{if the server is waiting for repair when failed during second optional phase service at time } t, \\
5, & \text{if the server is under essential repair when failed during first phase service at time } t, \\
5 + l, & \text{if the server is under } l^{th} (l = 1, 2, \ldots, m - 1) \text{ optional repair when failed during first phase service at time } t, \\
5 + m, & \text{if the server is under essential repair when failed during second optional phase service at time } t, \\
5 + m + l, & \text{if the server is under } l^{th} (l = 1, 2, \ldots, m - 1) \text{ optional repair when failed during second optional phase service at time } t. 
\end{cases}
\]

For analysis purpose, we consider the bivariate Markov process (cf. Choudhury et al., 2009), \( \{N_q(t), X(t)\} \) where \( X(t) \) takes values

\[
0, B^0_1(t), B^0_2(t), \psi_1(t), \psi_2(t), \varphi_{1,1}(t), \ldots, \varphi_{1,m}, \varphi_{2,1}(t), \ldots, \varphi_{2,m}(t) ;
\]

if \( \zeta(t) = 0, 1, \ldots, 4 + 2m \) respectively.
The limiting probabilities of the system states are defined as

\[ P_n^{(0)} = \lim_{t \to \infty} \text{Prob.}\{N_q(t) = 0, X(t) = 0\} \quad \text{for} \quad n = 0, 1, \ldots, N - 1 \]

\[ P_n^{(i)}(x)dx = \lim_{t \to \infty} \text{Prob.}\{N_q(t) = n, X(t) = B_i^0(t); x < B_i^0(t) \leq x + dx\}; \quad n \geq 1, x > 0, i \in \{1, 2\} \]

\[ B_n^{(i)}(x, y)dy = \lim_{t \to \infty} \text{Prob.}\{N_q(t) = n, X(t) = \psi_i(t); y < \psi_i(t) \leq y + dy / B_i^0(t) = x\}; \quad n \geq 1, (x, y) > 0, i \in \{1, 2\} \]

\[ R_{i,n}^{(i)}(x, y)dy = \lim_{t \to \infty} \text{Prob.}\{N_q(t) = n, X(t) = \varphi_{i,i}(t); y < \varphi_{i,i}(t) \leq y + dy / B_i^0(t) = x\}; \quad n \geq 1, (x, y) > 0, \quad i \in \{1, 2\}, l = 1, 2, \ldots, m \]

with

\[ B_i(0) = 0, B_i(\infty) = 1, D_i(0) = 0, D_i(\infty) = 1, G_{i,l}(0) = 0, G_{i,l}(\infty) = 1; \quad i = 1, 2. \]

We assume that \( B_i(x) \) is continuous at \( x = 0 \), \( G_{i,l}(y) \) and \( D_i(y) \) are continuous at \( y = 0 \).

The hazard rate functions for \( i^{th} (i = 1, 2) \) phase service are same as defined in equation (6.1) and for delay repair (under repair) states, the hazard rate functions are defined as follows:

\[ \eta_i(y) = \frac{d_i(y)}{1 - D_i(y)}; \quad i = 1, 2 \]  

(7.1)

\[ \xi_{i,l}(y) = \frac{g_{i,l}(y)}{1 - G_{i,l}(y)}; \quad i = 1, 2 \text{ and } l = 1, 2, \ldots, m \]  

(7.2)

so that

\[ d_i(y) = \eta_i(y) e^{-\int_0^y \eta_i(t) dt} \quad \text{and} \quad g_{i,l}(y) = \xi_{i,l}(y) e^{-\int_0^y \xi_{i,l}(t) dt} \]  

(7.3)

Further, we define the probability generating functions as follows:

\[ B_i^{(i)}(x, y, z) = \sum_{n=1}^{\infty} z^n B_i^{(i)}(x, y); \quad B_i^{(i)}(x, 0, z) = \sum_{n=1}^{\infty} z^n B_i^{(i)}(x, 0) \]  

(7.4)

\[ R_i^{(i)}(x, y, z) = \sum_{n=1}^{\infty} z^n R_i^{(i)}(x, y); \quad R_i^{(i)}(x, 0, z) = \sum_{n=1}^{\infty} z^n R_i^{(i)}(x, 0), l = 1, 2, \ldots, m \]  

(7.5)

\[ P_i^{(0)}(z) = \sum_{n=0}^{N-1} z^n P_i^{(0)} \]  

(7.6)

Let us further define \( \xi_n (n = 0, 1, 2, \ldots, N - 1) \) as the probability that a batch of unit finds at least \( n \) unit in the system during the idle period. Thus \( \xi_n \) satisfies the following recursive relation as given below.
\[ \xi_0 = 1, \quad \xi_n = \sum_{k=1}^{n} c_k \xi_{n-k}, \quad 1 \leq n \leq N - 1. \] (7.7)

### 7.3 Governing Equations

In this section, we construct Chapman-Kolmogorov steady state equations governing the system states (cf. Cox, 1955; Choudhury et al., 2009) by using the probability reasoning as follows:

\[
\frac{d}{dx} P_{n}^{(i)}(x) + [\lambda_i + \alpha_i + \mu_i(x)] P_{n}^{(i)}(x) = \sum_{j=1}^{n} \lambda_j c_j P_{n-j}^{(i)}(x) + \int_{0}^{\infty} \xi_{j,m}(y) P_{m,n}^{(i)}(x, y) dy
\]

\[ + \sum_{k=1}^{m-1} (1 - r_k) \int_{0}^{\infty} \xi_{j,k}(y) R_{k,n}^{(i)}(x, y) dy \quad ; \quad i = 1, 2; \quad n \geq 1 \] (7.8)

\[
\frac{d}{dy} B_{n}^{(i)}(x, y) + [\alpha_2 + \eta_i(y)] B_{n}^{(i)}(x, y) = \sum_{j=1}^{n} \lambda_j c_j B_{n-j}^{(i)}(x, y); \quad i = 1, 2; \quad n \geq 1 \] (7.9)

\[
\frac{d}{dy} R_{i,n}^{(i)}(x, y) + [\lambda_3 + \xi_{i,j}(y)] R_{i,n}^{(i)}(x, y) = \sum_{j=1}^{n} \lambda_j c_j R_{i,n-j}^{(i)}(x, y);
\]

\[ 1 \leq l \leq m, i = 1, 2; \quad n \geq 1 \] (7.10)

\[ \lambda_0 P_{0}^{(0)} = \int_{0}^{\infty} \mu_2(x) P_{1}^{(2)}(x) dx + q \int_{0}^{\infty} \mu_1(x) P_{1}^{(1)}(x) dx \] (7.11)

\[ \lambda_0 P_{n}^{(0)} = \lambda_0 \sum_{j=1}^{n} c_{n-j} P_{j}^{(0)} \quad n = 1, 2, ..., N - 1. \] (7.12)

The set of equations (7.8)-(7.10) are to be solved under the following boundary conditions at \( x = 0 \):

\[ P_{n}^{(1)}(0) = \int_{0}^{\infty} \mu_2(x) P_{n+1}^{(2)}(x) dx + q \int_{0}^{\infty} \mu_1(x) P_{n+1}^{(1)}(x) dx, \quad 1 \leq n \leq N - 1 \] (7.13)

\[ P_{n}^{(1)}(0) = \int_{0}^{\infty} \mu_2(x) P_{n+1}^{(2)}(x) dx + q \int_{0}^{\infty} \mu_1(x) P_{n+1}^{(1)}(x) dx + \lambda_0 \sum_{j=1}^{n} c_{n-j} P_{j}^{(0)}, \quad n \geq N \] (7.14)

\[ P_{n}^{(2)}(0) = p \int_{0}^{\infty} \mu_1(x) P_{n}^{(0)}(x) dx, \quad n \geq 1. \] (7.15)

Also, for fixed value of \( x \) and at \( y = 0 \) for \( i = 1, 2 \), we have

\[ B_{n}^{(i)}(x, 0) = \alpha_i P_{n}^{(i)}(x), \quad x > 0, n \geq 1 \] (7.16)

\[ R_{i,n}^{(i)}(x, 0) = \int_{0}^{\infty} \eta_i(y) B_{n}^{(i)}(x, y) dy, \quad x > 0, n \geq 1 \] (7.17)
The normalizing condition is
\[\sum_{n=0}^{N-1} P_n^{(0)} + \sum_{i=1}^{\infty} \left\{ \int_0^\infty P_n^{(i)}(x)dx + \int_0^\infty B_n^{(i)}(x,y)dxdy + \int_0^\infty \sum_{l=1}^m R_{l,n}^{(i)}(x,y)dxdy \right\} = 1. \quad (7.19)\]

