Chapter 2

Forced convective heat transfer from sphere under low $R_m$ approximation

In this chapter, the steady forced convective heat transfer from an isothermal sphere under the influence of an external magnetic field is considered up to $Re = 200$ above which the flow is known to be unsteady. A higher order compact scheme (HOCS) is employed with the combination of multigrid technique to solve the energy equation in spherical polar coordinates.

2.1 Basic equations

The forced convective heat transfer problem is formulated as steady, laminar flow in axis-symmetric spherical polar coordinates. The center of the sphere is chosen at origin and the flow is symmetric about $\theta = \pi$ (upstream) and $\theta = 0^o$ (downstream). The fluid is considered to be incompressible viscous and electrically conducting. A uniform stream from infinity, $U_\infty$ is imposed from left to right at far distances from the sphere. The magnetic Reynolds number is assumed to be small so that induced magnetic field can be neglected and a constant magnetic field

$$H = (-\cos \theta, \sin \theta, 0) \quad (2.1)$$
is imposed opposite to the flow. The governing equations are Navier-Stokes equations and Maxwell’s equations which are expressed in non-dimensional form as follows.

\[ \nabla \cdot q = 0 \]  \hspace{1cm} (2.2)

\[ (q \cdot \nabla) q = -\nabla p + \frac{2}{Re} \nabla^2 q + N(J \times H) \]  \hspace{1cm} (2.3)

\[ J = \nabla \times H \]  \hspace{1cm} (2.4)

\[ J = E + (q \times H) \]  \hspace{1cm} (2.5)

\[ \nabla \cdot H = 0 \]  \hspace{1cm} (2.6)

\[ \nabla \times E = 0 \]  \hspace{1cm} (2.7)

where \( q \) is the fluid velocity, \( H \) is the magnetic field, \( p \) is pressure, \( E \) is electric field, \( J \) is current density and \( N \) is the interaction parameter defined as \( N = \sigma H^2 \infty / \rho U^\infty \), where \( \sigma \) and \( \rho \) are electrical conductivity and density of the fluid and \( a \) is the radius of sphere. \( Re = 2aU^\infty / \nu \) is the Reynolds number based on the diameter \( (2a) \) of the sphere. A schematic diagram showing the flow configuration is depicted in Figure 2.1.

In order to have fine resolution near the surface of the sphere, we have used the transformation \( r = e^\xi \) along the radial direction, which provides the solution in the non-uniform physical plane while keeping the uniform grid in the computational plane as shown in Figure 1.2. The fluid motion is described by radial and transverse components of velocity \( q_r, q_\theta \) in a plane through axis of symmetry which are obtained by dividing the corresponding dimensional components by the main-stream velocity \( U^\infty \). The velocity components are expressed in terms of a dimensionless stream function \( \psi(\xi, \theta) \) such that the equation of continuity \( \nabla \cdot q = 0 \) is satisfied. They are

\[ q_r = \frac{e^{-2\xi}}{\sin \theta} \frac{\partial \psi}{\partial \theta}, \quad q_\theta = -\frac{e^{-2\xi}}{\sin \theta} \frac{\partial \psi}{\partial \xi}. \]  \hspace{1cm} (2.8)

Using the transformation \( r = e^\xi \) the equations that are governing the flow are written in vorticity-stream function formulation [51] as follows.
Figure 2.1: Schematic representation of the flow configuration.

Equation for stream function:

\[ \frac{\partial^2 \psi}{\partial \xi^2} - \frac{\partial \psi}{\partial \xi} + \frac{\sin \theta}{\sin \theta} \frac{\partial}{\partial \theta} \left( \frac{1}{\sin \theta} \frac{\partial \psi}{\partial \theta} \right) + \sin \theta e^{3\xi} \psi = 0 \quad (2.9) \]

Equation for solving Vorticity:

\[ \frac{\partial^2 \omega}{\partial \xi^2} + \frac{\partial \omega}{\partial \xi} + \cot \theta \frac{\partial \omega}{\partial \theta} + \frac{\partial^2 \omega}{\partial \theta^2} - \frac{\omega}{\sin^2 \theta} = \frac{Re}{2} e^{\xi} \left( q_r \frac{\partial \omega}{\partial \xi} + q_\theta \frac{\partial \omega}{\partial \theta} - q_r \omega - q_\theta \omega \cot \theta \right) \]

\[ - \frac{NRe}{2} e^{\xi} \left[ \frac{\sin 2\theta}{2} \left( - \frac{\partial u}{\partial \xi} + \frac{\partial v}{\partial \theta} + u \right) \right. \]

\[ + \sin^2 \theta \left( \frac{\partial u}{\partial \theta} - v \right) - \cos^2 \theta \frac{\partial v}{\partial \xi} \] \quad (2.10)

