Chapter 1

Introduction

Constructing and describing systems whose dynamical behavior can be modeled by either partial differential equations (PDE) or differential-difference equations or ordinary differential equations or difference equations [1–4]. In many cases, these are nonlinear and become closet to real type applications. One of the challenging task is to understand them by finding their exact solutions. In this framework, integrable systems are of main interest to study, a ubiquitous class of nonlinear equations by having rich mathematical structures and significant diverse applications in the area of physics. The celebrated systems in this field are soliton systems, a special class of integrable systems, admitting stable solution (soliton) as well as most properties which are often used to confirm the integrability of a given system [5–8]. Moreover, studying the underlying algebraic structures of integrable systems offers more and more to realize their way to modern physical theories. Therefore, the description and understanding of nonlinear phenomena is of great interest in both from the theoretical and applications point of view.

1.1 Lax pair

In 1967, Gardner, Greene, Kruskal and Miura [9] solved exactly the initial value problem of Korteweg-de Vries (KdV) equation through inverse scattering technique (IST). Later in 1968, Peter Lax [10] formulated this theory in the frame work of finding a pair of linear operators (possibly depending on spatial and temporal variables) called Lax pair. Thus, solving the initial value problem of a nonlinear soliton system is facilitated by finding
a suitable Lax pair for a given system. Further, this approach has been generalized to solve other partial differential equations including (2+1) dimensional Kadomtsev-Petviashvili (KP) equation [11–13], the Davey-Stewartson (DS) equation [14], etc. In particular, the existence of Lax pair becomes a common feature for soliton equations. Here, we briefly explain Lax method by considering the two linear operators $L$ and $M$, called Lax pair associated with the following spectral problem:

\begin{align}
L \Psi &= \lambda \Psi, \\
\Psi_t &= M \Psi,
\end{align}

where $\Psi$ is the eigenfunction, $\lambda$ is the spectral parameter, $L$ is the spectral operator, and $M$ is the operator governing the associated time evolution of the eigenfunctions.

Differentiating the equation (1.1.1a) with respect to $t$, we obtain

\begin{equation}
L_t \Psi + L \Psi_t = \lambda_t \Psi + \lambda \Psi_t.
\end{equation}

Using (1.1.1a) and (1.1.1b) in (1.1.2), we get

\begin{align*}
L_t \Psi + LM \Psi &= \lambda_t \Psi + \lambda M \Psi, \\
&= \lambda_t \Psi + M \lambda \Psi, \\
&= \lambda_t \Psi + ML \Psi.
\end{align*}

Therefore, we obtain

\begin{equation}
(L_t + [L, M]) \Psi = \lambda_t \Psi.
\end{equation}

Hence, in order to solve the spectral problem (1.1.1), the operators $L$ and $M$ have to satisfy

\begin{equation}
L_t + [L, M] = 0,
\end{equation}

where

\begin{equation}
[L, M] = LM - ML,
\end{equation}

with $\lambda_t = 0$. Equation (1.1.4) is called Lax’s equation which contains a nonlinear evolution equation for the suitable choice of the operators $L$ and $M$. For example, consider the
following linear operators

\[
L = \partial^2 + u, \quad \text{(1.1.5a)}
\]

\[
M = 4\partial^3 + 3u_x + 6u\partial, \quad \text{(1.1.5b)}
\]

and using (1.1.5) in the Lax equation (1.1.4), we arrive

\[u_t = 6uu_x + u_{xxx}. \quad \text{(1.1.6)}\]

