Chapter 4

Bulk Queue with Second Optional Service

4.1 Introduction

Queueing systems with the provision of second optional service, which is rendered to the units after the first essential service is completed, has played an important role in the modelling of numerous congestion problems of real life activities. To illustrate, we cite the educational institute where a group of students may join the institution to complete their basic courses but only some of them may be interested to take admission in further advance courses. At the doctor’s clinic, all patients who have serious diseases arrive to consult the doctor regarding their health problems, however only some of them admit for further treatment. In many industrial congestion situations, the arriving units may also require the optional services along with essential services. The queueing models with state dependent rate and bulk arrival have been frequently used by many researchers to predict the performance indices of manufacturing systems, computer networks, production systems, etc.

Madan (2000) considered a model in which he has assumed that the units arrive one by one with the homogeneous arrival rate. He obtained the probability generating functions for queue length distribution with the provision of second optional service available in the system. In manufacturing systems or automobile repair service station, the customers may join the service station to get the essential repair of their vehicles; however due to shortage of time or high cost, only a few of them would like to replace the spare part or to get some additional service available in the station. The customers may also join the service station in a batch and the arrival rate of customers may differ with the status of the service station, if it is busy in rendering essential service or optional service to other customers. These facts motivate us to develop a model dealing with queueing situation wherein the units arrive in bulk with state dependent arrival rates and may demand for optional secondary service apart from essential service.

In this chapter, we extend the model of Madan (2000) for the situation wherein the units arrive in bulk with different arrival rates depending upon server status. By incorporating the bulk arrivals and heterogeneous arrival rates of the units, present model depicts more versatile queueing problem of many real life systems. It is assumed
that the arrival rates may be different in idle state and operating state when the system is busy in essential or optional service. The rest of the paper is structured as follows: By stating the requisite assumptions and notations, the mathematical model is described in section 4.2. In section 4.3, the set of governing equations along with boundary conditions are constructed by introducing the supplementary variables corresponding to elapsed service time of the essential service. In section 4.4, we analyse the model to study the transient and steady state behaviour by using probability generating functions. The performance measures of the model are obtained in section 4.5. The maximum entropy principle to find the queue size distribution is discussed in section 4.6. Some special cases are discussed in section 4.7. In section 4.8, the numerical illustration and sensitivity analysis are presented to examine the effect of system parameters on various performance indices.

4.2 Model Description

The units arrive in batches of random size $X$ according to the Poisson process with different rates $\lambda_1$ or $\lambda_2 (\lambda_3)$ depending upon the status of the system which may be in idle state or busy in providing essential service (optional service). First essential service is compulsory to all arriving units; as soon as essential service is completed either units remain in the system to get optional service immediately with probability $p$ or may leave the system with probability $(1 - p)$.

The inter-arrival time and service time of essential and optional services are assumed to be independent. The service time of first essential service is general distributed with distribution function and density function $B_1(v)$ and $b_1(v)$, respectively. The second optional service follows the exponential distribution with mean service time $1/\mu_2 (\mu_2 > 0)$. It is assumed that there are $n \geq 0$ units in the queue excluding one unit being in the service. It is further assumed that initially there is no unit in the system and the server is in idle state. The conditional probability of completion of first essential service is $\mu_1(x) dx$ with elapsed time $x$.

$$\mu_1(x) = \frac{b_1(x)}{1 - B_1(x)}$$  \hspace{1cm} (4.1)

$$b_1(x) = \mu_1(x)e^{-\int_0^x \mu_1(t) dt}$$  \hspace{1cm} (4.2)
The Laplace transform of a function \( f(t) \) is defined as
\[
\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) \, dt, \quad \text{Re}(s) > 0.
\]

### 4.3 Governing Equations

In this chapter, we investigate a non-markovian \( M^X/G/1 \) model by assuming that the service time distribution is general distributed. For analyzing the non-markovian queueing system, the supplementary variable technique is used to find the queue length distribution. The non-markovian queue can be transformed into markovian one by introducing the supplementary variable corresponding to the elapsed service time or remaining service time. By considering the elapsed service time as supplementary variable, we construct the Chapman-Kolmogorov equations governing the model (cf. Gross and Harris, 1985):