The boundary conditions to be satisfied are

On the surface of the sphere (\( \xi = 0 \)):

\[ \psi = \frac{\partial \psi}{\partial \xi} = 0, \quad \omega = -\frac{1}{\sin \theta} \frac{\partial^2 \psi}{\partial \xi^2} \]

At large distances from the sphere (\( \xi \to \infty \)):

\[ \psi \sim \frac{1}{2} e^{2\xi} \sin^2 \theta, \quad \omega \to 0, \]

Along the axis of symmetry (\( \theta = 0 \) and \( \theta = \pi \)):

\[ \psi = 0, \quad \omega = 0. \]
The velocity field is obtained by solving equations (2.9 - 2.10) using a second order based finite difference method combined with accelerated multigrid method as explained in the introduction chapter. The grid independent velocity components thus obtained over a high resolution grid $512 \times 512$ are used to solve the energy equation. If the physical properties of the fluid are assumed to be constant and the internal generation of heat by friction is neglected, the energy equation

$$\frac{\partial^2 T}{\partial \xi^2} + \cot \theta \frac{\partial T}{\partial \theta} + \frac{\partial^2 T}{\partial \theta^2} = \frac{Re Pr}{2} e^{\xi} \left( q_r \frac{\partial T}{\partial \xi} + q_0 \frac{\partial T}{\partial \theta} \right)$$  \hspace{1cm} (2.11)$$

where $T(\xi, \theta)$ is the non-dimensionalized temperature, defined by subtracting the main-flow temperature $T_\infty$ from the temperature and dividing by $T_s - T_\infty$. $Re$ is the Reynolds number based on the diameter $(2a)$ of the sphere and $Pr$ is the Prandtl number defined as the ratio between kinematic viscosity ($\nu$) and thermal diffusivity ($\kappa$). The boundary conditions for temperature are $T = 1$ on the surface of the sphere, $T \to 0$ as $\xi \to \infty$ and $\partial T/\partial \theta = 0$ along the axis of symmetry. The details of the fourth order compact scheme to solve equation 2.11 are given in the following section.

### 2.2 Fourth order scheme with MG method

Both the fluid motion and temperature field are axially symmetric and hence all computations have been performed only in one of the symmetric regions. The discretization of the governing equation (2.11) in the symmetric region is made using the compact stencil. By combining the convection terms $\frac{\partial \psi}{\partial \theta} \frac{\partial T}{\partial \xi}$ and $\frac{\partial \psi}{\partial \xi} \frac{\partial T}{\partial \theta}$ on the right hand side of equation (2.11) with the terms $\frac{\partial T}{\partial \xi}$ and $\cot \theta \frac{\partial T}{\partial \theta}$ respectively we obtain

$$- \frac{\partial^2 T}{\partial \xi^2} - \frac{\partial^2 T}{\partial \theta^2} + u \frac{\partial T}{\partial \xi} + v \frac{\partial T}{\partial \theta} = 0 \hspace{1cm} (2.12)$$

where

$$u = \frac{Re Pr}{2} e^{\xi} q_r - 1, \quad v = \frac{Re Pr}{2} e^{\xi} q_0 - \cot \theta.$$  \hspace{1cm} (2.13)
The velocity components $q_r$ and $q_\theta$ in the equation (2.13) are obtained using usual fourth order approximations from the stream function $\psi$. Applying standard central difference operators to equation (2.12) gives,

$$-\delta_\xi^2 T_{i,j} - \delta_\theta^2 T_{i,j} + u_{i,j} \delta_\xi T_{i,j} + v_{i,j} \delta_\theta T_{i,j} - \tau_{i,j} = 0. \quad (2.14)$$

The truncation error of equation (2.14) is given by

$$\tau_{i,j} = \left[ 2 \left( \frac{h^2}{12} \frac{\partial^3 T}{\partial \xi^3} + \frac{k^2}{12} \frac{\partial^3 T}{\partial \theta^3} \right) - \left( \frac{h^2}{12} \frac{\partial^4 T}{\partial \xi^4} + \frac{k^2}{12} \frac{\partial^4 T}{\partial \theta^4} \right) \right]_{i,j} + O(h^4, k^4), \quad (2.15)$$

where $h$ and $k$ are grid spacing ($h \neq k$) in the radial and angular directions, respectively. From equation (2.12), we get

$$\frac{\partial^3 T}{\partial \xi^3} = -\frac{\partial^3 T}{\partial \xi \partial \theta^2} + u \frac{\partial^2 T}{\partial \xi^2} + \frac{\partial u}{\partial \xi} \frac{\partial T}{\partial \xi} + v \frac{\partial^2 T}{\partial \xi \partial \theta} + \frac{\partial v}{\partial \xi} \frac{\partial T}{\partial \theta}$$