Equation (1.1.6) is called KdV equation which is the prototype example for the IST problem in one spatial dimension. Moreover, Lax [10] demonstrated a way to construct \(M\) when \(L\) is given. It is important to observe that there is no systematic method to find the Lax pair of a given nonlinear evolution equation. In particular, there is no guarantee that a given evolution equation admits a genuine Lax pair. Suppose, a system associated with a spectral problem in the form of (1.1.1), then there exists two sets of evolution equations in general, called the isospectral flows/generalized symmetries and nonisospectral flows/master symmetries. Isospectral equations are derived from a spectral problem with time independent spectral parameter, whereas for nonisospectral equations the spectral parameter is time-dependent. In the case of soliton systems, the solutions of the isospectral equations explain the behavior of solitary waves in the lossless and uniform media and the corresponding solutions of nonisospectral equations reveal the nature of solitary waves in a certain type of nonuniform media [15–18]. It is also known that the isospectral flows of an integrable system commute with each other. This means that every member of the isospectral flows is a symmetry to all others. However, it is interesting to observe that there are other kind of symmetries which do not commute between themselves, called additional symmetries. Nonisospectral flows/master symmetries and time-dependent symmetries belong to this class. Moreover, the study of isospectral flows with additional symmetries of an integrable system is often related with its underlying algebraic structure. The Lax equation (1.1.4) is called isospectral Lax equation since we have chosen \(\lambda_t = 0\). If we take \(\lambda_t \neq 0\) with appropriately defined \(\lambda_t\), we could arrive the associated nonisospectral Lax equation. Next, we recall the iso and nonisospectral flows of KdV equation [19, 20].
1.1.1 Isospectral flows of KdV equation

The isospectral flows of KdV equation can be derived from the following equation

\[ u_t = T^n u_x, \quad (n = 0, 1, 2, \ldots) \tag{1.1.7} \]

where \( T = \partial^2 + 4u + 2u_x \partial^{-1} \) is the recursion operator. Equation (1.1.7) is associated with the following spectral problem with \( \lambda_t = 0 \)

\[
\Psi_{xx} = (\lambda - u)\Psi, \quad (1.1.8a)
\]

\[
\Psi_t = A\Psi + B\Psi_x, \quad (1.1.8b)
\]

where the functions \( A \) and \( B \) are given by [19]

\[
B = (4\lambda)^n + \sum_{j=1}^{n} 2(4\lambda)^{n-j} \partial^{-1} T^{j-1} u_x, \quad (1.1.9a)
\]

\[
A = -\frac{1}{2} B_x, \quad (1.1.9b)
\]

From (1.1.7), we arrive the isospectral flows of KdV equation as

\[
\begin{align*}
    u_t &= K_0 = u_x, \quad (1.1.10a) \\
    u_t &= K_1 = 6uu_x + u_{xxx}, \quad (1.1.10b) \\
    u_t &= K_2 = u_{xxxx} + 10uu_{xxx} + 20u_xu_{xx} + 30u^2u_x, \quad (1.1.10c) \\
    &\vdots
\end{align*}
\]

For \( n = 1 \), the spectral problem (1.1.8) reduces to spectral problem of KdV equation with Lax pairs \( L \) and \( M \) given in (1.1.5).

1.1.2 Nonisospectral flows of KdV equation

In order to derive the nonisospectral flows of KdV equation, we need to consider the spectral problem (1.1.8) with \( \lambda_t = \frac{1}{2}(4\lambda)^{n+1} \). Here, the functions \( A \) and \( B \) are given by

\[
\begin{align*}
    B &= (4\lambda)^n x + \sum_{j=1}^{n} 2(4\lambda)^{n-j} \partial^{-1} T^{j-1}(ux_x + 2u), \quad (1.1.11a) \\
    A &= -\frac{1}{2} B_x, \quad (1.1.11b)
\end{align*}
\]
and the nonsospectral flows can be derived from the following equation

\[ u_t = T^n(xu_x + 2u), \quad (n = 0, 1, 2, \cdots). \]  \tag{1.1.12}

From (1.1.12), we list first few nonsospectral flows of KdV equation as

\[ u_t = \sigma_0 = xu_x + 2u, \]  \tag{1.1.13a}

\[ u_t = \sigma_1 = xK_1 + 4uxx + 8u^2 + 2ux^{-1}u, \]  \tag{1.1.13b}

\[ u_t = \sigma_2 = xK_2 + 6K_{1,x} + 12uu_{xx} + 32u^3 + 2K_1u^{-1}u + 6ux^{-1}u^2, \]  \tag{1.1.13c}

\[ \vdots. \]

The equation (1.1.13b) is called nonisospectral KdV equation.