1. \[
\frac{\partial}{\partial x} P_n^{(1)}(x,t) + \frac{\partial}{\partial t} P_n^{(1)}(x,t) + (\lambda_2 + \mu_1)(x)P_n^{(1)}(x,t) = \lambda_2 \sum_{j=1}^{n} c_j P_{n-j}^{(1)}(x,t); \quad n \geq 1
\]

2. \[
\frac{\partial}{\partial x} P_0^{(1)}(x,t) + \frac{\partial}{\partial t} P_0^{(1)}(x,t) + (\lambda_2 + \mu_1)(x)P_0^{(1)}(x,t) = 0
\]

3. \[
\frac{d}{dt} P_n^{(2)}(t) + (\lambda_3 + \mu_2)P_n^{(2)}(t) = \lambda_3 \sum_{j=1}^{n} c_j P_{n-j}^{(2)}(t) + p \int_0^\infty P_n^{(1)}(x,t)\mu_1(x)dx; \quad n \geq 1
\]

4. \[
\frac{d}{dt} P_0^{(2)}(t) + (\lambda_3 + \mu_2)P_0^{(2)}(t) = p \int_0^\infty P_0^{(1)}(x,t)\mu_1(x)dx
\]

5. \[
\frac{d}{dt} Q(t) + \lambda_1 Q(t) = \mu_2 P_0^{(2)}(t) + (1-p) \int_0^\infty P_0^{(1)}(x,t)\mu_1(x)dx.
\]

The above set of equations is to be solved subject to the following boundary and initial conditions:

- \[
P_n^{(1)}(0,t) = (1-p) \int_0^\infty P_n^{(1)}(x,t)\mu_1(x)dx + \mu_2 P_{n+1}^{(2)}(t) + \lambda_1 c_{n+1}Q(t); \quad n \geq 1
\]

- \[
P_0^{(1)}(0,t) = (1-p) \int_0^\infty P_0^{(1)}(x,t)\mu_1(x)dx + \mu_2 P_1^{(2)}(t) + \lambda_1 c_1Q(t)
\]

- \[
Q(0) = 1;
\]

- \[
P_n^{(1)}(x,0) = P_n^{(2)}(0) = 0; \quad n \geq 0.
\]
4.4 Mathematical Analysis

In this section, the probability generating function of the number of units in the queue is obtained to discuss the transient and steady state behaviours of the system. For brevity of notations, we denote

\[ p_x = (s + \lambda_z + \mu_t(x)); \quad q_x = (s + \lambda_3 + \mu_z); \quad r_x = (s + \lambda_z - \lambda_z X(z) + \mu_t(x)); \]

\[ m_x = (s + \lambda_3 (1 - X(z)) + \mu_z); \quad u_x = (s + \lambda_z (1 - X(z))); \quad v_x = (s + \lambda_3 (1 - X(z))); \]

\[ w_x = (s + \lambda_1 (1 - X(z))); \quad E_1(X) = (1 - \lambda_2 E(X) E(B_1)); \quad E_2(X) = (\lambda_1 - \lambda_3) E(X). \]

By taking Laplace transforms of equations (4.4)-(4.8) and using initial conditions given in equations (4.11) and (4.12), we get

\[ \frac{\partial}{\partial x} \overline{P}_1^n (x, s) + p_x \overline{P}_1^n (x, s) = \lambda_z \sum_{j=1}^{n} c_j \overline{P}_n^{(j)} (x, s); \quad n \geq 1 \quad (4.13) \]

\[ \frac{\partial}{\partial x} \overline{P}_0^n (x, s) + p_x \overline{P}_0^n (x, s) = 0 \quad (4.14) \]

\[ q_x \overline{P}_2^n (s) = \lambda_3 \sum_{j=1}^{n} c_j \overline{P}_n^{(j)} (s) + p \int_{0}^{\infty} \overline{P}_1^n (x, s) \mu_1(x) dx; \quad n \geq 1 \quad (4.15) \]

\[ q_x \overline{P}_0^n (s) = p \int_{0}^{\infty} \overline{P}_0^n (x, s) \mu_1(x) dx \quad (4.16) \]

\[ (s + \lambda_1) \overline{Q}(s) = 1 + \mu_2 \overline{P}_2^n (s) + (1 - p) \int_{0}^{\infty} \overline{P}_1^n (x, s) \mu_1(x) dx. \quad (4.17) \]