$$\frac{\partial^4 T}{\partial \xi^4} = -\frac{\partial^4 T}{\partial \xi^2 \partial \theta^2} + v \frac{\partial^3 T}{\partial \xi^3} - u \frac{\partial^3 T}{\partial \xi \partial \theta^2} + \left( 2 \frac{\partial v}{\partial \xi} + uv \right) \frac{\partial^2 T}{\partial \xi \partial \theta} + \left( \frac{\partial^2 u}{\partial \xi^2} + u \right) \frac{\partial T}{\partial \xi} + \left( \frac{\partial^2 v}{\partial \xi^2} + u \right) \frac{\partial T}{\partial \theta} + \left( \frac{\partial^2 u}{\partial \xi \partial \theta} + u \right) \frac{\partial T}{\partial \xi}$$

$$\frac{\partial^3 T}{\partial \theta^3} = -\frac{\partial^3 T}{\partial \xi \partial \theta^2} + u \frac{\partial^2 T}{\partial \xi \partial \theta} + \frac{\partial u}{\partial \theta} \frac{\partial T}{\partial \xi} + v \frac{\partial^2 T}{\partial \theta^2} + \frac{\partial v}{\partial \theta} \frac{\partial T}{\partial \theta}$$

$$\frac{\partial^4 T}{\partial \theta^4} = -\frac{\partial^4 T}{\partial \xi^2 \partial \theta^2} + u \frac{\partial^3 T}{\partial \xi^3 \partial \theta} - v \frac{\partial^3 T}{\partial \xi \partial \theta^2} + \left( 2 \frac{\partial u}{\partial \theta} + uv \right) \frac{\partial^2 T}{\partial \xi \partial \theta} + \left( \frac{\partial^2 u}{\partial \theta^2} + u \right) \frac{\partial T}{\partial \theta} + \left( \frac{\partial^2 v}{\partial \theta^2} + v \right) \frac{\partial T}{\partial \theta} + \left( \frac{\partial^2 u}{\partial \xi \partial \theta} + u \right) \frac{\partial T}{\partial \xi}.$$
On substituting in equation (2.15) and hence in (2.14) gives,

\[-e_{i,j} \delta_\xi^2 T_{i,j} - f_{i,j} \delta_\theta^2 T_{i,j} + g_{i,j} \delta_\xi T_{i,j} + o_{i,j} \delta_\theta T_{i,j} - \frac{h^2 + k^2}{12} (\delta_\xi^2 \delta_\theta^2 T_{i,j} -
\quad u_{i,j} \delta_\xi^2 T_{i,j} - v_{i,j} \delta_\theta^2 T_{i,j}) + w_{i,j} \delta_\xi \delta_\theta T_{i,j} = 0\]

where the coefficients \(e_{i,j}, f_{i,j}, g_{i,j}, o_{i,j}\) and \(w_{i,j}\) are given by

\[
e_{i,j} = 1 + \frac{h^2}{12} (u_{i,j}^2 - 2 \delta_\xi u_{i,j})
\]

\[
f_{i,j} = 1 + \frac{k^2}{12} (v_{i,j}^2 - 2 \delta_\theta v_{i,j})
\]

\[
g_{i,j} = u_{i,j} + \frac{h^2}{12} (\delta_\xi^2 u_{i,j} - u_{i,j} \delta_\xi u_{i,j}) + \frac{k^2}{12} (\delta_\theta^2 u_{i,j} - v_{i,j} \delta_\theta u_{i,j})
\]

\[
o_{i,j} = v_{i,j} + \frac{h^2}{12} (\delta_\xi^2 v_{i,j} - u_{i,j} \delta_\xi v_{i,j}) + \frac{k^2}{12} (\delta_\theta^2 v_{i,j} - v_{i,j} \delta_\theta v_{i,j})
\]

\[
w_{i,j} = \frac{h^2}{6} \delta_\xi v_{i,j} + \frac{k^2}{6} \delta_\theta u_{i,j} - \left(\frac{h^2 + k^2}{12}\right) u_{i,j} v_{i,j}
\]

and two-dimensional cross derivative \(\delta\) operators on a uniform anisotropic mesh \((h \neq k)\) are given by

\[
\delta_\xi \delta_\theta T_{i,j} = \frac{T_{i+1,j+1} - T_{i+1,j-1} - T_{i-1,j+1} + T_{i-1,j-1}}{4hk}
\]

\[
\delta_\xi^2 \delta_\theta T_{i,j} = \frac{T_{i+1,j+1} - T_{i+1,j-1} + T_{i-1,j+1} - T_{i-1,j-1} - 2T_{i,j+1} + 2T_{i,j-1}}{2h^2k}
\]

\[
\delta_\xi \delta_\theta^2 T_{i,j} = \frac{T_{i+1,j+1} - T_{i-1,j+1} + 2T_{i+1,j-1} - 2T_{i-1,j-1} - 2T_{i+1,j} - 2T_{i-1,j}}{2h^2k}
\]

\[
\delta_\xi^2 \delta_\theta^2 T_{i,j} = \frac{1}{h^2k^2} \left( T_{i+1,j+1} + T_{i+1,j-1} + T_{i-1,j+1} + T_{i-1,j-1} -
\quad 2T_{i,j+1} - 2T_{i,j-1} - 2T_{i+1,j} - 2T_{i-1,j} + 4T_{i,j} \right)
\]