7.4 Mathematical Analysis

To obtain the queue size distribution and other performance characteristics of the system with the assumption that the server may start the service to the unit when the queue size becomes at least \(N(N \geq 1)\), we analyse the queueing problem by solving the set of governing equations under given conditions and using generating functions (cf. Choudhury et al., 2009). For brevity of notations, we denote
\[\phi_k(z) = \lambda_k (1 - X(z)) ; \quad \sigma_i^{(2)} = E(X) \left\{ \lambda_1 + \alpha_1 (\lambda_2 \gamma_i^{(1)} + \lambda_3 g_{il}^{(1)}) + \lambda_3 \sum_{l=2}^m \left( \prod_{j=1}^{l-1} r_j \right) g_{il}^{(1)} \right\} \]
\[\tau_i(z) = \phi_i(z) + \alpha \left[ 1 - D_i(\phi_i(z)) \right] G_{i,1}(\phi_i(z)) + \sum_{l=2}^m \left( \prod_{j=1}^{l-1} r_j \right) G_{i,l}(\phi_i(z)) \left( G_{i,l}(\phi_i(z)) - 1 \right) \]
\[\sigma_i^{(2)} = \lambda_i E(X^{(2)}) + \alpha \left[ 2 \lambda_3 \lambda_2 (E(X))^2 \gamma_i^{(1)} g_{il}^{(1)} + 2 \lambda_3 \lambda_2 (E(X))^2 \gamma_i^{(2)} + \frac{m}{r_i} \sum_{l=2}^m \left( \prod_{j=1}^{l-1} r_j \right) g_{il}^{(1)} \right] + 2(\lambda_3 E(X))^2 \sum_{l=2}^m \left( \prod_{j=1}^{l-1} r_j \right) \left( \prod_{j=1}^{l-1} r_j \right) g_{il}^{(1)} + \lambda_3 E(X^{(2)}) g_{il}^{(1)} + \lambda_3 E(X)^2 g_{il}^{(2)} \]
\[\rho = E(B_i) \sigma_i^{(1)} + p E(B_2) \sigma_2^{(1)} \]
\[\rho = \lambda_0 E(B_0) E(X) \left[ 1 + \alpha_1 (\gamma_1^{(1)} + g_{11}^{(1)}) + \sum_{l=2}^m \left( \prod_{j=1}^{l-1} r_j \right) g_{1l}^{(1)} \right] + p \lambda_0 E(B_2) E(X) \left[ 1 + \alpha_2 (\gamma_2^{(1)} + g_{21}^{(1)}) + \sum_{l=2}^m \left( \prod_{j=1}^{l-1} r_j \right) g_{2l}^{(1)} \right] \]
On multiplying the equations (7.9) and (7.10) by suitable powers of \(z\) and taking summation over all possible values of \(n\) and after simplification, we have
\[B^{(i)}(x,y,z) = B^{(i)}(x,0,z) e^{-\phi_i(z)} [1 - D_i(y)] ; i = 1,2 ; x,y > 0 \quad (7.20)\]
\[R_l^{(i)}(x,y,z) = R_l^{(i)}(x,0,z) e^{-\phi_i(z)} [1 - G_{l,y}(y)] ; i = 1,2 ; 1 \leq l \leq m. \quad (7.21)\]
Now from equation (7.16), we have
\[ B^{(i)}(x,0,z) = \alpha_i P^{(i)}(x,z). \] (7.22)

From equations (7.17) and (7.18), we obtain
\[ R_{i}^{(i)}(x,0,z) = B^{(i)}(x,0,z)\overline{D}_i(\phi_2(z)) \] (7.23)
\[ R_{i}^{(i)}(x,0,z) = r_{i-1}R_{i-1}^{(i)}(x,0,z)\overline{G}_{i,i-1}(\phi_i(z)), \quad 2 \leq l \leq m. \] (7.24)

Similarly solving equation (7.8), and using (7.23) and (7.24), we get
\[ P^{(i)}(x,z) = P^{(i)}(0,z)[1 - B_j(x)]e^{-z\phi_j(y)}. \] (7.25)

Multiplying equations (7.13) and (7.14) by \( z^n \) and then taking summation over all possible values of \( n \) and after simplification, we get
\[ zP^{(i)}(0,z) = -z\phi_i(z)P_N^{(i)}(z) + qP^{(i)}(0,z)\overline{B}_1(\tau_1(z)) + P^{(2)}(0,z)\overline{B}_2(\tau_2(z)). \] (7.26)

Similarly, from equation (7.15), we get
\[ P^{(2)}(0,z) = pP^{(i)}(0,z)\overline{B}_1(\tau_1(z)). \] (7.27)

Utilizing the value of equation (7.27) in equation (7.26), we get
\[ P^{(i)}(0,z) = \frac{z\phi_i(z)P_N^{(i)}(z)}{[(q + p\overline{B}_2(\tau_2(z)))\overline{B}_1(\tau_1(z)) - z]} \] (7.28)

From equations (7.20), (7.22) and (7.25), we get
\[ B^{(i)}(x,y,z) = \alpha_i P^{(i)}(0,z)[1 - B_j(x)]e^{-z\phi_j(y)}[1 - D_j(y)] e^{-\phi_j(y)} \] (7.29)

Similarly using equations (7.22)-(7.25) in equation (7.21), we get
\[ R_{i}^{(i)}(x,y,z) = \alpha_i P^{(i)}(0,z)[1 - B_j(x)]e^{-y\phi_j(y)} D_i(\phi_2(z))[1 - G_{i,1}(y)] e^{-\phi_j(y)}; \quad i = 1,2 \] (7.30)
\[ R_{i}^{(i)}(x,y,z) = \alpha_i P^{(i)}(0,z)[1 - B_j(x)]e^{-y\phi_j(y)} D_i(\phi_2(z))[\prod_{j=1}^{l-1} r_{i,j} \overline{G}_{i,1}(\phi_i(z))] \times[1 - G_{i,1}(y)] e^{-\phi_j(y)}; \quad 2 \leq l \leq m \quad i = 1,2. \] (7.31)

**Theorem 7.1:** The probability generating functions for the joint distributions of the number of units in the queue under different server states are given by
\[ P_N^{(0)}(z) = \frac{(1 - \rho) \sum_{n=0}^{N-1} \xi_n z^n}{\sum_{n=0}^{N-1} \xi_n} \] (7.32)
\[ P^{(1)}(x, z) = \frac{(1 - \rho) \sum_{n=0}^{N-1} \xi_n z^{n+1} \phi_0(z)[1 - B_1(x)e^{-\tau_1(z)}]}{\sum_{n=0}^{N-1} \xi_n [(q + pB_2(\tau_2(z)))B_1(\tau_1(z)) - z]} \quad (7.33) \]

\[ P^{(2)}(x, z) = \frac{p(1 - \rho) \sum_{n=0}^{N-1} \xi_n z^{n+1} \phi_0(z)B_1(\tau_1(z))[1 - B_2(x)e^{-\tau_2(z)}]}{\sum_{n=0}^{N-1} \xi_n [(q + pB_2(\tau_2(z)))B_1(\tau_1(z)) - z]} \quad (7.34) \]

\[ B^{(1)}(x, y, z) = \frac{(1 - \rho) \alpha_1 \sum_{n=0}^{N-1} \xi_n z^{n+1} \phi_0(z)[1 - B_1(x)e^{-\tau_1(z)}][1 - D_1(y)]e^{-\phi_2(z)y}}{\sum_{n=0}^{N-1} \xi_n [(q + pB_2(\tau_2(z)))B_1(\tau_1(z)) - z]} \quad (7.35) \]

\[ B^{(2)}(x, y, z) = \frac{(1 - \rho) \alpha_2 p \sum_{n=0}^{N-1} \xi_n z^{n+1} \phi_0(z)B_1(\tau_1(z))[1 - B_2(x)e^{-\tau_2(z)}][1 - D_2(y)]e^{-\phi_2(z)y}}{\sum_{n=0}^{N-1} \xi_n [(q + pB_2(\tau_2(z)))B_1(\tau_1(z)) - z]} \quad (7.36) \]

\[ R^{(1)}_l(x, y, z) = \frac{[(1 - \rho) \alpha_1 \sum_{n=0}^{N-1} \xi_n z^{n+1} \phi_0(z)[1 - B_1(x)e^{-\tau_1(z)}]\bar{D}_1(\phi_2(z))]}{\sum_{n=0}^{N-1} \xi_n [(q + pB_2(\tau_2(z)))B_1(\tau_1(z)) - z]} \times [1 - G_{1,1}(y)]e^{-\phi_2(z)y} \quad (7.37) \]

\[ R^{(1)}_l(x, y, z) = \frac{[(1 - \rho) \alpha_1 \sum_{n=0}^{N-1} \xi_n z^{n+1} \phi_0(z)[1 - B_1(x)e^{-\tau_1(z)}]\bar{D}_1(\phi_2(z))]}{\sum_{n=0}^{N-1} \xi_n [(q + pB_2(\tau_2(z)))B_1(\tau_1(z)) - z]} \times \prod_{j=1}^{l-1} G_{1,j}(\phi_2(z)) \quad (7.38) \]

\[ R^{(2)}_l(x, y, z) = \frac{[(1 - \rho) \alpha_2 p \sum_{n=0}^{N-1} \xi_n z^{n+1} \phi_0(z)B_1(\tau_1(z))[1 - B_2(x)e^{-\tau_2(z)}]\bar{D}_2(\phi_2(z))]}{\sum_{n=0}^{N-1} \xi_n [(q + pB_2(\tau_2(z)))B_1(\tau_1(z)) - z]} \times [1 - G_{2,1}(y)]e^{-\phi_2(z)y} \quad (7.39) \]
\[(1 - \rho)\alpha_2 p \sum_{n=0}^{N-1} \xi_n z^{n+1} \phi_0(z) \bar{B}_1(\tau_1(z))[1 - B_2(x)] e^{-\tau_1(z)x} \bar{D}_2(\phi_2(z))\]

\[R_i^{(2)}(x, y, z) = \frac{\times [\prod_{j=1}^{l-1} r_j \bar{G}_{2,j}(\phi_j(z))][1 - G_{2,j}(y)] e^{-\phi_j(y)}}{\sum_{n=0}^{N-1} \xi_n [(q + p \bar{B}_2(\tau_2(z))) \bar{B}_1(\tau_1(z)) - z]} \]

\[2 \leq l \leq m\]

where

The utilization factor is given by \(\rho = \frac{\rho_2}{1 - \rho_1 + \rho_2}\).