\section{1.2 Dispersionless equations}

In recent years, intense research activities are going on in investigating various aspects of dispersionless integrable equations [21–47]. The study of dispersionless integrable systems is an essential area of research due to their wide connection with various physical problems such as low-dimensional quantum field theory, two dimensional topological field theory, systems of hyperbolic type, etc. Initially, dispersionless systems are considered by Lebedev and Manin [21] and Zakharov [22]. This class of systems can be derived by taking quasiclassical limit in the dispersionfull integrable systems. Let us illustrate this construction by applying quasiclassical limit in the KdV equation.

Taking \( \epsilon x = X \), \( \epsilon t = T \) and thinking of \( u(x, t) = U(X, T) + O(\epsilon) \), and using this in (1.1.6) when \( \epsilon \to 0 \), we obtain

\[ U_T = 6UU_X. \]  \tag{1.2.1}

The above equation is called dispersionless KdV (Riemann) equation and it belongs to hydrodynamic type. In particular, this equation is the prototype for the hyperbolic systems. The most interesting feature of this system is also integrable and admits Lax representation, which is given by

\[ \mathcal{L}_T = \{ \mathcal{L}, \mathcal{B} \}, \]  \tag{1.2.2}
where $\mathcal{L} = p^2 + U$, $\mathcal{B} = 4p^3 + 6Up$, and \{..\} is the dispersionless Poisson bracket defined as

\[
\{A(p, X), B(p, X)\} = \frac{\partial A}{\partial p} \frac{\partial B}{\partial X} - \frac{\partial A}{\partial X} \frac{\partial B}{\partial p}.
\] (1.2.3)

Here, \( p \) is called the momentum function. To understand the solution behavior of dispersive and dispersionless equations, consider the linear portion of (1.1.6) as an example. From this, we obtain the linear dispersive equation as

\[
u_t = \nu_{xxx}.
\] (1.2.4)

The above equation admits the solution of the form

\[u(x, t) = \int dk A(k)e^{i(kx - \omega(k)t)},\]

and by looking this solution at \( t = 0 \) and \( t > 0 \) graphically, we have the following type of dispersive wave profile

\[\text{Figure 1.1: Dispersion Wave}\]

Observe that as the time goes on from the initial data \( t = 0 \), the wave gets dispersed. Next, consider the dispersionless KdV equation by removing the dispersive term from (1.1.6)

\[
u_t = 6\nu \nu_x,
\] (1.2.5)

admits the implicit solution of the form

\[u(x, t) = f(x + 6ut),\] (1.2.6)
for arbitrary $f$. From this solution, we identify the breaking wave. This is due to the fact that the velocity of a point of the wave with constant amplitude $u$, is proportional to its amplitude. Thus, we have the following wave profile:

As time evolves, the wave allows discontinuities, which is indicated by the vertical dashed line in Figure 1.2. By combining breaking and dispersive wave, we get soliton wave for the KdV equation, preserving its shape even when time evolves. This peculiar property arises because of the compensation between linear and nonlinear terms in the KdV equation. As we mentioned earlier, the soliton equations admit rich mathematical structures and interesting physical characteristics. Though, the dispersionless equations posses wave breaking type of solutions, they are also considered equally well due to their own right of importance in both mathematical and physical perspective. Several methods exist for finding solutions of dispersionless equations, among which hodograph and generalized hodograph transformations are commonly followed [31–33].