For evaluating boundary conditions, along the axis of symmetry, the derivative \(\frac{\partial T}{\partial \theta}\) is approximated by fourth order forward difference along \(\theta = 0\) (i.e., \(j = 1\)) and
fourth order backward difference along θ = π (or j = m + 1) as follows.

\[
T(i, 1) = \frac{1}{25} [48T(i, 2) - 36T(i, 3) + 16T(i, 4) - 3T(i, 5)]
\]
\[
T(i, m + 1) = \frac{1}{25} [48T(i, m) - 36T(i, m - 1) + 16T(i, m - 2) - 3T(i, m - 3)]
\]

The algebraic system of equations obtained using the fourth order compact scheme described as above are solved using a multigrid scheme with coarse grid correction. Point Gauss-Seidal relaxation is used as pre-smoothers and post-smoothers. Please note that as the grid independent solutions of the flow (like \(ψ\) and \(ω\)) are obtained from the finest grid of 512 × 512, the same finest grid is used in solving heat transfer equation although such a high resolution grid is not necessary. The other coarser grids used are 256 × 256, 128 × 128, 64 × 64 and the coarsest grid 32 × 32. The injection operator and 9-point prolongation operators are used to move from finer to coarser and coarser to finer grids, respectively [50]. For a two-grid problem, to solve \(Lu = f\), one iteration implies the following steps.

1. Let the initial solution be \(u_o\) in the finest grid.

2. Apply point Gauss Seidel iterations on \(u_o\) in the finer grid a few times as pre-smoother to get an approximate solution \(u_1\).

3. Calculate residue \(r\) in the finest grid, \(r = f - \mathcal{L}f u_1\)

4. To get residue in coarser grid \(r_c\), restrict the residue \(r\) from finer to coarser grid and then multiply with a residual scaling parameter \(β\), that is, \(r_c = β \mathcal{R}r\). Here \(\mathcal{R}\) represents the restriction operator.

5. Setup the error equation \(\mathcal{L}_c e_c = r_c\) on the coarser grid and solve for error \(e\) using point Gauss Seidel method. This gives the error on the coarser grid.

6. To get error \(e\) in finer grid, prolongate the error \(e_c\) to finer grid and then multiply with residual weighting parameter \(α\). Add this error \(e\) to the solution \(u_1\) obtained in step-2 to get an improved solution \(u_2\). That is, \(u_2 = u_1 + α \mathcal{P} e_c\), where \(\mathcal{P}\) is the 9-point prolongation operator.
7. Perform a few point Gauss-Seidal iterations on the solution $u_2$ in the finer grid to obtain much better solution $u_3$.

8. Consider $u_3$ as $u_o$ and go to step-2.

The steps 1 – 8 above constitute one iteration of the two-grid problem. The iterations are continued until the norm of the dynamic residuals is less than $10^{-5}$. In the algorithm given above, if $\alpha = \beta = 1$, then it is a standard two-grid method with coarse grid correction. The parameters $\alpha$ and $\beta$ are used to accelerate the convergence rate. In this study, we used $0 < \alpha < 2$ and $0 < \beta < 2$.

### 2.3 Results and Discussion

The higher order compact scheme combined with accelerated multigrid method is applied to the problem of heated sphere which is immersed in an incompressible, viscous and electrically conducting fluid. An external magnetic field is applied in the opposite direction of the uniform stream. In this study, mainly four values of $Re$ namely $Re = 5, 40, 100$ and $200$ are considered. The results are discussed for the range of $N$ for $0 \leq N \leq 20$ and the Prandtl numbers $Pr = 0.065, 0.73, 1, 2, 5, 8$.

The local Nusselt number $N_u(\theta)$ and the mean Nusselt number $N_m$ are calculated as follows:

$$Nu(\theta) = \frac{2aq(T)}{k(T_s - T_\infty)} = -2 \left( \frac{\partial T}{\partial \xi} \right)_{\xi=0}$$

and

$$N_m = -\int_0^\pi \left( \frac{\partial T}{\partial \xi} \right)_{\xi=0} \sin \theta d\theta.$$  \hspace{1cm} (2.17)

In equations (2.16) and (2.17) the derivative $\frac{\partial T}{\partial \xi}$ is approximated with usual fourth order forward finite differences and the integration is evaluated using the Simpson’s rule.