**Proof:** For proof see Appendix-VI.A.

**Theorem 7.2:** The marginal probability generating functions of the queue size distribution under different server states are given by

\[P^{(1)}(z) = \frac{(1 - \rho) \sum_{n=0}^{N-1} \xi_n z^{n+1} \phi_0(z) [1 - \bar{B}_1(\tau_1(z))]}{\sum_{n=0}^{N-1} \xi_n [(q + p \bar{B}_2(\tau_2(z))) \bar{B}_1(\tau_1(z)) - z] \{\tau_1(z)\}} \]

(7.41)

\[P^{(2)}(z) = \frac{p(1 - \rho) \sum_{n=0}^{N-1} \xi_n z^{n+1} \phi_0(z) [1 - \bar{B}_2(\tau_2(z))] \bar{B}_1(\tau_1(z))}{\sum_{n=0}^{N-1} \xi_n [(q + p \bar{B}_2(\tau_2(z))) \bar{B}_1(\tau_1(z)) - z] \{\tau_2(z)\}} \]

(7.42)

\[B^{(1)}(z) = \frac{(1 - \rho)\alpha_1 \sum_{n=0}^{N-1} \xi_n z^{n+1} \phi_0(z) [1 - \bar{B}_1(\tau_1(z))] [1 - \bar{D}_1(\phi_2(z))]}{\sum_{n=0}^{N-1} \xi_n [(q + p \bar{B}_2(\tau_2(z))) \bar{B}_1(\tau_1(z)) - z] \{\tau_1(z)\} \{\phi_2(z)\}} \]

(7.43)

\[B^{(2)}(z) = \frac{p(1 - \rho)\alpha_2 \sum_{n=0}^{N-1} \xi_n z^{n+1} \phi_0(z) [1 - \bar{B}_2(\tau_2(z))] \bar{B}_1(\tau_1(z)) [1 - \bar{D}_2(\phi_2(z))]}{\sum_{n=0}^{N-1} \xi_n [(q + p \bar{B}_2(\tau_2(z))) \bar{B}_1(\tau_1(z)) - z] \{\tau_2(z)\} \{\phi_2(z)\}} \]

(7.44)

\[R_i^{(1)}(z) = \frac{(1 - \rho)\alpha_1 \sum_{n=0}^{N-1} \xi_n z^{n+1} \phi_0(z) [1 - \bar{B}_1(\tau_1(z))] \bar{D}_1(\phi_2(z)) [1 - \bar{G}_{1,1}(\phi_3(z)))]}{\sum_{n=0}^{N-1} \xi_n [(q + p \bar{B}_2(\tau_2(z))) \bar{B}_1(\tau_1(z)) - z] \{\tau_1(z)\} \{\phi_3(z)\}} \]

(7.45)
\[ R^{(1)}_l(z) = \frac{(1 - \rho) \alpha \sum_{n=0}^{N-1} \xi_n z^{n+1} \phi_0(z)[1 - \overline{B}_1(\tau_1(z))]\overline{D}_1(\phi_2(z)) \prod_{j=1}^{l-1} \overline{G}_{1,j}(\phi_3(z))}{\sum_{n=0}^{N-1} \xi_n[(q + p\overline{B}_2(\tau_2(z)))\overline{B}_1(\tau_1(z)) - z]\{\tau_1(z)\} \{\phi_3(z)\}} \; ; \quad 2 \leq l \leq m \]  

(7.46)

\[ R^{(2)}_l(z) = \frac{(1 - \rho) \alpha z \sum_{n=0}^{N-1} \xi_n z^{n+1} \phi_0(z)[1 - \overline{B}_2(\tau_2(z))]\overline{B}_1(\tau_1(z))\overline{D}_2(\phi_2(z))[1 - \overline{G}_{2,l}(\phi_3(z))]}{\sum_{n=0}^{N-1} \xi_n[(q + p\overline{B}_2(\tau_2(z)))\overline{B}_1(\tau_1(z)) - z]\{\tau_2(z)\} \{\phi_3(z)\}} \]  

(7.47)

\[ R^{(2)}_l(z) = \frac{(1 - \rho) \alpha z \sum_{n=0}^{N-1} \xi_n z^{n+1} \phi_0(z)[1 - \overline{B}_2(\tau_2(z))]}{\sum_{n=0}^{N-1} \xi_n[(q + p\overline{B}_2(\tau_2(z)))\overline{B}_1(\tau_1(z)) - z]\{\tau_2(z)\} \{\phi_3(z)\}} \; ; \quad 2 \leq l \leq m. \]  

(7.48)

**Proof:** For proof see Appendix-VI.B.

**Theorem 7.3:** The probability generating function of the stationary queue size at departure epoch under the stability condition is given by

\[ \omega(z) = \frac{(1 - \rho) \sum_{n=0}^{N-1} \xi_n z^n (1 - X(z))(q + p\overline{B}_2(\tau_2(z)))\overline{B}_1(\tau_1(z))}{E(X) \sum_{n=0}^{N-1} \xi_n[(q + p\overline{B}_2(\tau_2(z)))\overline{B}_1(\tau_1(z)) - z]} \]  

(7.49)

**Proof:** For proof see Appendix-VI.C.

### 7.5 Performance Measures

To study the behaviour of the queueing system, in this section we obtain various performance measures by setting the appropriate parameters in the expressions of the probability generating functions of the queue size distributions.
(a) Long run probabilities of the server states

We derive the expressions for the long run probabilities of the server states by taking limiting values when $z \to 1$ of the marginal probability generating functions established in theorem 7.2.

(i) The probability that the server being busy in rendering the $i^{th}$ ($i = 1, 2$) phase of service is given by

$$P(B_i) = \lim_{z \to 1} P^{(i)}(z) = \frac{\lambda_0 E(X) E(B_i)}{1 - \rho_1 + \rho_2}. \quad (7.50)$$

(ii) The probability that the server being busy in rendering the second optional phase of service is given by

$$P(B_2) = \lim_{z \to 1} P^{(2)}(z) = \frac{p \lambda_0 E(X) E(B_2)}{1 - \rho_1 + \rho_2}. \quad (7.51)$$

(iii) The probability that the server is waiting for repair while broken down during the first essential phase of service is given by

$$P(D_1) = \lim_{z \to 1} B^{(1)}(z) = \frac{\alpha_1 \lambda_0 E(X) E(B_1) \gamma_1^{(1)}}{1 - \rho_1 + \rho_2}. \quad (7.52)$$

(iv) The probability that the server is waiting for repair when failed during the second optional phase service is given by

$$P(D_2) = \lim_{z \to 1} B^{(2)}(z) = \frac{\alpha_2 p \lambda_0 E(X) E(B_2) \gamma_2^{(1)}}{1 - \rho_1 + \rho_2}. \quad (7.53)$$

(v) The probabilities that the server is under first phase and $l^{th}$ ($l = 2, \ldots, m$) phase repair when failed during the first essential phase service are given by

$$P(R_1^{(1)}) = \lim_{z \to 1} R_1^{(1)}(z) = \frac{\alpha_1 \lambda_0 E(X) E(B_1) g_{11}^{(1)}}{1 - \rho_1 + \rho_2} \quad (7.54)$$

and

$$P(R_l^{(1)}) = \lim_{z \to 1} R_l^{(1)}(z) = \frac{\alpha_1 \lambda_0 E(X) E(B_1) \prod_{j=1}^{l-1} r_j g_{11}^{(1)}}{1 - \rho_1 + \rho_2}; \quad 2 \leq l \leq m. \quad (7.55)$$

(vi) The probabilities that the server failed during the second optional phase service under first phase and $l^{th}$ ($l = 2, \ldots, m$) phase repair are given by

$$P(R_1^{(2)}) = \lim_{z \to 1} R_1^{(2)}(z) = \frac{\alpha_2 p \lambda_0 E(X) E(B_2) g_{21}^{(1)}}{1 - \rho_1 + \rho_2} \quad (7.56)$$
\[ P(R_i^{(2)}) = \lim_{z \to 1} R_i^{(2)}(z) = \frac{\alpha_p \lambda_p E(X) E(B_2)[\prod_{j=1}^{l-1} r_j]g_{ij}^{(1)}}{1 - \rho_1 + \rho_2} \quad 2 \leq i \leq m. \] (7.57)

The probability that the server is idle is given by

\[ P(I) = 1 - \sum_{i=1}^{2} \left( (P(B_i) + P(W_i)) + P(R_i^{(1)}) \right) = 1 - \rho \] (7.58)

with

\[ P(R_i^{(1)}) = \sum_{i=1}^{m} P(R_i^{(1)}); \quad i = 1, 2. \]

(b) **Average number of units in system at departure epoch**

In order to obtain the average number of units in system at departure epoch \( L_s \), the derivatives of the numerator and denominator of right hand side of equation (7.49) at \( z = 1 \) are obtained as

\[ N'(1) = -(1 - \rho_1) \] (7.59)

\[ N''(1) = (1 - \rho_1) \left( -\frac{E(X^{(2)})}{E(X)} - 2 \left[ \sum_{n=0}^{N-1} \frac{n \sigma_n}{\sigma_n} + \rho_1 \right] \right) \] (7.60)

\[ D'(1) = (\rho_1 - 1) \] (7.61)

\[ D''(1) = 2pE(B_1)E(B_2)\sigma_1^{(1)}\sigma_2^{(1)} + E(B_1^2)\{\sigma_1^{(1)}\}^2 + E(B_1)\sigma_1^{(2)} + pE(B_2^2)\{\sigma_2^{(2)}\}^2 \] (7.62)

By using the equations (7.59)-(7.62) in equation (C.2.1), we get the required result of average number of units in system at departure epoch \( L_s \).

(c) **Average waiting time of the units in the system**

The average waiting time \( W_s \) of the units in the system can be obtained by using the equation (2.19) with

\[ \lambda_s = \lambda_p P_N^{(0)}(1) + \lambda_1 \sum_{i=1}^{3} P_i^{(i)}(1) + \lambda_2 \sum_{i=1}^{3} B_i^{(i)}(1) + \lambda_3 \sum_{i=1}^{3} \left( \sum_{i=1}^{m} R_i^{(i)}(1) \right). \]
(d) Reliability indices

Let $A_s(t)$ be the system availability at the time $t$. Then the steady state availability $A_s$, which is the probability that the server is either busy in rendering service or in an idle state, is obtained using

$$A_s = \lim_{z \to 1} \{P_N^{(0)}(z) + P^{(1)}(z) + P^{(2)}(z)\}$$

(7.63)

$$\lambda_0 E(X)E(B_1)\alpha_1 (\gamma_1^{(1)} + g_1^{(1)}) + \sum_{i=2}^{m} \prod_{j=1}^{l-1} r_j g_{2l}^{(1)} \alpha_1 (\gamma_1^{(1)} + g_2^{(1)}) + \lambda_0 E(X)E(B_2)\alpha_2 (\gamma_2^{(1)} + g_2^{(1)})$$

$$+ \sum_{i=2}^{m} \prod_{j=1}^{l-1} r_j g_{2l}^{(1)} = 1 - \frac{1 - \rho_1 + \rho_2}{1 - \rho_1 + \rho_2}.$$  

(7.64)

The steady state failure frequency is determined using

$$F_f = \alpha_1 \int_0^\infty P^{(1)}(x,1)dx + \alpha_2 \int_0^\infty P^{(2)}(x,1)dx = \frac{\alpha_1 \lambda_0 E(B_1)E(X) + p \alpha_2 \lambda_0 E(B_2)E(X)}{1 - \rho_1 + \rho_2}.$$  

(7.65)

7.6 Maximum Entropy Results

In this section, we describe the maximum entropy approach for estimating the steady state probabilities of the queueing system under the assumptions described in previous sections.