Laplace transforms of equations (4.9) and (4.10) provide

\[ \overline{P}_1^n (0, s) = (1 - p) \int_{0}^{\infty} \overline{P}_{n+1}^{(1)} (x, s) \mu_1(x) dx + \mu_2 \overline{P}_{n+1}^{(2)} (s) + \lambda_1 c_{n+1} \overline{Q}(s); \quad n \geq 1 \quad (4.18) \]

\[ \overline{P}_0^n (0, s) = (1 - p) \int_{0}^{\infty} \overline{P}_1^{(1)} (x, s) \mu_1(x) dx + \mu_2 \overline{P}_1^{(2)} (s) + \lambda_1 c_1 \overline{Q}(s). \quad (4.19) \]

**Theorem 4.1:** The transient probability generating function of the number of units in the queue when the server is busy to provide essential service or optional service is given by

\[ \overline{P}_x(z, s) = \frac{[p u_x \overline{B}_t(u_x) + m_x \{1 - \overline{B}_t(u_x)\}](1 - w_x \overline{Q}(s))}{u_x [m_x z - \overline{B}_t(u_x) \{\mu_2 + (1 - p) v_x\}].} \quad (4.20) \]
**Proof:** The probability generating function of the number of units in the queue can be obtained as

\[ P_q(z,s) = \bar{P}^{(1)}(z,s) + \bar{P}^{(2)}(z,s) \]

where, \( \bar{P}^{(1)}(z,s) \) and \( \bar{P}^{(2)}(z,s) \) denote the Laplace transforms of the probability generating functions of the number of units in the queue when the server is busy in providing essential and optional services, respectively. Now we obtain \( \bar{P}^{(1)}(z,s) \) and \( \bar{P}^{(2)}(z,s) \) as follows:

\[
\bar{P}^{(1)}(z,s) = \frac{(1 - \bar{B}_1(u_s))m_s(1 - w_s\bar{Q}(s))}{u_s\{m_s z - \bar{B}_1(u_s)(\mu_z + (1 - p)v_s)\}} \tag{4.21}
\]
\[
\bar{P}^{(2)}(z,s) = \frac{p\bar{B}_1(u_s)(1 - w_s\bar{Q}(s))}{\{m_s z - \bar{B}_1(u_s)(\mu_z + (1 - p)v_s)\}}. \tag{4.22}
\]

For detailed proof see Appendix-III.A.

**Corollary:** For \( z = 1 \), equation (4.20) gives

\[
\lim_{z \to 1} \bar{P}_q(z,s) = \frac{1 - s\bar{Q}(s)}{s}
\]

It can be easily verified the condition \( \bar{Q}(s) + \bar{P}_q(1,s) = \frac{1}{s} \) from normalizing condition. \( \bar{Q}(s) \) may be obtained from the equation (4.20) by using the Rouche’s theorem (cf. Gross and Harris, 1985) inside the circle \( |z| = 1 \).

**Theorem 4.2:** In steady state, the probability generating function of the number of units in the queue when the server is busy in providing the essential service or optional service is

\[
P_q(z) = \left[ \frac{(1 - \rho)(\lambda_1/\lambda_2)[m_o(1 - \bar{B}_1(u_o)) + pu_o \bar{B}_1(u_o)]}{\bar{B}_1(u_o)\{\mu_z + (1 - p)v_o\} - m_o z} \right]. \tag{4.23}
\]

**Proof:** For proof see Appendix-III.B.

**4.5 Performance Measures**

To predict the behaviour of any queueing system, it is of vital importance to establish explicit formulae for the system size distribution, average waiting time etc. so that these results can be easily applied to the real life congestion situations. For this
purpose, we derive the expressions for various performance measures to predict the behaviour of the system by using the probability generating functions derived in the previous section as follows:

(i) Average number of units in the queue (system)

On evaluating the first and second derivatives of \( N(z) \) and \( D(z) \), the numerator and denominator respectively of right hand side of equation (4.23), with respect to \( z \) at \( z = 1 \) we get

\[
N'(l) = -(1 - \rho) \lambda_1 E(X) \{ \mu_2 E(B_1) + p \}
\]

\[
N''(l) = (1 - \rho) \lambda_2 [2 \lambda_1 (E(X))^2 E(B_1) - \mu_2 \{ E(X^2) E(B_1) + \lambda_1 (E(X))^2 E(B_1^2) \} - 2 p \lambda_2 (E(X))^2 E(B_1) - p E(X^2)]
\]

\[
D'(l) = \mu_2 \{ \lambda_2 E(X) E(B_1) - 1 \} + p \lambda_3 E(X)
\]

\[
D''(l) = 2(1 - p) \lambda_2 \lambda_3 (E(X))^2 E(B_1) + \mu_2 (\lambda_1 E(X^2) E(B_1) + \lambda_2 E(X))^2 E(B_1^2)) + p \lambda_3 E(X^2) + 2 \lambda_3 E(X).
\]

By using the equations (4.24)-(4.27) in equation (C.2.1), we get the required value of \( L_q \). The average number of units in the system \( (L_s) \) can be obtained by using the value of \( L_q \) and \( \rho \) obtained from equation (B.4.6), in equation (2.18).

(ii) Average waiting time of the units in the queue

The average waiting time \( (W_q) \) in the queue can be determined by using equation (2.19) with \( \lambda_c = \lambda_1 Q + \lambda_2 P^{(1)}(1) + \lambda_3 P^{(2)}(1) \).

4.6 Maximum Entropy Results

To obtain the approximate values of system probabilities, the maximum entropy principle is applied. On formulating the maximum entropy model, we have the entropy function \( Z \) of the form

\[
Z = -Q \log Q - \sum_{n=0}^{\infty} P_n^{(1)} \log P_n^{(1)} - \sum_{n=0}^{\infty} P_n^{(2)} \log P_n^{(2)}.
\]

The maximum entropy solutions for the model are obtained by maximizing (4.28) subject to the following constraints.
\[ Q + \sum_{n=0}^{\infty} P_n^{(1)} + \sum_{n=0}^{\infty} P_n^{(2)} = 1; \quad Q = \eta_0 = 1 - \rho \]  

(4.29)

\[ \sum_{n=0}^{\infty} P_n^{(1)} = \eta_1 = P^{(1)}(1) \]  

(4.30)

\[ \sum_{n=0}^{\infty} P_n^{(2)} = \eta_2 = P^{(2)}(1) \]  

(4.31)

\[ \sum_{n=1}^{\infty} nP_n^{(1)} + \sum_{n=1}^{\infty} nP_n^{(2)} = L_q \]  

(4.32)

**Theorem 4.3:** The steady state probabilities of the state dependent bulk arrival queue for different states are given as

\[ P_n^{(1)} = \eta_1 \left( \frac{\eta_1 + \eta_2}{\eta_1 + \eta_2 + L_q} \right)^n ; \quad n = 0, 1, 2, \ldots \]  

(4.33)

\[ P_n^{(2)} = \eta_2 \left( \frac{\eta_1 + \eta_2}{\eta_1 + \eta_2 + L_q} \right)^n ; \quad n = 0, 1, 2, \ldots \]  

(4.34)

**Proof:** For proof see Appendix-III.C.

4.7 Special Cases

In this section, some special cases are deduced by setting the appropriate parameters, in the performance measures of present investigation. To validate the present model with those which are studied by other researchers, we consider the following cases:

**Case (i) M/G/1 queue with homogeneous arrival rates.**

By setting \( \lambda_1 = \lambda_2 = \lambda_3 = \lambda \), \( E(B_1) = 1/\mu_1 \), \( E(B_1^2) = E(v^2) \), \( E(X) = 1 \), \( E(X^{(2)}) = 0 \) in equations (4.24)-(4.27) and using equation (C.2.1), we get

\[ L_q = \left[ \frac{\lambda^2 \{2p((\mu_1/\mu_1 + 1) + \mu_2^2E(v^2))\}}{2\{1-(\lambda/\mu_1)\}^2} \right] \frac{(1-(\lambda/\mu_1))\mu_2 - p\lambda_2}{\mu_2^2}. \]  

(4.35)

This result coincides with that of Madan (2000).
Case (ii) $M/G/1$ queue with heterogeneous arrival rates.

In this case, the average number of units can be obtained by setting $E(X) = 1, E(X^{(2)}) = 0$ in equations (4.24)-(4.27) and using equation (C.2.1)

$$L_q = \frac{\mu_2 (\lambda_2 E(B_1) - 1) + p\lambda_3 [2\lambda_3 E(B_1) - \mu_2 \lambda_2 E(B_1^2) - 2p\lambda_1 \lambda_2 E(B_1)]}{2[\mu_2 (1 - \lambda_2 E(B_1)) - p\lambda_3]^2} \times \frac{\mu_2 (1 - \lambda_2 E(B_1)) - p\lambda_3}{\mu_2 (1 - \lambda_2 E(B_1)) + p(\lambda_1 - \lambda_3) + \mu_2 \lambda_4 E(B_1)}.$$  (4.36)

Case (iii) $M^x/G/1$ queue when no unit opt the second service.

Now putting, $p = 0$ in equations (4.24)-(4.27) and using (C.2.1), we obtain

$$L_q = \frac{\lambda_3 [E(X^{(2)}) E(B_1)] + \lambda_4 (E(X))^2 E(B_1^2)]}{[1 - \lambda_2 E(X) E(B_1)][1 + E(X) E(B_1)(\lambda_1 - \lambda_2)]}. \quad (4.37)$$

4.8 Numerical Illustration and Sensitivity Analysis

In this section, we discuss the numerical tractability of the present model for evaluating the system size distribution and other performance measures by taking the illustration with the help of different parameters and observe the effect of various parameters on the performance measures.

Consider a doctor’s clinic where the family members (i.e. patients) may arrive in batches for regular check-up. The arrival of the units is considered as Poisson distributed with rates $\lambda_1$ and $\lambda_2$ (or $\lambda_3$) depending upon the status of the doctor who may be idle or busy in providing essential (optional) service, respectively. The doctor starts the check-up (i.e. essential service) of all the arriving patients one by one according to the FCFS discipline in four stages (i.e. $k = 4$) which includes physical examination, blood pressure test, blood test and urine/stool test. Sometimes the patients demand special tests (i.e. optional service) like X-ray, MRI, ECG, etc. Keeping this in mind, the provision of second optional service rendered by the doctor is quite obvious. The service times of the patients during essential service and optional service are Erlangian and exponentially distributed respectively. The doctor would like to determine the average number of patients and the average waiting time spend with the patients at the clinic. It is assumed that the batch size of the units follows a geometric distribution with
first and second moments given by equation (2.30). The distribution of essential service is taken as \( k \)-Erlangian \( M/E_k/1 \), so that the numerical results are obtained from \( M/E_k/1 \) queueing model with

\[
E(B_1) = \frac{1}{\mu_1}, \quad E(B_1^2) = \frac{k + 1}{k\mu_1^2}.
\]

Now, we present the sensitivity analysis to show the effects of different parameters on various performance measures. The numerical results are summarized in tables (4.1)-(4.3) for default parameters fixed as follows:

Table 4.1: \( E(X) = 2, p = 0.5, \lambda_1 = 1.4\lambda, \lambda_2 = 0.6\lambda, \lambda_3 = \lambda, \mu_2 = 9, \mu_1 = \mu \).

Table 4.2: \( E(X) = 2, p = 0.5, \mu_2 = 9, \mu_1 = 7.1 \).