#### 2.3.1 Validation

In the absence of the magnetic field ($N = 0$) the basic hydrodynamic problem is equivalent to the steady viscous flow around a sphere. Therefore, for $N = 0$, the
developed scheme is validated with the available theoretical results (Table 2.1) for $Re = 5$ and 40. It is clear from the table that the present results agree well with the numerical results of Dennis [19] with variation of $0.35\%$ and recent results of Feng and Michaelides [24] with variation of $0.20 - 4.78\%$. The results also agree with experimental results of Ranz and Marshall [14] with variation of $0.47 - 11.21\%$ and Whitaker [18] with variation of $3.06 - 6.75\%$. The simulations have been carried out over $32 \times 32$, $64 \times 64$, $128 \times 128$, $256 \times 256$ and $512 \times 512$ grids and the mean Nusselt number for $Re = 5$ and 40 for selected $Pr$ and $N$ are presented in Tables 2.2, 2.3, 2.4 and 2.5. It is clear from these tables that (i) the solutions obtained from the present numerical scheme exhibit grid independence, and (ii) it is possible to obtain grid independence in the smaller $64 \times 64$ grid for low $Pr$. For higher $Pr$ values, a grid finer than $64 \times 64$ but less than $128 \times 128$ is necessary for grid independence. Clearly solutions obtained from high resolution grids such as $256 \times 256$ and $512 \times 512$ are not required as mentioned in the previous section.

The fourth order compact scheme is combined with accelerated multigrid technique to achieve fast convergence so that CPU time can be minimized. Although, multigrid methods are well established with first and second order discretization methods its combination with higher order compact schemes are not found much in the literature. To verify the effect of the multigrid method and accelerated multigrid method on the convergence of the Point Gauss-Seidal iterative method while solving resulting algebraic system of equations, the solution is obtained from different multigrids starting with five grids $32 \times 32$, $64 \times 64$, $128 \times 128$, $256 \times 256$ and $512 \times 512$ and by omitting each coarser grid until it reaches single-grid $512 \times 512$. This experiment is done with $Re = 40$, $Pr = 0.73$ and four values of interaction parameters 0, 0.5, 2 and 7. The simulations are also made for $Re = 40$, $N = 2$ and for selected $Pr$ values 0.73, 2 and 5. The computations are carried out on AMD quad core Phenom-II X4 965 (3.4 GHz) desktop computer. The number of iterations and CPU time (in hours) taken in different multi-grids and single-grid are illustrated in Figures 2.2 and 2.3. From these figures it is clear that, the multigrid method with coarse grid correction is very effective in enhancing the convergence rate of the solutions. It enhances the convergence rate at least $94\%$ in comparison with single-grid. The accelerated
Table 2.1: Comparison of the mean Nusselt number with literature in the absence of the magnetic field.

<table>
<thead>
<tr>
<th></th>
<th>$N_m$ for $Re = 5$</th>
<th>$N_m$ for $Re = 40$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$Pr = 0.73$</td>
<td>$Pr = 5$</td>
</tr>
<tr>
<td>Present simulation</td>
<td>2.8518</td>
<td>4.3101</td>
</tr>
<tr>
<td>Ranz &amp; Marshall (1952)</td>
<td>3.2080</td>
<td>4.2942</td>
</tr>
<tr>
<td>Whitaker (1972)</td>
<td>2.9433</td>
<td>4.0366</td>
</tr>
<tr>
<td>Dennis (1973)</td>
<td>2.86</td>
<td>—</td>
</tr>
<tr>
<td>Feng &amp; Michaelides (2000)</td>
<td>2.7250</td>
<td>4.3460</td>
</tr>
</tbody>
</table>

Table 2.2: Effect of grid size on the results for $Re = 5$ and $Pr = 0.73$.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$N_m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$32 \times 32$</td>
<td>$64 \times 64$</td>
</tr>
<tr>
<td>$0.5$</td>
<td>2.849</td>
</tr>
<tr>
<td>$1$</td>
<td>2.858</td>
</tr>
<tr>
<td>$3$</td>
<td>2.885</td>
</tr>
<tr>
<td>$5$</td>
<td>2.900</td>
</tr>
<tr>
<td>$8$</td>
<td>2.915</td>
</tr>
</tbody>
</table>

Table 2.3: Effect of grid size on the results $Re = 40$ and $Pr = 0.73$.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$N_m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$32 \times 32$</td>
<td>$64 \times 64$</td>
</tr>
<tr>
<td>$0.5$</td>
<td>4.928</td>
</tr>
<tr>
<td>$1$</td>
<td>4.913</td>
</tr>
<tr>
<td>$3$</td>
<td>4.920</td>
</tr>
<tr>
<td>$5$</td>
<td>4.948</td>
</tr>
<tr>
<td>$8$</td>
<td>5.004</td>
</tr>
</tbody>
</table>
Table 2.4: Effect of grid size on the results $Re = 5$ and $N = 2$.