The maximum entropy function is defined as

$$Z = -\sum_{n=0}^{N-1} P_n^{(0)} \log P_n^{(0)} - \sum_{i=1}^{m} \sum_{n=1}^{\infty} P_n^{(i)} \log P_n^{(i)} - \sum_{i=1}^{m} \sum_{n=1}^{\infty} B_n^{(i)} \log B_n^{(i)} - \sum_{i=1}^{m} \sum_{n=1}^{\infty} R_n^{(i)} \log R_n^{(i)}.$$  

(7.66)

The maximum entropy results are obtained by maximizing the entropy function (7.66) subject to the following constraints.

$$\sum_{n=0}^{N-1} P_n^{(0)} + \sum_{i=1}^{m} \sum_{n=1}^{\infty} P_n^{(i)} + \sum_{i=1}^{m} \sum_{n=1}^{\infty} B_n^{(i)} + \sum_{i=1}^{m} \sum_{n=1}^{\infty} R_n^{(i)} = 1$$

(7.67)

$$\sum_{n=1}^{\infty} P_n^{(1)} = \eta_1 \equiv P^{(1)}(1)$$

(7.68)

$$\sum_{n=1}^{\infty} P_n^{(2)} = \eta_2 \equiv P^{(2)}(1)$$

(7.69)

$$\sum_{n=1}^{\infty} B_n^{(1)} = \eta_3 \equiv B^{(1)}(1)$$

(7.70)
\[ \sum_{n=1}^{\infty} B_n^{(2)} = \eta_4 \equiv B^{(2)}(1) \quad (7.71) \]
\[ \sum_{n=1}^{\infty} R_n^{(1)} = \eta_5 \equiv R^{(1)}(1) \quad (7.72) \]
\[ \sum_{n=1}^{\infty} R_n^{(2)} = \eta_6 \equiv R^{(2)}(1) \quad (7.73) \]
\[ \sum_{n=0}^{N-1} nP_n^{(0)} + \sum_{n=1}^{\infty} np_n^{(1)} + \sum_{n=1}^{\infty} nP_n^{(2)} + \sum_{n=1}^{\infty} nb_n^{(2)} + \sum_{n=1}^{\infty} nB_n^{(1)} + \sum_{n=1}^{\infty} nR_n^{(1)} + \sum_{n=1}^{\infty} nR_n^{(2)} = L_s. \quad (7.74) \]

**Theorem 7.4:** The steady state probabilities of the system states for bulk arrival queue with repair for different states are obtained as
\[ P_{n}^{(0)} = \frac{(1 - \eta)(1 - \tau_0)(\tau_0)^n}{(1 - \tau_0)} \quad n = 0, 1, 2, ..., N - 1 \quad (7.75) \]
\[ P_{n}^{(1)} = \eta_1(1 - \tau_0)(\tau_0)^{n-1} \quad n = 1, 2, 3, ... \quad (7.76) \]
\[ P_{n}^{(2)} = \eta_2(1 - \tau_0)(\tau_0)^{n-1} \quad n = 1, 2, 3, ... \quad (7.77) \]
\[ B_{n}^{(1)} = \eta_3(1 - \tau_0)(\tau_0)^{n-1} \quad n = 1, 2, 3, ... \quad (7.78) \]
\[ B_{n}^{(2)} = \eta_4(1 - \tau_0)(\tau_0)^{n-1} \quad n = 1, 2, 3, ... \quad (7.79) \]
\[ R_{n}^{(1)} = \eta_5(1 - \tau_0)(\tau_0)^{n-1} \quad n = 1, 2, 3, ... \quad (7.80) \]
\[ R_{n}^{(2)} = \eta_6(1 - \tau_0)(\tau_0)^{n-1} \quad n = 1, 2, 3, ... \quad (7.81) \]

**Proof:** For proof see Appendix-VI.D.

**Theorem 7.5:** The approximate average waiting time of the units is obtained as
\[ W_s^* = \left( E(B_1) + pE(B_2) \right) \frac{E(X^{(2)})}{2E(X)} + \frac{1}{\lambda_0 E(X)} \left( N - 1 - \frac{E(X^{(2)})}{2E(X)} \right) \eta_0 + \left( \frac{\gamma_2^{(2)}}{2\gamma_1^{(0)}} + g_1^{(1)} \right) \eta_3 \]
\[ + \left( \frac{\gamma_2^{(2)}}{2\gamma_2^{(1)}} + g_2^{(1)} \right) \eta_4 + \frac{g_{11}^{(2)}}{2g_{11}^{(1)}} \eta_3 + \frac{g_{21}^{(2)}}{2g_{21}^{(1)}} \eta_6 + (E(B_1) + pE(B_2))L_s - \frac{1}{\lambda_0 E(X)} \left( \sum_{n=0}^{N-1} \hat{P}_n^{(0)} \right) \quad (7.82) \]
**Proof:**

We find the approximate average waiting time of a test unit $C$ by considering the server states when the server is idle ($I$); busy ($B_1, B_2$); waiting for repair ($D_1, D_2$) and under the repair ($R_1, R_2$) respectively.

- When the server is in idle state ($I$) and a batch arrives in the system then the idle state switched to busy state and the server starts the service if there is $N$ or more units accumulate in the system. Thus the unit $C$ will be in service only if
  
  $$(N - n - \frac{E(X^{(2)})}{2E(X)} - 1) \text{ units arrive in the system and } (n + \frac{E(X^{(2)})}{2E(X)}) \text{ units are in front of him waiting for service.}$$

  The average waiting time in idle state is
  
  $$\frac{1}{\lambda_0 E(X)} (N - n - \frac{E(X^{(2)})}{2E(X)} - 1) + (E(B_1) + pE(B_2))(n + \frac{E(X^{(2)})}{2E(X)}) .$$

- When the server is in the busy state to provide essential (optional) service, then the unit $C$ will wait only for service time of those $n$ units in front of him and additional waiting time due to unit preceding him in the same group. Thus average waiting time in essential (optional) service is
  
  $$(E(B_1) + pE(B_2))(n + \frac{E(X^{(2)})}{2E(X)}).$$

- When the server is waiting for repair i.e. in state $D_1(D_2)$ after failure during the essential (optional) service of the unit, the unit $C$ will wait residual delay time, mean repair time, service time of those $n$ units in front of him and additional waiting time due to unit preceding him in the same group. Thus average waiting time for $D_1$ and $D_2$ states are obtained as
  
  $$\frac{g_1^{(2)}}{2\gamma_1^{(1)}} + \frac{E(X^{(2)})}{2E(X)} + (E(B_1) + pE(B_2))(n + \frac{E(X^{(2)})}{2E(X)}) \text{ and}$$

  $$\frac{g_2^{(2)}}{2\gamma_2^{(1)}} + \frac{E(X^{(2)})}{2E(X)} + (E(B_1) + pE(B_2))(n + \frac{E(X^{(2)})}{2E(X)}),$$

  respectively.

- If the server is under repair i.e. in the state $R_1(R_2)$, when it failed during the essential (optional) service of the unit, then the unit $C$ will wait residual repair time, service time of those $n$ units in front of him and additional waiting time due to the
unit preceding him in the same group. Thus the average waiting time for the states $R_1$ and $R_2$ are

\[
\frac{g_{11}^{(2)}}{2g_{11}^{(1)}} + (E(B_1) + pE(B_2)) (n + \frac{E(X^{(2)})}{2E(X)}) \quad \text{and}
\]

\[
\frac{g_{21}^{(2)}}{2g_{21}^{(1)}} + (E(B_1) + pE(B_2)) (n + \frac{E(X^{(2)})}{2E(X)})
\]

respectively.

Thus the approximate average waiting time is

\[
W_s^* = \sum_{n=0}^{\infty} \left( \frac{1}{\lambda_n E(X)} (N - n - \frac{E(X^{(2)})}{2E(X)} - 1) + (E(B_1) + pE(B_2)) (n + \frac{E(X^{(2)})}{2E(X)}) \right) P_n^{(0)}
\]

\[
+ \sum_{n=1}^{\infty} \left( (E(B_1) + pE(B_2)) (n + \frac{E(X^{(2)})}{2E(X)}) \right) P_n^{(1)}
\]

\[
+ \sum_{n=1}^{\infty} \left( (E(B_1) + pE(B_2)) (n + \frac{E(X^{(2)})}{2E(X)}) \right) P_n^{(2)}
\]

\[
+ \sum_{n=1}^{\infty} \left( \gamma_1^{(2)} \frac{g_{11}^{(2)}}{2g_{11}^{(1)}} + (E(B_1) + pE(B_2)) (n + \frac{E(X^{(2)})}{2E(X)}) \right) B_n^{(1)}
\]

\[
+ \sum_{n=1}^{\infty} \left( \gamma_2^{(2)} \frac{g_{21}^{(2)}}{2g_{21}^{(1)}} + (E(B_1) + pE(B_2)) (n + \frac{E(X^{(2)})}{2E(X)}) \right) B_n^{(2)}
\]

\[
+ \sum_{n=1}^{\infty} \left( \gamma_1^{(2)} \frac{g_{11}^{(2)}}{2g_{11}^{(1)}} + (E(B_1) + pE(B_2)) (n + \frac{E(X^{(2)})}{2E(X)}) \right) R_n^{(1)}
\]

\[
+ \sum_{n=1}^{\infty} \left( \gamma_2^{(2)} \frac{g_{21}^{(2)}}{2g_{21}^{(1)}} + (E(B_1) + pE(B_2)) (n + \frac{E(X^{(2)})}{2E(X)}) \right) R_n^{(2)}
\]

which is equivalent to the required result as given in equation (7.82).

The deviation of approximate average waiting time ($W_s^*$) from average waiting time ($W_s$) is given by

\[
\text{error(\%)} = \frac{|W_s - W_s^*|}{W_s} \times 100 \quad (7.83)
\]

7.7 Special Cases

To validate the results obtained in the present investigation with some existing results obtained by various researchers and available in the literature of queueing theory, we deduce some special cases by setting the appropriate parameters.
Case (i) Bulk queue with uniform arrival rate and single phase repair.