1.3 Noncommutative equations

In recent past, noncommutative (NC) integrable systems have been widely studied by various authors due to their several physical applications [48–51]. These systems are broadly classified into two types. In the first type, noncommutativity of the systems arises due to their underlying space become noncommutative, that is, the noncommutativity in the space-space and/or space-time variables, wherein the products of the variables are defined by Groenewold-Moyal product/star-product [52, 53]. In the second type, noncommutativity of the systems arises by defining the field variables over associative algebra.
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or by simply considering the field variables as they do not commute. Moreover, systems over associative algebra, in particular linear associative algebra $A$ are much simpler than systems of first type, since it carries easy computations especially in the directions of searching additional symmetries. The important examples of linear associative algebra are algebras of $n \times n$ matrices, Clifford algebras and the group algebras appearing in the representation theory of finite dimensional groups. We list few examples of NC integrable systems [54, 55] as

\[ u_t = 3uu_x + 3u_xu + u_{xxx}, \quad \text{NCKdV}. \]  \hspace{1cm} (1.3.1)

\[ u_t = 3u^2u_x + 3u_xu^2 + u_{xxx}, \quad \text{NC modified KdV}. \]  \hspace{1cm} (1.3.2)

\[ \begin{align*}
    v^k_n u^k_n &= u^{k+1}_{n+1} v^k_{n+1}, \\
    v^k_{n+1,t} + v^k_{n+1} - u^k_n &= 0,
\end{align*} \quad \text{NC semi-discrete Toda equation.} \]  \hspace{1cm} (1.3.3)

In this thesis, we consider the noncommutative equations from the second type only.

1.4 Thesis outline

This thesis is organized as follows. Chapter 2 is devoted to the study of iso and nonisospectral flows of dispersionless Kadomtsev-Petviashvili (dKP) equation. To achieve this, Lax triad approach is followed wherein the operators are replaced by phase space functions and Lie brackets by Poisson brackets. By finding suitable value of $A_m$ (phase-space function, treated as a dispersionless analogue of the nonisospectral operator $A_m$ for KP equation) at $U = 0$, the nonisospectral flows of dKP equation are obtained. Through the construction of the implicit flow representations on the iso and nonisospectral flows, the underlying infinite dimensional Lie algebraic structure of dKP equation is found with Virasoro type. The material of Chapter 2 is the original result, based on the joint work with K.M. Tamizhmani, W. Fu and D.J. Zhang [56].

In Chapter 3, we use the ideas of Chapter 2 for a dispersionless modified Kadomtsev-Petviashvili (dmKP) equation. In addition, we show that the dispersionless Miura map connects the nonisospectral flows of dKP and dmKP equations. The content of this Chapter is original and again based on the joint work with K.M. Tamizhmani, W. Fu and D.J. Zhang [56].
Chapter 4 discusses the various symmetries of (1+1) dimensional dispersionless system, called extended classical long wave system [57]. By performing Lie point symmetry analysis, different similarity reductions of this system are obtained. From the derivation of generalized and master symmetries of this system, the infinite dimensional Virasoro type Lie algebraic structure is found. Using the property of uniformity in rank, the negative ranking generalized symmetries are derived. From the construction of casimir of Poisson pencil, the conserved quantities are also obtained for this system. The content of Chapter 4 is based on the joint work with K.M. Tamizhmani and B. Dubrovin and is original [41, 58].

Chapter 5 presents the derivation of nonisospectral flows of noncommutative Kadomtsev-Petviashvili (NCKP) equation. Here, the pseudo-differential Lax operator of KP equation is considered over linear associative algebra $\mathcal{A}$. Using Lax triad approach, iso and nonisospectral flows of NCKP equation are obtained. To study the algebraic structure, implicit flow representations are presented and thus the infinite dimensional Lie algebraic structure of Virasoro type for NCKP equation is identified. The material of this Chapter is the original result, based on the joint work with K.M. Tamizhmani [59].

Chapter 6 deals the study of algebraic structure associated with the noncommutative differential-difference Kadomtsev-Petviashvili (NCD$\Delta$KP) equation from the derivation of iso and nonisospectral flows. Here, we take the pseudo-difference Lax operator of D$\Delta$KP equation over linear associative algebra $\mathcal{A}$. By setting the implicit flow representation, the infinite dimensional Lie algebraic structure is identified for NCD$\Delta$KP equation. The result of this Chapter is original and based on the joint work with L. Haung, K.M. Tamizhmani and D.J. Zhang [60].

Chapter 7 is devoted to conclude the results of the thesis and discuss the scope of further research work.