<table>
<thead>
<tr>
<th>$Pr$</th>
<th>$N_m$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>32 $\times$ 32</td>
</tr>
<tr>
<td>0.065</td>
<td>2.142</td>
</tr>
<tr>
<td>0.73</td>
<td>2.873</td>
</tr>
<tr>
<td>1</td>
<td>3.051</td>
</tr>
<tr>
<td>2</td>
<td>3.530</td>
</tr>
<tr>
<td>5</td>
<td>4.369</td>
</tr>
<tr>
<td>8</td>
<td>4.899</td>
</tr>
</tbody>
</table>

Table 2.5: Effect of grid size on the results $Re = 40$ and $N = 2$.

<table>
<thead>
<tr>
<th>$Pr$</th>
<th>$N_m$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>32 $\times$ 32</td>
</tr>
<tr>
<td>0.065</td>
<td>2.780</td>
</tr>
<tr>
<td>0.73</td>
<td>4.910</td>
</tr>
<tr>
<td>1</td>
<td>5.330</td>
</tr>
<tr>
<td>5</td>
<td>8.510</td>
</tr>
</tbody>
</table>
multigrid technique further enhances the convergence rate by reducing 21% of the
time taken by multigrid (5 grids). In this study, the acceleration parameters which
are found suitable for enhancement of the convergence rate for one \( Pr \), in the absence
of the magnetic field, is suitable for all values of non-zero interaction parameters.
The results are also obtained for some parameters using a second order accurate
scheme combined with the multigrid method [51]. The CPU time (in hours) taken
for both the methods in each grid as well as multigrid are tabulated in Table 2.6.
It can be noted from the table that HOCS is more computationally efficient when
compared to second order accurate scheme in each grid as well as when it is combined
with the multigrid method.

2.3.2 Local and mean Nusselt numbers

The angular variation of the local Nusselt number \( Nu \) on the surface of the sphere for
different Prandtl numbers and for different interaction parameters are presented in
Figures 2.5 to 2.9. In the absence of the magnetic field, for all \( Pr \) it is found that the
local Nusselt number decreases along the surface of the sphere for Reynolds number
\( Re = 40 \) (Fig. 2.5), whereas the local Nusselt number decreases along the surface
of the sphere until it reaches near point of separation beyond which it increases in
the far downstream [52, 27] in the case of \( Re = 100, 200 \) (Fig. 2.6, for \( Re = 100, [53] \)). While studying the dependence of \( Nu \) on \( Pr \), when external magnetic field
is not present, the maximum heat transfer takes place near the front stagnation
point \( \theta = \pi \) (Figs. 2.5, 2.6) whereas, when the magnetic field is increased, the peak
heat transfer region is shifted towards \( \theta = \pi/2 \). The changes in velocity gradients
due to the applied field damp the flow instability or oscillations of the flow leading
to the steady flow. A similar effect of magnetic field is reported in MHD mixed
convection [54], in the silicon crystal growth simulation [55] and recently in the
study of semiconductor \( Si_{1-x}Ge_x \) crystal growth by 3D simulations [56]. In the
upstream region (Figs. 2.7, 2.8, 2.9), the viscous boundary layer thickens with the
application of magnetic field. All the curves in these figures meet at one critical point
after which an inverse effect is exhibited, that is, the boundary layer gets thinner
with magnetic field. In the upstream region (Fig. 2.9), up to \( \theta \approx 118^\circ \), the viscous
boundary layer thickens with the application of magnetic field. All the curves in Fig. 2.9 meet near the point $\theta \approx 118^\circ$ after which an inverse effect is exhibited, that is, in the region $118^\circ$ to $35^\circ$, the boundary layer gets thinner with magnetic field. The curves meet once again in the far downstream. These features are attributed to changes in radial and transverse velocity gradients of the fluid which resulted due to the application of magnetic field to the flow [57].