Table 4.3: \( E(X) = 2, p = 0.5, \lambda_1 = 3.85, \lambda_2 = 1.65, \lambda_3 = 1.75, \mu_2 = 8, \mu_1 = 6.1, k = 5 \).

The effects of essential service rate (\( \mu \)) and arrival rate (\( \lambda \)) on the average number of units (\( L_q \)) are displayed in table 4.1. We observe that \( L_q \) decreases with the increase in service rates; on the contrary, it increases with the increase in the arrival rate of the units. It is also noticed that the \( L_q \) decreases significantly with the increase in the number of service phases (\( k \)). It shows that the average number of units in the queue may decrease because they will get the service in different phases. Table 4.2 shows the effect of different state dependent arrival rates on the average number of units (\( L_q \)) and the waiting time (\( W_q \)). The queue size distribution by using the maximum entropy principle is presented in table 4.3. Finally, we conclude that

1. When the arrival rate (service rate) increases, the average number of units in the queue increases (decreases); this fact can be observed in various congestion problems.
2. The average number of units in the queue decreases with the increase in the number of service phases.

**Conclusion**

In the present chapter, we have obtained the queue length distribution for the number of units with second optional service for more realistic bulk arrival queueing model. Such models may be helpful in tackling queueing situations and can be encountered in many manufacturing system and production system where the
production of the items has to be done in different phases namely assembling, testing and packing, etc. and some service may be optional depending upon customer’s demand.
Table 4.1: Effects of \( \lambda \) and \( \mu \) on \( L_q \)

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Table 4.2: Effects of \( \lambda_1 \), \( \lambda_2 \) and \( \lambda_3 \) on \( L_q \) and \( W_q \)

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<td>84.07</td>
<td>15.80</td>
</tr>
<tr>
<td>4.65</td>
<td>2.40</td>
<td>3.30</td>
<td>99.79</td>
<td>22.28</td>
<td>91.99</td>
<td>17.33</td>
</tr>
</tbody>
</table>

Table 4.3: Steady state probabilities \( P_n^{(1)} \) and \( P_n^{(2)} \)

<table>
<thead>
<tr>
<th>( n )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P_n^{(1)} )</td>
<td>0.1339</td>
<td>0.1075</td>
<td>0.0863</td>
<td>0.0693</td>
<td>0.0556</td>
<td>0.0447</td>
<td>0.0358</td>
<td>0.0288</td>
<td>0.0231</td>
<td>0.0185</td>
<td>0.0149</td>
</tr>
<tr>
<td>( P_n^{(2)} )</td>
<td>0.0511</td>
<td>0.0410</td>
<td>0.0329</td>
<td>0.0264</td>
<td>0.0212</td>
<td>0.0170</td>
<td>0.0137</td>
<td>0.0110</td>
<td>0.0088</td>
<td>0.0071</td>
<td>0.0057</td>
</tr>
<tr>
<td>( n )</td>
<td>11</td>
<td>12</td>
<td>13</td>
<td>14</td>
<td>15</td>
<td>16</td>
<td>17</td>
<td>18</td>
<td>19</td>
<td>20</td>
<td>21</td>
</tr>
<tr>
<td>( P_n^{(1)} )</td>
<td>0.0120</td>
<td>0.0096</td>
<td>0.0077</td>
<td>0.0062</td>
<td>0.0050</td>
<td>0.0040</td>
<td>0.0032</td>
<td>0.0026</td>
<td>0.0021</td>
<td>0.0017</td>
<td>0.0013</td>
</tr>
<tr>
<td>( P_n^{(2)} )</td>
<td>0.0046</td>
<td>0.0037</td>
<td>0.0029</td>
<td>0.0024</td>
<td>0.0019</td>
<td>0.0015</td>
<td>0.0012</td>
<td>0.0010</td>
<td>0.0008</td>
<td>0.0006</td>
<td>0.0005</td>
</tr>
<tr>
<td>( n )</td>
<td>22</td>
<td>23</td>
<td>24</td>
<td>25</td>
<td>26</td>
<td>27</td>
<td>28</td>
<td>29</td>
<td>30</td>
<td>31</td>
<td>32</td>
</tr>
<tr>
<td>( P_n^{(1)} )</td>
<td>0.0011</td>
<td>0.0009</td>
<td>0.0007</td>
<td>0.0006</td>
<td>0.0004</td>
<td>0.0004</td>
<td>0.0003</td>
<td>0.0002</td>
<td>0.0002</td>
<td>0.0001</td>
<td>0.0001</td>
</tr>
<tr>
<td>( P_n^{(2)} )</td>
<td>0.0004</td>
<td>0.0003</td>
<td>0.0002</td>
<td>0.0002</td>
<td>0.0002</td>
<td>0.0001</td>
<td>0.0001</td>
<td>0.0001</td>
<td>0.0001</td>
<td>0.0001</td>
<td>0.0000</td>
</tr>
</tbody>
</table>
Appendix-III