By setting $\lambda_0 = \lambda_1 = \lambda_2 = \lambda_3 = \lambda$ and $r_1 = r_1 = \ldots = r_{m-1} = 0$; equation (7.49) gives

$$o(z) = \frac{(1 - \rho_1) \sum_{n=0}^{N-1} \xi_n z^n (1 - X(z)) (q + pB_2(\tau_2(z))) B_1(\tau_1(z))}{E(X) \sum_{n=0}^{N-1} \xi_n [(q + pB_2(\tau_2(z))) B_1(\tau_1(z)) - z]} \quad (7.84)$$

where $\tau_i(z) = \lambda(1 - X(z)) + \alpha_i[1 - D_i(\lambda(1 - X(z))) G_{i,1}(\lambda(1 - X(z)))$]

and $\rho_1 = \lambda E(X)(B_1)(1 + \alpha_1(\gamma_1^{(i)} + g_{i,1}^{(i)}) + p\lambda E(X)(B_2)(1 + \alpha_2(\gamma_2^{(i)} + g_{2,1}^{(i)})].$

This result is same as obtained by Choudhury et al. (2009).

Case (ii) Single arrival queue with uniform arrival rate and without $N$–policy.

For $\lambda_0 = \lambda_1 = \lambda_2 = \lambda_3 = \lambda$ , $r_1 = r_1 = \ldots = r_{m-1} = 0$ and $X(z) = z$, $E(X) = 1$, $N = 1$; equation (7.49) provides

$$o(z) = \frac{(1 - \rho_1)(1 - z)(q + pB_2(\tau_2(z))) B_1(\tau_1(z))}{[(q + pB_2(\tau_2(z))) B_1(\tau_1(z)) - z]} \quad (7.85)$$

where $\tau_i(z) = \lambda(1 - z) + \alpha_i[1 - D_i(\lambda(1 - z)) G_{i,1}(\lambda(1 - z))$]

and $\rho_1 = \lambda E(B_1)(1 + \alpha_1(\gamma_1^{(i)} + g_{i,1}^{(i)}) + p\lambda E(B_2)(1 + \alpha_2(\gamma_2^{(i)} + g_{2,1}^{(i)}].$

The present model reduces to the model studied by Choudhury and Tadj (2009).

Case (iii) Single arrival queue with uniform arrival rate and reliable server.

By substituting $\lambda_0 = \lambda_1 = \lambda_2 = \lambda_3 = \lambda$ , $r_1 = r_1 = \ldots = r_{m-1} = 0$, $\alpha_1 = \alpha_2 = 0$ in equation (7.49), we get

$$o(z) = \frac{(1 - \rho_1) \sum_{n=0}^{N-1} \xi_n z^n (1 - X(z)) (q + pB_2(\tau_2(z))) B_1(\tau_1(z))}{E(X) \sum_{n=0}^{N-1} \xi_n [(q + pB_2(\tau_2(z))) B_1(\tau_1(z)) - z]} \quad (7.86)$$

where $\tau_i(z) = \lambda(1 - X(z))$, $\rho_1 = \lambda E(X)(B_1) + pE(B_2))$.

The model reduced in this case was studied by Choudhury and Paul (2004) for single optional service.
Case (iv) Bulk queue with state dependent arrival rate and compulsory repair.

By setting \( \lambda_0 = \lambda_1 \) and \( r_1 = r_2 = \cdots = r_{m-1} = 1, N = 1 \); equation (7.49) gives

\[
\omega(z) = \frac{(1 - \rho_1)(1 - X(z))(q + p\tilde{B}_2(\tau_2(z)))\tilde{B}_1(\tau_1(z))}{E(X)[(q + p\tilde{B}_2(\tau_2(z)))\tilde{B}_1(\tau_1(z)) - z]} \quad (7.87)
\]

where

\[
\tau_i(z) = \phi_i(z) + \alpha_i(1 - \bar{D}_i(\phi_i(z))\prod_{j=1}^{m} G_{i,j}(\phi_i(z))) \quad i = 1, 2
\]

and

\[
r_i = E(B_i)E(X)[\lambda_i + \alpha_i(\lambda_2\gamma_i^{(1)} + \lambda_3\sum_{j=1}^{m} g_{ij}^{(1)})] + pE(B_2)E(X)[\lambda_i + \alpha_i(\lambda_2\gamma_i^{(1)} + \lambda_3\sum_{j=1}^{m} g_{ij}^{(1)})].
\]

7.8 Numerical Illustration and Sensitivity Analysis

To facilitate numerical results, first of all we assume that the batch size of the units follows a geometric distribution with first and second moments given by equation (2.30). The distribution of essential and optional service time are taken as \( k \)-Erlangian, so that first and second moments of the service time distribution are obtained using

\[
E(B_i) = \frac{1}{\mu_i}, E(B_i^2) = \frac{k + 1}{k\mu_i^2}; \quad (7.88)
\]

where \( \mu_i \) denotes the \( i^{th} \) phase service rate.

The delay time distribution is assumed to be \( 2 \)-Erlangian distributed with parameters \( \gamma_i (i = 1, 2) \). The first and second moments of the delay time distribution are

\[
\gamma_i^{(1)} = 1, \gamma_i^{(2)} = \frac{3}{2\gamma_i^2}. \quad (7.89)
\]

Furthermore, we assume that the repair time follows an exponential distribution with parameter \( g_{ij} \) and has first two moments as

\[
g_{ij}^{(1)} = \frac{1}{g_{ij}}, g_{ij}^{(2)} = \frac{2}{g_{ij}}; \quad i = 1, 2; \quad j = 1, 2, \ldots, m. \quad (7.90)
\]

To develop the computer program, the coding is done in MATLAB. Now we display the numerical results in figures (7.1)-(7.4) and tables (7.1)-(7.6).

For figures (7.1)-(7.4), we set the default parameters as follows:
$E(X) = 3, N = 5, r_1 = 0.8, r_2 = 0, \mu_1 = \mu_2 = \mu = 8, \lambda_0 = 1.4\lambda, \lambda_1 = \lambda, \lambda_2 = 0.9\lambda, \lambda_3 = 0.7\lambda, \rho = 0.6$

In tables (7.1)-(7.5), we summarize the performance indices by setting the fixed values of default parameters as follows:

Table 7.1: $E(X) = 3, \mu_1 = \mu_2 = 8, N = 5, p = 0.6, \alpha = 0.1, \alpha_1 = \alpha, \alpha_2 = 0.8\alpha$

$\lambda = 3, \lambda_0 = 1.4\lambda, \lambda_1 = \lambda, \lambda_2 = 0.7\lambda, \lambda_3 = 0.9\lambda.$

Table 7.2: $E(X) = 3, \mu_1 = \mu_2 = 8, p = 0.6, \alpha = 0.1, \alpha_1 = \alpha, \alpha_2 = 0.8\alpha$

$\lambda = 3, \lambda_0 = 1.4\lambda, \lambda_1 = \lambda, \lambda_2 = 0.7\lambda, \lambda_3 = 0.9\lambda, r_1 = 0.8, r_2 = 0.$

Table 7.3: $E(X) = 3, \mu_1 = \mu_2 = 8, N = 5, p = 0.6, \alpha = 0.1, \alpha_1 = \alpha, \alpha_2 = 0.8\alpha$

$\lambda = 3, \lambda_0 = 1.4\lambda, \lambda_1 = \lambda, \lambda_2 = 0.7\lambda, \lambda_3 = 0.9\lambda, r_1 = 0.8, r_2 = 0.$

Table 7.4: $E(X) = 3, N = 5, p = 0.6, \alpha = 0.1, \alpha_1 = \alpha, \alpha_2 = 0.8\alpha, r_1 = 0.8, r_2 = 0, k = 3$

$\lambda_0 = 1.4\lambda, \lambda_1 = \lambda, \lambda_2 = 0.7\lambda, \lambda_3 = 0.9\lambda, \lambda = 1.3.$

Table 7.5: $E(X) = 3, \mu_1 = \mu_2 = 8, N = 5, \alpha = 0.1, \alpha_1 = \alpha, \alpha_2 = 0.8\alpha, r_1 = 0.8, r_2 = 0, k = 3$

$\lambda_0 = 1.4\lambda, \lambda_1 = \lambda, \lambda_2 = 0.7\lambda, \lambda_3 = 0.9\lambda, \lambda = 1.3.$

Table 7.6(a): $E(X) = 3, r_1 = r_2 = 0, \mu_1 = \mu_2 = 8, \alpha_1 = 0.1, \alpha_2 = 0.8\alpha_1, \lambda_0 = 1.4\lambda, \lambda_1 = \lambda,$

$\lambda_2 = 0.7\lambda, \lambda_3 = 0.9\lambda.$

Table 7.6(b): $E(X) = 3, r_1 = r_2 = 0, \lambda = 1.55, \lambda_0 = 1.4\lambda, \lambda_1 = \lambda, \lambda_2 = 0.7\lambda, \lambda_3 = 0.9\lambda,$

$\alpha_1 = 0.1, \alpha_2 = 0.8\alpha_1.$

Table 7.6(c): $E(X) = 3, r_1 = r_2 = 0, \mu_1 = \mu_2 = 8, \lambda = 1.5, \lambda_0 = 1.4\lambda, \lambda_1 = \lambda, \lambda_2 = 0.7\lambda,$

$\lambda_3 = 0.9\lambda.$

Table 7.1 displays the effect of arrival rate on the average number of units for the different repair phases; we observe that when the arrival rate $\lambda$ increases, the queue length ($L_s$) also increases. Table 7.2 exhibits the effect of arrival rate ($\lambda$) and $N$ on the $L_s$ and $W_s$. It is noticed that as $N$ increases, the $L_s$ and $W_s$ increase. Tables 7.3 and 7.4 summarize the effect of arrival rate, failure rate and service rates on the long run probabilities of the server states. From table 7.3, we conclude that as $\alpha$ increases, the long run probabilities $P(I), P(B_1)$ and $P(B_2)$ decrease while $P(D_1), P(D_2), P(R^{(1)})$ and $P(R^{(2)})$ increase. From table 7.4, we see that as $\mu_1$ increases, $P(I)$ increases but $P(B_1), P(D_1), P(R^{(1)})$ decrease. Further, it is noticed that $P(B_2), P(D_2), P(R^{(2)})$ remain almost constant. Also as $\mu_2$ increases, $P(B_2), P(D_2), P(R^{(2)})$ decrease but
$P(B_i), P(D_i), P(R(i))$ remain almost constant. Table 7.5 displays the effect of failure rate and optional probability on the steady state availability ($A_i$) and failure frequency($F_j$). We see that for fixed values of $\alpha_1(\alpha_2)$, the values of $A_i(F_j)$ decreases (increases) when $\alpha_1(\alpha_2)$ increases. Finally, in table 7.6(a-c) we display the approximate average waiting time ($W^*_x$) using the maximum entropy principle and to facilitate a comparison between the average waiting time ($W_x$) obtained analytically and approximate average waiting time ($W^*_x$).