Irrespective of the magnetic field, when $Pr$ is increased, the local Nusselt number increased along the surface of the sphere. However, the heat transfer rate of a fluid with higher $Pr$ is affected more by external magnetic field when compared to a fluid with lower $Pr$. The higher values of $Nu$ for higher $Re$, as seen from Figs. 2.7, 2.8 and 2.9, is as expected since fluids with larger Reynolds number indicate dominant convection, wherein the viscous and thermal boundary layers get thinner with growing $Re$. When these local values of heat flux is surface averaged over the sphere, we get mean Nusselt number $N_m$. In particular we have carried out forced convection heat transfer simulation for Lithium ($Pr = 0.065, Re = 40, 100, 200$) to compare with the available experimental results. The comparisons have been made in Fig. 2.10, Fig. (2.11-a). In this case, the mean Nusselt number behaves non-monotonically with applied magnetic field. From Fig. 2.11, for $Re = 40$ we observe that there is a degradation in the mean Nusselt number when $0 \leq N \leq 2$. In Figs. 2.12, 2.13, when $Re = 100, 200$ degradation is observed in $0 \leq N \leq 3$ beyond which it increases leading to a non-linear behavior. These observation are in line with the recent experimental findings of Uda et al. [34] and Yokomine et al. [35]. In particular, to compare our results with the recent experimental results of Yokomine et al. at low values of $N$ up to 0.1, simulations are made with KOH solution with $Pr = 5$ for $Re = 5$ and 40 and the mean Nusselt number is presented in Fig. 2.14. The degradation of heat transfer found in this study, at low values of $N$, is also in agreement with experimental results [35]. The increased $N_m$ with $Pr$, as observed here, is in agreement with [58] in their study without magnetic field.

To understand this non-monotonic behavior of $N_m$, we analyze the local Nusselt number ($Nu$) at stagnation points and also on the top of the surface of the sphere (Fig. 2.15). At the front stagnation point ($\theta = \pi$) the magnetic field reduces
the heat transfer effectively, and at the rear stagnation point the heat transfer is drastically reduced for low magnetic fields \((N \leq 1)\) beyond which the reduction is not appreciable. The percentage of reduction of heat transfer due to magnetic field is found to be high for a fluid with larger Prandtl number. From the variation of velocity gradients of the fluid velocity, upon applying magnetic field, on the surface of the sphere we find that the fluid is decelerated near the stagnation points while it is accelerated near the top surface of the sphere. The velocity gradients are affected to a greater extent by the magnetic field when \(N \leq 2\) and higher fields are ineffective in changing the velocity gradients. In fact, in the magnetohydrodynamic problem [57] it is found that the suppression of the boundary layer separation is effective only for \(N \leq 2\) and thereafter the boundary layer control is less significant. When \(N > 2\), an increase in separation length and separation angle is observed (Fig. 2.4). Such non-monotonic behavior is also reported in the study of cylinder wake in an aligned magnetic field [59]. The non-monotonic behavior is due to the damping of transverse (angular direction) velocity component monotonically and the incapability of pushing down the adverse pressure gradient towards the downstream. In the numerical simulation of silicon melts Akamatsu et al., observed that a horizontal magnetic field suppresses the flow strongly in the circumferential direction [60]. The same is responsible for the ineffective decrease in the Nusselt number in the vicinity of the rear stagnation point at higher values of \(N\) (Figs. 2.9 and 2.15 c). In contrast to the observation at stagnation points, the local \(Nu\) increases with magnetic field (2.15 b) around \(\theta = \pi/2\).

2.3.3 Temperature field

The effect of \(Re, Pr\) and \(N\) on the temperature distribution have been presented in Figs. 2.16 to 2.18 in the flow field. In the upstream region, near the surface of the sphere, the isotherms are relatively dense implying that high thermal gradients are present, leading to higher local Nusselt numbers in comparison to the wake region of the sphere. For flows with higher \(Re\), the density of isotherms near the surface of the sphere is higher (hence larger local \(Nu\)). Consequently, the thermal boundary layer thickness decreases for higher \(Re\). In the absence of the magnetic field, as \(Re\)
increases, the isotherms are bent towards the surface of the sphere in the downstream which is due to the increase in recirculation region of the bubble (Fig. 2.16). As \( Pr \) increases, the isotherms with large \( T \) are shifted towards the surface of the sphere while those with small \( T \) spread farther and are pushed towards the axis of symmetry, thereby increasing the thermal fluctuations (Fig. 2.17). When the magnetic field is imposed for a fixed \( Pr \), the isotherms are moving away from the sphere in the downstream region so as to nullify the thermal fluctuations that are otherwise present when \( N = 0 \). This behavior is observed until \( N = 3 \), beyond which non-monotonic behavior is observed wherein the thermal gradients are higher (Fig. 2.18). As \( Pr \) increases in the presence of the magnetic field (Fig. 2.19), the isotherms are moving away from the surface of the sphere in addition to arresting the thermal fluctuations. Fig. 2.20 show the variation of temperature with radial distance along \( \theta = \pi/2 \) where we observe that the temperature and thermal gradients increase with increase of \( N \) as well as of \( Pr \). Even though the applied magnetic field and Prandtl number affects the thermal gradients of the flow, the latter influences more strongly.