Appendix-III.A

Proof of Theorem 4.1:
Multiplying equations (4.13) and (4.14) by suitable powers of \( z \) and on summation, we get

\[
\frac{\partial}{\partial x} \bar{P}^{(1)}(x,z,s) + r_j \bar{P}^{(1)}(x,z,s) = 0. \tag{A.4.1}
\]

Similarly, equations (4.15), (4.16) and equations (4.18), (4.19) provide

\[
m_x \bar{P}^{(2)}(z,s) = p \int_0^{\infty} \bar{P}^{(1)}(x,z,s) \mu_i(x) dx \tag{A.4.2}
\]

\[
z \bar{P}^{(1)}(0,z,s) = (1 - p) \int_0^{\infty} \bar{P}^{(1)}(x,z,s) \mu_i(x) dx - (1 - p) \int_0^{\infty} \bar{P}_0^{(1)}(x,s) \mu_i(x) dx
+ \mu_2 \bar{P}^{(2)}(z,s) - \mu_2 \bar{P}_0^{(2)}(s) + \lambda_i X(z) \bar{Q}(s). \tag{A.4.3}
\]

Using equation (4.17), equation (A.4.3) gives

\[
z \bar{P}^{(1)}(0,z,s) = (1 - p) \int_0^{\infty} \bar{P}^{(1)}(x,z,s) \mu_i(x) dx + \mu_2 \bar{P}^{(2)}(z,s) + 1
+ (\lambda_i (X(z) - 1) - s) \bar{Q}(s). \tag{A.4.4}
\]

Integrating equation (A.4.1) with the limit 0 and \( x \), we get

\[
\bar{P}^{(1)}(x,z,s) = \bar{P}^{(1)}(0,z,s) e^{-\mu_i x} e^{-\int_a^{t_i} dt \mu_i(t)} \tag{A.4.5}
\]

where

\[
\bar{P}^{(1)}(0,z,s) \text{ is given in equation (A.4.4).}
\]

Further, on integrating the equation (A.4.5) with limit 0 to \( \infty \) and using equations (4.1) and (4.2), we obtain

\[
\bar{P}^{(1)}(z,s) = \bar{P}^{(1)}(0,z,s) \frac{1 - B_1(u_s)}{u_s} \tag{A.4.6}
\]

where

\[
B_1(u_s) = \int_0^{\infty} e^{-u_s x} b_1(x) dx.
\]

Multiplying (A.4.5) by \( \mu_i(x) \) and integrating with limit 0 to \( \infty \), we have
Equations (A.4.7) and (A.4.2) yield
\[ m_1 \overline{P}^{(2)}(z,s) = p \overline{P}^{(1)}(0,z,s) \overline{B}_1(u_1). \]  
(A.4.8)

Similarly, from equations (A.4.7) and (A.4.4), we obtain
\[ \{z - (1 - p)\overline{B}_1(u_1)\} \overline{P}^{(1)}(0,z,s) = \mu_2 \overline{P}^{(2)}(z,s) + 1 - w_1 \overline{Q}(s). \]  
(A.4.9)

On solving equations (A.4.8) and (A.4.9), we get
\[ \overline{P}^{(1)}(0,z,s) = \frac{m_1(1 - w_1 \overline{Q}(s))}{\{m_1 z - \overline{B}_1(u_1) (\mu_2 + (1 - p)v_0)\}}. \]  
(A.4.10)

By using the value of equation (A.4.10) in equations (A.4.6) and (A.4.8), we get the required equations (4.21) and (4.22).