In figure 7.1, we examine the effect of arrival rates on the average number of units ($L_x$) by increasing the value of $\lambda$. It is found that initially $L_x$ increases gradually and then after sharply. Furthermore, as the optional probability($p$) increases, the $L_x$ increases. Figure 7.2 shows the effect of service rate on the average queue length for different failure rates; a significant increase in the average queue length with the increase in the failure rate is found. Figure 7.3 exhibits the effect of the number of phases of service and arrival rate on the average queue length. It is found that with the increase in the number of phases of repair, the average queue length goes on increasing; the increasing trend is more prominent for higher values of $\lambda$. Figure 7.4 shows the effect of optional probability ($p$) on the average queue length; the increasing trend of $L_x$ is quite significant for higher value.

**Conclusion**

The stochastic model with unreliable server, wherein the failed server may undergo for $m$ phases repair as per requirement, is investigated. The explicit formulae obtained for various measures can be treated as performance evaluation tools which may be prepared easily for congestion situations arising in many practical applications. The present model plays an important role in depicting the practical blocking and delay scenarios encountered in flexible manufacturing/production process, computer and communication system, etc. Bulk queueing models with unreliable server are widely used to study the behaviour of various types of telecommunication and mainframe systems wherein a group of jobs joins the system to get the service. Sometimes exact results are too complicated to be impracticable for use; in such cases maximum entropy principle used to obtain approximate probability distributions facilitates a simple
approach for determining the probability distribution of all possible states. The numerical results obtained reveal that our study may be helpful to control the parameters based on sensitivity analysis
Table 7.1: Effects of $\lambda$ and $k$ on $L_s$ for different repair phases

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$k = 1$</th>
<th>$k = 3$</th>
<th>$k \to \infty$</th>
<th>$k = 1$</th>
<th>$k = 3$</th>
<th>$k \to \infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.20</td>
<td>45.01</td>
<td>43.84</td>
<td>43.26</td>
<td>45.61</td>
<td>44.42</td>
<td>43.82</td>
</tr>
<tr>
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<td>55.69</td>
<td>54.11</td>
<td>53.32</td>
<td>56.56</td>
<td>54.95</td>
<td>54.14</td>
</tr>
<tr>
<td>1.30</td>
<td>70.99</td>
<td>68.79</td>
<td>67.70</td>
<td>72.34</td>
<td>70.08</td>
<td>68.95</td>
</tr>
<tr>
<td>1.35</td>
<td>94.11</td>
<td>90.94</td>
<td>89.35</td>
<td>96.28</td>
<td>93.02</td>
<td>91.38</td>
</tr>
<tr>
<td>1.40</td>
<td>131.53</td>
<td>126.72</td>
<td>124.31</td>
<td>135.35</td>
<td>130.36</td>
<td>127.87</td>
</tr>
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</table>

Table 7.2: Effects of $\lambda$ and $N$ on $L_s(W_s)$

<table>
<thead>
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<th>$\lambda$</th>
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<th>$k = 1$</th>
<th>$k = 3$</th>
<th>Uniform arrival rates</th>
</tr>
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<tbody>
<tr>
<td>L_s</td>
<td>W_s</td>
<td>L_s</td>
<td>W_s</td>
<td>$L_s$</td>
</tr>
<tr>
<td>1.20</td>
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<td>9.85</td>
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<td>48.31</td>
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<td>64.83</td>
<td>15.62</td>
<td>62.58</td>
<td>15.08</td>
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<td>87.63</td>
<td>20.52</td>
<td>84.36</td>
<td>19.76</td>
</tr>
<tr>
<td>1.40</td>
<td>125.09</td>
<td>28.51</td>
<td>120.11</td>
<td>27.37</td>
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<table>
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<th>$k = 1$</th>
<th>$k = 3$</th>
<th>Uniform arrival rates</th>
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<td>L_s</td>
<td>W_s</td>
<td>$L_s$</td>
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<tr>
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<td>13.64</td>
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<th>Uniform arrival rates</th>
</tr>
</thead>
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<td>L_s</td>
<td>W_s</td>
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</tr>
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</table>
Table 7.3: Effects of $\lambda$ and $\alpha$ on the long run probabilities of the server states

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<th>$\lambda$</th>
<th>$\alpha$</th>
<th>$P(I)$</th>
<th>$P(B_1)$</th>
<th>$P(B_2)$</th>
<th>$P(D_1)$</th>
<th>$P(D_2)$</th>
<th>$P(R^{(1)})$</th>
<th>$P(R^{(2)})$</th>
</tr>
</thead>
<tbody>
<tr>
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<td>0.1</td>
<td>0.2126</td>
<td>0.4878</td>
<td>0.2927</td>
<td>0.0010</td>
<td>0.0005</td>
<td>0.0037</td>
<td>0.0018</td>
</tr>
<tr>
<td></td>
<td>0.2</td>
<td>0.2078</td>
<td>0.4865</td>
<td>0.2919</td>
<td>0.0020</td>
<td>0.0010</td>
<td>0.0073</td>
<td>0.0035</td>
</tr>
<tr>
<td></td>
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<td>0.4852</td>
<td>0.2911</td>
<td>0.0030</td>
<td>0.0015</td>
<td>0.0109</td>
<td>0.0052</td>
</tr>
<tr>
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<td>0.4</td>
<td>0.1983</td>
<td>0.4839</td>
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<td>0.0040</td>
<td>0.0019</td>
<td>0.0145</td>
<td>0.0070</td>
</tr>
<tr>
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<td>0.4826</td>
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<td>0.0087</td>
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<td>0.1627</td>
<td>0.5187</td>
<td>0.3112</td>
<td>0.0011</td>
<td>0.0005</td>
<td>0.0039</td>
<td>0.0019</td>
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<tr>
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<td>0.1578</td>
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<td>0.3103</td>
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<td>0.0010</td>
<td>0.0078</td>
<td>0.0037</td>
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<td>0.0056</td>
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<td>0.3086</td>
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<td>0.0154</td>
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<td>0.5129</td>
<td>0.3077</td>
<td>0.0053</td>
<td>0.0026</td>
<td>0.0192</td>
<td>0.0092</td>
</tr>
<tr>
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<td>0.1</td>
<td>0.1146</td>
<td>0.5485</td>
<td>0.3291</td>
<td>0.0011</td>
<td>0.0005</td>
<td>0.0041</td>
<td>0.0020</td>
</tr>
<tr>
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<td>0.2</td>
<td>0.1095</td>
<td>0.5468</td>
<td>0.3281</td>
<td>0.0023</td>
<td>0.0011</td>
<td>0.0082</td>
<td>0.0039</td>
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<td>0.5452</td>
<td>0.3271</td>
<td>0.0034</td>
<td>0.0016</td>
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<td>0.0078</td>
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Table 7.4: Effects of $\mu_i$ and $\mu_z$ on the long run probabilities of server states

<table>
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<tr>
<th>$(\mu_1, \mu_2)$</th>
<th>$P(I)$</th>
<th>$P(B_1)$</th>
<th>$P(B_2)$</th>
<th>$P(D_1)$</th>
<th>$P(D_2)$</th>
<th>$P(R^{(1)})$</th>
<th>$P(R^{(2)})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(8,0.8,0)</td>
<td>0.1627</td>
<td>0.5187</td>
<td>0.3112</td>
<td>0.0011</td>
<td>0.0005</td>
<td>0.0039</td>
<td>0.0019</td>
</tr>
<tr>
<td>(8,5,8,0)</td>
<td>0.1865</td>
<td>0.4926</td>
<td>0.3140</td>
<td>0.0010</td>
<td>0.0005</td>
<td>0.0035</td>
<td>0.0019</td>
</tr>
<tr>
<td>(9,0,8,0)</td>
<td>0.2080</td>
<td>0.4690</td>
<td>0.3166</td>
<td>0.0009</td>
<td>0.0005</td>
<td>0.0031</td>
<td>0.0019</td>
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<tr>
<td>(9,5,8,0)</td>
<td>0.2275</td>
<td>0.4475</td>
<td>0.3189</td>
<td>0.0008</td>
<td>0.0005</td>
<td>0.0028</td>
<td>0.0019</td>
</tr>
<tr>
<td>(10,8,0)</td>
<td>0.2453</td>
<td>0.4280</td>
<td>0.3210</td>
<td>0.0007</td>
<td>0.0005</td>
<td>0.0026</td>
<td>0.0019</td>
</tr>
<tr>
<td>(8,0,8,0)</td>
<td>0.1627</td>
<td>0.5187</td>
<td>0.3112</td>
<td>0.0011</td>
<td>0.0005</td>
<td>0.0039</td>
<td>0.0019</td>
</tr>
<tr>
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<td>0.2945</td>
<td>0.0011</td>
<td>0.0005</td>
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<td>0.0011</td>
<td>0.0004</td>
<td>0.0039</td>
<td>0.0015</td>
</tr>
<tr>
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<td>0.0004</td>
<td>0.0039</td>
<td>0.0013</td>
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<tr>
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<td>0.2115</td>
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<td>0.0003</td>
<td>0.0040</td>
<td>0.0012</td>
</tr>
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Table 7.5: Effects of $(\alpha_1, \alpha_2)$ and $p$ on $A_v$ and $F_f$