### 2.4 Conclusions

A higher-order compact scheme is combined with accelerated multigrid method in spherical polar coordinates to simulate the steady forced convective heat transfer from a sphere under the influence of an external magnetic field. The speedy convergence of the solution using the multigrid method and accelerated multigrid method is illustrated. The computational efficiency of the higher order compact scheme over second order accurate scheme is presented. The angular variation of the Nusselt number and mean Nusselt number are calculated and compared with recent experimental results. Upon applying the magnetic field, a slight degradation of heat transfer is found for moderate values of the interaction parameter (\( N \)) and for high values of \( N \), an increase in heat transfer is observed, leading to a nonlinear behavior.
Figure 2.2: Effect of the multigrid and acceleration parameter on convergence factor for $Re = 40$ and $N = 2$, where the range between 5 to 6 on X scale indicates 5 grids with acceleration parameters $\alpha = 1.5$, $\beta = 0.5$ for $Pr = 0.73$ and $\alpha = 1.4$, $\beta = 0.5$ for $Pr = 2$.

Table 2.6: Comparison of CPU times (in hours) with second order accurate scheme.

<table>
<thead>
<tr>
<th>Grid</th>
<th>CPU time in hours</th>
<th>Second order</th>
<th>HOCS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$64 \times 64$</td>
<td>0.00083</td>
<td>0.00027</td>
<td></td>
</tr>
<tr>
<td>$128 \times 128$</td>
<td>0.01055</td>
<td>0.00277</td>
<td></td>
</tr>
<tr>
<td>$256 \times 256$</td>
<td>0.1575</td>
<td>0.07527</td>
<td></td>
</tr>
<tr>
<td>$512 \times 512$</td>
<td>2.9108</td>
<td>2.2291</td>
<td></td>
</tr>
<tr>
<td>$32^2 - 128^2$</td>
<td>—</td>
<td>0.0011*</td>
<td></td>
</tr>
<tr>
<td>$32^2 - 512^2$</td>
<td>1.29*</td>
<td>0.1158</td>
<td></td>
</tr>
</tbody>
</table>

* CPU time on achieving grid independence
Figure 2.3: Effect of the multigrid and acceleration parameter on convergence factor 
$Re = 40$ and $Pr = 0.73$, where the range between 5 to 6 on X scale indicates 5 grids 
with acceleration parameters $\alpha = 1.5$, $\beta = 0.5$
Figure 2.4: (a) Recirculation length [top] and (b) Separation angle [bottom] as a function of interaction parameter for $Re = 200$. 
Figure 2.5: Angular variation of the Nusselt number for $Re = 40$ when $N = 0$, $N = 2$ and $N = 7$
Figure 2.6: Angular Nusselt number for Re = 100 and for different Prandtl numbers with (top) $N = 0$ (middle) $N = 4$ (bottom) $N = 8$. 
Figure 2.7: Angular variation of the Nusselt number for $Re = 5$ when $Pr = 0.73$ and $Pr = 8$
Figure 2.8: Angular variation of the Nusselt number for $Re = 40$ when $Pr = 0.73$ and $Pr = 8$
Figure 2.9: Angular variation of the Nusselt number for $Re = 100$ when (top) $Pr = 0.73$ and (bottom) $Pr = 8$. 
Figure 2.10: Variation of mean Nusselt number $N_m$ with magnetic field $N$. The case with $Pr = 0.065$ corresponds to liquid metal (Li).
Figure 2.11: Variation of mean Nusselt number $N_m$ with magnetic field $N$ for different $Pr$ when $Re = 40$. 
Figure 2.12: Variation of mean Nusselt number $N_m$ with magnetic field $N$ for different $Pr$ for $Re = 100$. 
Figure 2.13: Variation of mean Nusselt number $N_m$ with magnetic field $N$ for different $Pr$ for $Re = 200$. 
Figure 2.14: Mean Nusselt number versus interaction parameter \((N < 0.1)\) for aqueous KOH solution \((Pr \approx 5)\) when \(Re = 5\) and \(Re = 40\)
Figure 2.15: Variation of local Nusselt number (a) $Nu(\theta = \pi)$, (b) $Nu\left(\theta = \frac{\pi}{2}\right)$ and (c) $Nu(\theta = 0)$ with $N$ for $Re = 100$. 

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Figure 2.16: Isotherms of the flow in the absence of the magnetic field for $Re = 100$ and 200 with $Pr = 0.73$. 
Figure 2.17: Isotherms of the flow in the absence of the magnetic field for $Re = 100$ and $Pr = 0.73, 2, 4$ and 8.
Figure 2.18: Isotherms of the flow for $Re = 100$ with $Pr = 8$ and $N = 0, 0.5, 3, 8$. 
Figure 2.19: Isotherms of the flow for $Re = 100$ with $N = 3$ and $Pr = 0.73, 2, 4$ and 8.
Figure 2.20: Variation of temperature with radial distance from the center of the sphere along $\theta = \frac{\pi}{2}$.  

$Re = 100, Pr = 0.73$

$Re = 100, N = 1$