**Appendix-III.B**

**Proof of Theorem 4.2:**

On multiplying the equation (4.20) by \( s \), with taking \( s \to 0 \) and by using the Tauberian property, \( \lim_{s \to 0} \overline{f}(s) = \lim_{t \to \infty} f(t) \), we obtain the value of \( P_q(z) \) as given by
\[ P_q(z) = \frac{\lambda_1 \int [1 - \overline{B}_1(u_0)] m_0 + p u_0 \overline{B}_1(u_0) \overline{Q} \overline{B}_1(u_0) \{ \mu_2 + (1 - p)v_0 \} - m_0 z}{\lambda_2 \{ \overline{B}_1(u_0) \{ \mu_2 + (1 - p)v_0 \} - m_0 z \}}. \]  
(B.4.1)

Using the basic properties \( \overline{B}(0) = 1 \) and \( \overline{B}'(0) = \frac{1}{\mu_1} \); for \( z = 1 \), we get
\[ P_q(1) = \lim_{z \to 1} P_q(z) = \frac{\lambda_1 E(X) (\mu_2 E(B_1) + p) \overline{Q}}{(\mu_2 E(X) - p \lambda_3 E(X))}. \]  
(B.4.2)

Normalizing condition \( Q + P_q(1) = 1 \) yields
\[ Q = \frac{\mu_2 E(X) - p \lambda_3 E(X)}{\mu_2 E(X) + \lambda_1 \mu_2 E(X) E(B_1) + p E_2(X)} \]  
(B.4.3)

where
\[ P_q(1) = P^{(1)}(1) + P^{(2)}(1) \text{ with} \]
\[ P^{(1)}(1) = \frac{\lambda_1 E(X) E(B_1) \mu_2}{(\mu_2 E(X) - p \lambda_3 E(X))} Q \]  
(B.4.4)
\[ P^{(2)}(1) = \frac{p \lambda_1 E(X)}{(\mu_2 E(X) - p \lambda_3 E(X))} Q \]  
(B.4.5)
The value of \( \rho \) can be obtained as
\[
\rho = 1 - Q = \frac{\lambda_i E(X)(p + \mu_i E(B_i))}{\mu_i (\lambda_i E(X)E(B_i) + E_i(X)) + pE_2(X)}.
\]
(B.4.6)

From equations (B.4.1) and (B.4.6), we get equations (4.23).

**Appendix-III.C**

**Proof of Theorem 4.3:**

By using Lagrange’s multipliers method, the entropy function (4.28) and set of constraints (4.29)-(4.32) give
\[
Z = -Q \log Q - \sum_{n=0}^{\infty} P_n^{(1)} \log P_n^{(1)} - \sum_{n=0}^{\infty} P_n^{(2)} \log P_n^{(2)} - \alpha_1 \left( Q + \sum_{n=0}^{\infty} P_n^{(1)} + \sum_{n=0}^{\infty} P_n^{(2)} - 1 \right) \\
- \alpha_2 \left( \sum_{n=0}^{\infty} P_n^{(1)} - \eta_1 \right) - \alpha_3 \left( \sum_{n=0}^{\infty} P_n^{(2)} - \eta_2 \right) - \alpha_4 \left( \sum_{n=1}^{\infty} nP_n^{(1)} + \sum_{n=1}^{\infty} nP_n^{(2)} - L_4 \right)
\]
(C.4.1)

where \( \alpha_1, \alpha_2, \alpha_3 \) and \( \alpha_4 \) are the Lagrangian multipliers corresponding to constraints (4.29)-(4.32), respectively.

As discussed in Appendix-I.D, on solving the equation (C.4.1) subject to boundary conditions (4.29)-(4.32), we get the required results as given in equations (4.33) and (4.34).