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<tr>
<th>$(\alpha_1, \alpha_2)$</th>
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<th>$p = 0.5$</th>
<th>$p = 0.7$</th>
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<tr>
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<td>$A_v$</td>
<td>$F_f$</td>
<td>$A_v$</td>
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<td>(0.1,0.4)</td>
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<td>0.1193</td>
<td>0.9849</td>
</tr>
<tr>
<td>(0.2,0.4)</td>
<td>0.9834</td>
<td>0.1731</td>
<td>0.9799</td>
</tr>
<tr>
<td>(0.3,0.4)</td>
<td>0.9783</td>
<td>0.2268</td>
<td>0.9749</td>
</tr>
<tr>
<td>(0.4,0.4)</td>
<td>0.9732</td>
<td>0.2802</td>
<td>0.9700</td>
</tr>
<tr>
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<td>0.9651</td>
</tr>
<tr>
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<td>0.2321</td>
<td>0.9774</td>
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<tr>
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<td>0.9749</td>
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<tr>
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<td>0.2642</td>
<td>0.9725</td>
</tr>
<tr>
<td>(0.4,0.4)</td>
<td>0.9732</td>
<td>0.2802</td>
<td>0.9700</td>
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<tr>
<td>(0.4,0.5)</td>
<td>0.9716</td>
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Table 7.6 (a): Effects of $\lambda$ and $N$ on $W_s$ and $W_s^*$

<table>
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<th>$W_s$</th>
<th>$W_s^*$</th>
<th>error(%)</th>
<th>$W_s$</th>
<th>$W_s^*$</th>
<th>error(%)</th>
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<td>20.26</td>
<td>11.44</td>
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<td>25.32</td>
<td>9.45</td>
<td>30.84</td>
<td>27.82</td>
<td>9.82</td>
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<td>38.08</td>
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</tbody>
</table>

Table 7.6 (b): Effects of $(\mu_1,\mu_2)$ and $N$ on $W_s$ and $W_s^*$

<table>
<thead>
<tr>
<th>$(\mu_1,\mu_2)$</th>
<th>$W_s$</th>
<th>$W_s^*$</th>
<th>error(%)</th>
<th>$W_s$</th>
<th>$W_s^*$</th>
<th>error(%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(8.0,8.0)</td>
<td>140.93</td>
<td>134.22</td>
<td>4.76</td>
<td>147.10</td>
<td>140.06</td>
<td>4.79</td>
</tr>
<tr>
<td>(8.5,8.0)</td>
<td>63.06</td>
<td>58.85</td>
<td>6.68</td>
<td>67.24</td>
<td>62.68</td>
<td>6.79</td>
</tr>
<tr>
<td>(9.0,8.0)</td>
<td>38.22</td>
<td>35.04</td>
<td>8.31</td>
<td>41.51</td>
<td>37.97</td>
<td>8.52</td>
</tr>
<tr>
<td>(9.5,8.0)</td>
<td>26.84</td>
<td>24.23</td>
<td>9.74</td>
<td>29.63</td>
<td>26.64</td>
<td>10.09</td>
</tr>
<tr>
<td>(10,8.0)</td>
<td>20.55</td>
<td>18.29</td>
<td>11.02</td>
<td>23.02</td>
<td>20.37</td>
<td>11.52</td>
</tr>
<tr>
<td>(8.0,8.0)</td>
<td>140.93</td>
<td>134.22</td>
<td>4.76</td>
<td>147.10</td>
<td>140.06</td>
<td>4.79</td>
</tr>
<tr>
<td>(8.0,8.5)</td>
<td>79.04</td>
<td>74.39</td>
<td>5.88</td>
<td>83.58</td>
<td>78.60</td>
<td>5.95</td>
</tr>
<tr>
<td>(8.0,9.0)</td>
<td>52.21</td>
<td>48.66</td>
<td>6.79</td>
<td>55.84</td>
<td>51.97</td>
<td>6.92</td>
</tr>
<tr>
<td>(8.0,9.5)</td>
<td>37.96</td>
<td>35.09</td>
<td>7.55</td>
<td>41.01</td>
<td>37.83</td>
<td>7.75</td>
</tr>
<tr>
<td>(8.0,10)</td>
<td>29.38</td>
<td>26.98</td>
<td>8.18</td>
<td>32.03</td>
<td>29.32</td>
<td>8.46</td>
</tr>
</tbody>
</table>

Table 7.6 (c): Effects of $(\alpha_1,\alpha_2)$ and $N$ on $W_s$ and $W_s^*$

<table>
<thead>
<tr>
<th>$(\alpha_1,\alpha_2)$</th>
<th>$W_s$</th>
<th>$W_s^*$</th>
<th>error(%)</th>
<th>$W_s$</th>
<th>$W_s^*$</th>
<th>error(%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0.10,0.10)</td>
<td>69.39</td>
<td>64.91</td>
<td>6.45</td>
<td>73.78</td>
<td>68.95</td>
<td>6.55</td>
</tr>
<tr>
<td>(0.15,0.10)</td>
<td>71.47</td>
<td>66.79</td>
<td>6.55</td>
<td>75.92</td>
<td>70.88</td>
<td>6.64</td>
</tr>
<tr>
<td>(0.20,0.10)</td>
<td>73.65</td>
<td>68.75</td>
<td>6.65</td>
<td>78.17</td>
<td>72.90</td>
<td>6.74</td>
</tr>
<tr>
<td>(0.25,0.10)</td>
<td>75.93</td>
<td>70.81</td>
<td>6.75</td>
<td>80.52</td>
<td>75.02</td>
<td>6.83</td>
</tr>
<tr>
<td>(0.30,0.10)</td>
<td>78.33</td>
<td>72.96</td>
<td>6.85</td>
<td>82.99</td>
<td>77.24</td>
<td>6.93</td>
</tr>
<tr>
<td>(0.10,0.10)</td>
<td>69.39</td>
<td>64.91</td>
<td>6.45</td>
<td>73.78</td>
<td>68.95</td>
<td>6.55</td>
</tr>
<tr>
<td>(0.10,0.15)</td>
<td>70.79</td>
<td>66.18</td>
<td>6.52</td>
<td>75.23</td>
<td>70.26</td>
<td>6.61</td>
</tr>
<tr>
<td>(0.10,0.20)</td>
<td>72.24</td>
<td>67.48</td>
<td>6.58</td>
<td>76.72</td>
<td>71.61</td>
<td>6.67</td>
</tr>
<tr>
<td>(0.10,0.25)</td>
<td>73.72</td>
<td>68.83</td>
<td>6.64</td>
<td>78.26</td>
<td>73.00</td>
<td>6.73</td>
</tr>
<tr>
<td>(0.10,0.30)</td>
<td>75.25</td>
<td>70.21</td>
<td>6.70</td>
<td>79.84</td>
<td>74.42</td>
<td>6.79</td>
</tr>
</tbody>
</table>
Fig. 7.1: $L_s$ vs. $\lambda$ for different optional probabilities

Fig. 7.2: $L_s$ vs. $\mu$ for different failure rates

Fig. 7.3: $L_s$ vs. $\lambda$ for different service phases

Fig. 7.4: $L_s$ vs. $p$ for different failure rates
Appendix-VI

Appendix-VI.A

Proof of theorem 7.1:

By taking the limit \( z \to 1 \) in equations (7.28), (7.25), (7.29), (7.30) and (7.31) for \( i = 1,2 \); we get

\[
P^{(1)}(0,1) = \frac{\lambda_0 E(X) \sum_{n=0}^{N-1} P_n^{(0)}}{1 - \rho_1}
\]

(A.7.1)

\[
P^{(1)}(x,1) = P^{(1)}(0,1)[1 - B_1(x)]
\]

(A.7.2)

\[
P^{(2)}(x,1) = pP^{(1)}(0,1)[1 - B_2(x)]
\]

(A.7.3)

\[
B^{(1)}(x,y,1) = \alpha_1 P^{(1)}(0,1)[1 - B_1(x)][1 - D_1(y)]
\]

(A.7.4)

\[
B^{(2)}(x,y,1) = \alpha_2 P^{(1)}(0,1)[1 - B_2(x)][1 - D_2(y)]
\]

(A.7.5)

\[
R^{(1)}_i(x,y,1) = \alpha_1 P^{(1)}(0,1)[1 - B_1(x)][1 - G_{1,i}(y)]; \quad 2 \leq i \leq m
\]

(A.7.6)

\[
R^{(2)}_i(x,y,1) = \alpha_2 P^{(1)}(0,1)[1 - B_2(x)][1 - G_{2,i}(y)]; \quad 2 \leq i \leq m.
\]

(A.7.7)

(A.7.8)

The \( P^{(0)}_n \), \( 0 \leq n \leq N - 1 \) satisfies the following relation

\[
P^{(0)}_n = C_0 \xi_n \quad n = 0,1,...,N - 1
\]

(A.7.9)

where \( \xi_n \) is given by (7.7) and \( C_0 \) is constant.

Then \( P^{(0)}_N(z) = C_0 \sum_{n=0}^{N-1} \xi_n z^n \).

(A.7.10)

(A.7.11)

Using equations (A.7.2)-(A.7.9) and (A.7.11) in normalising condition (7.19), we have

\[
C_0 = \left( 1 - \frac{\rho_2}{1 - \rho_1 + \rho_2} \right) \frac{1}{\sum_{n=0}^{N-1} \xi_n}
\]

(A.7.12)
Utilizing the result of equation (A.7.12) in equation (A.7.11), we get equation (7.32).
From equations (7.28) and (7.32), we have

\[ P^{(1)}(0, z) = \frac{z \phi_0(z)(1 - \rho) \sum_{n=0}^{\infty} \xi_n z^n}{\sum_{n=0}^{\infty} \xi_n [(q + p \tilde{B}_2(\tau_2(z)))\tilde{B}_1(\tau_1(z)) - z]} \]  \hspace{1cm} (A.7.13)

Using the result of equation (A.7.13) in equations (7.27), (7.25), (7.29), (7.30) and (7.31), for \( i = 1, 2 \) we get the equations (7.33)-(7.40).

Appendix-VI.B

Proof of theorem 7.2:
Integrating the equations (7.33)-(7.34) with respect to \( x \) with limit 0 to \( \infty \) and using the result

\[ \int_0^\infty e^{-sx}(1 - M(x))dx = \frac{1 - \bar{M}(s)}{s}, \]  \hspace{1cm} (B.7.1)

we get the required results as given in equations (7.41) and (7.42). Similarly, on integrating the equations (7.35)-(7.40) with respect to \( y \) with limit 0 to \( \infty \) and using the equation (B.7.1) and repeating the same procedure of integration with variable \( x \), we have the required results as given in equations (7.43)-(7.48).

Appendix-VI.C

Proof of Theorem 7.3:
To obtain the queue size distribution at the departure epoch (cf. Wolff, 1982; Choudhury et al., 2009), we have

\[ \omega_j = k_0 \left\{ q \int_0^\infty \mu_1(x)P_{j+1}^{(1)}(x)dx + \int_0^\infty \mu_2(x)P_{j+1}^{(2)}(x)dx \right\} \]  \hspace{1cm} (C.7.1)

where

\( k_0 \) is the normalising constant and \( \{\omega_j; j = 0, 1, 2, \ldots\} \) as the probability that there are \( j \) units in the queue at a departure epoch.

Multiplying equation (C.7.1) by \( z^j \) and using \( \omega(z) = \sum_{j=0}^\infty \omega_j z^j \) and after simplification, we get
\[ \omega(z) = k_0 P_n^{(0)}(z) \phi_0(z) \frac{[(q + p \bar{B}(\tau_2(z))) \bar{B}_1(\tau_1(z))]}{[(q + p \bar{B}(\tau_2(z))) \bar{B}_1(\tau_1(z)) - z]} . \]  
\hspace{1cm} \text{(C.7.2)}

Utilizing the normalizing condition \( \omega(1) = 1 \), we get

\[ k_0 = \frac{1 - \rho_i}{\lambda_0 E(X) \sum_{n=0}^{N-1} \xi_n} . \]  
\hspace{1cm} \text{(C.7.3)}

Using equations (C.7.3) and (C.7.2), we get result given in equation (7.49).

**Appendix-VI.D**

**Proof of Theorem 7.4:**

The entropy function is

\[ Z = \sum_{n=0}^{N-1} P_n^{(0)} \log P_n^{(0)} - \sum_{i=1}^{\infty} \sum_{n=1}^{N-1} P_n^{(i)} \log P_n^{(i)} - \sum_{i=1}^{\infty} \sum_{n=1}^{N-1} B_n^{(i)} \log B_n^{(i)} - \sum_{i=1}^{\infty} \sum_{n=1}^{N-1} R_n^{(i)} \log R_n^{(i)} \]

\[ - \alpha_1 (\sum_{n=0}^{N-1} P_n^{(0)} + \sum_{i=1}^{\infty} \sum_{n=1}^{N-1} P_n^{(i)} + \sum_{i=1}^{\infty} \sum_{n=1}^{N-1} B_n^{(i)} + \sum_{i=1}^{\infty} \sum_{n=1}^{N-1} R_n^{(i)}) - 1 - \alpha_2 (\sum_{n=1}^{\infty} P_n^{(1)} - \eta_1) \]

\[ - \alpha_3 (\sum_{n=1}^{\infty} B_n^{(2)} - \eta_2) - \alpha_4 (\sum_{n=1}^{\infty} B_n^{(3)} - \eta_3) - \alpha_5 (\sum_{n=1}^{\infty} B_n^{(4)} - \eta_4) - \alpha_6 (\sum_{n=1}^{\infty} R_n^{(5)} - \eta_5) \]

\[ - \gamma N \sum_{n=0}^{N-1} n P_n^{(0)} - \sum_{n=1}^{\infty} n P_n^{(1)} + \sum_{n=1}^{\infty} n P_n^{(2)} + \sum_{n=1}^{\infty} n B_n^{(1)} + \sum_{n=1}^{\infty} n B_n^{(2)} + \sum_{n=1}^{\infty} n R_n^{(1)} - L_s \]

\hspace{1cm} \text{(D.7.1)}

where

\( \alpha_i \)'s (1 \leq i \leq 7) and \( \gamma \) are the Langangian multipliers corresponding to equation (7.67)-(7.74), respectively. On differentiating partially equation (D.7.1) with respect to \( P_n^{(0)}, P_n^{(i)}, B_n^{(i)} \) and \( R_n^{(i)} \) for \( i = 1, 2 \) and setting equal to zero, we get

\[ - (1 + \log P_n^{(0)}) - \alpha_1 - n \gamma = 0 \quad \Rightarrow \quad P_n^{(0)} = e^{-(1+\alpha_1)} (e^{-\gamma})^n \quad 0 \leq n \leq N-1 \]  
\hspace{1cm} \text{(D.7.2)}

\[ - (1 + \log P_n^{(1)}) - \alpha_1 - \alpha_2 - n \gamma = 0 \quad \Rightarrow \quad P_n^{(1)} = e^{-(1+\alpha_1)} e^{-\alpha_2} (e^{-\gamma})^n \quad n = 1, 2, 3, \ldots \]  
\hspace{1cm} \text{(D.7.3)}

\[ - (1 + \log P_n^{(2)}) - \alpha_1 - \alpha_3 - n \gamma = 0 \quad \Rightarrow \quad P_n^{(2)} = e^{-(1+\alpha_1)} e^{-\alpha_3} (e^{-\gamma})^n \quad n = 1, 2, 3, \ldots \]  
\hspace{1cm} \text{(D.7.4)}

\[ - (1 + \log B_n^{(1)}) - \alpha_1 - \alpha_4 - n \gamma = 0 \quad \Rightarrow \quad B_n^{(1)} = e^{-(1+\alpha_1)} e^{-\alpha_4} (e^{-\gamma})^n \quad n = 1, 2, 3, \ldots \]  
\hspace{1cm} \text{(D.7.5)}
\[-(1 + \log B_n^{(2)}) - \alpha_1 - \alpha_5 - n\gamma = 0 \quad \Rightarrow \quad B_n^{(2)} = e^{-(1+\alpha_1)} e^{-\alpha_5 (e^{-\gamma})^n} \quad n = 1,2,3,\ldots \quad (D.7.6)\]

\[-(1 + \log R_n^{(1)}) - \alpha_1 - \alpha_6 - n\gamma = 0 \quad \Rightarrow \quad R_n^{(1)} = e^{-(1+\alpha_1)} e^{-\alpha_6 (e^{-\gamma})^n} \quad n = 1,2,3,\ldots \quad (D.7.7)\]

\[-(1 + \log R_n^{(2)}) - \alpha_1 - \alpha_7 - n\gamma = 0 \quad \Rightarrow \quad R_n^{(2)} = e^{-(1+\alpha_1)} e^{-\alpha_7 (e^{-\gamma})^n} \quad n = 1,2,3,\ldots \quad (D.7.8)\]

Let us denote
\[
\tau_1 = e^{-(1+\alpha_1)}, \quad \tau_2 = e^{-\alpha_2}, \quad \tau_3 = e^{-\alpha_3}, \quad \tau_4 = e^{-\alpha_4}, \quad \tau_5 = e^{-\alpha_5}, \quad \tau_6 = e^{-\alpha_6}, \quad \tau_7 = e^{-\alpha_7}, \quad \tau_8 = e^{-\gamma}.
\]

Now utilizing above notations in equations (D.7.2)-(D.7.8), we get
\[
P_n^{(0)} = \tau_1 \tau_8^n, \quad (D.7.9)
\]

\[
P_n^{(1)} = \tau_1 \tau_2 \tau_8^n, \quad P_n^{(2)} = \tau_1 \tau_3 \tau_8^n, \quad B_n^{(1)} = \tau_1 \tau_4 \tau_8^n, \quad B_n^{(2)} = \tau_1 \tau_5 \tau_8^n, \quad R_n^{(1)} = \tau_1 \tau_6 \tau_8^n, \quad R_n^{(2)} = \tau_1 \tau_7 \tau_8^n.
\]

Using above values in equations (7.68)-(7.73) we get
\[
\tau_1 \tau_2 = \eta_1 (1 - \tau_8), \quad \tau_1 \tau_3 = \eta_2 (1 - \tau_8), \quad \tau_1 \tau_4 = \eta_3 (1 - \tau_8), \quad \tau_1 \tau_5 = \eta_4 (1 - \tau_8), \quad \tau_1 \tau_6 = \eta_5 (1 - \tau_8), \quad \tau_1 \tau_7 = \eta_6 (1 - \tau_8).
\]

Using equations (7.68)-(7.73) and (D.7.9) in (7.67), we get
\[
\tau_1 = \frac{(1 - \eta)(1 - \tau_8)}{(1 - \tau_8^N)}, \quad (D.7.12)
\]

where
\[
\eta = \sum_{i=1}^{6} \eta_i.
\]

Further using (D.7.9) and (D.7.10) in equation (7.74), we get
\[
\tau_1 \tau_8^N \left[ 1 - N(\tau_8) N^{-1} + (N - 1)(\tau_8)^N + \tau_2 + \tau_3 + \tau_4 + \tau_5 + \tau_6 + \tau_7 \right] = L_s. \quad (D.7.13)
\]

Now equations (D.7.11)-(D.7.13) yield
\[
\tau_1 = \frac{(1 - \eta)(1 - \tau_0)}{(1 - \tau_0^N)}, \quad \tau_2 = \frac{\eta_1 (1 - \tau_0^N)}{(1 - \eta) \tau_0}, \quad (D.7.14)
\]

\[
\tau_3 = \frac{\eta_2 (1 - \tau_0^N)}{(1 - \eta) \tau_0}, \quad \tau_4 = \frac{\eta_3 (1 - \tau_0^N)}{(1 - \eta) \tau_0}, \quad (D.7.15)
\]

\[
\tau_5 = \frac{\eta_4 (1 - \tau_0^N)}{(1 - \eta) \tau_0}, \quad \tau_6 = \frac{\eta_5 (1 - \tau_0^N)}{(1 - \eta) \tau_0}, \quad (D.7.16)
\]
\[ \tau_7 = \frac{\eta_s(1 - \tau_0^N)}{(1 - \eta)\tau_0} \]  

(D.7.17)

where

\( \tau_0 \) is one of the real roots with \(|\tau_0| < 1\) of the equation

\[ [(1 - \eta)(N-1) - L_s]e^{N+1} - [(N(1-\eta) + \eta - L_s)e^N + [(1-\eta) + L_s]e + (\eta - L_s) = 0. \]  

(D.7.18)

From equations (D.7.14)-(D.7.17), (D.7.9) and (D.7.10), we get the required results as given in equations (7.75)-(7.81).