Chapter 6

Nonisospectral Flows of
Noncommutative D∆KP Equation

6.1 Introduction

In a series of papers [117–121], Date et al systematically derived soliton system and their Lax pairs for both continuous and discrete equations by using group theoretic approach. It is well-known that for KP hierarchy, Sato theory [122] is a powerful mathematical approach to derive Lax pairs, soliton solutions, conservation laws, etc. This approach has been further extended to differential-difference setup and obtained differential-difference KP (D∆KP) equation, generalized symmetries and conservation laws in [123–126]. Zhang et al [127–129] continued this analysis and found the conservation laws and solutions of D∆KP equation in a much simpler way. However, searching for various symmetries and Lie algebraic structures for continuous and lattice equations is an important topic for many years [61–68].

Recently, many studies have been carried out in finding the symmetries and their Lie algebraic structures for differential-difference systems [123–129]. Moreover, not enough is known about various integrability properties of noncommutative differential-difference soliton equations. Darboux transformation and quasideterminant solutions of non-Abelian Hirota-Miwa equation have been obtained by Nimmo [114]. In [116], Li et al derived NCD∆KP equation from the non-Abelian Hirota-Miwa equation through a cascade of
continuous limits and obtained its solution in quasideterminant form. Many further results have been obtained in [55, 130] for number of noncommutative differential-difference equations. In this Chapter, using Sato’s approach NCDΔKP equation is obtained by assuming the coefficients in the Lax operator of DΔKP equation over linear associative algebra \( A \) or simply, they do not commute. Also, from the implicit flow representations on the iso and nonisospectral flows, the infinite dimensional Lie algebraic structure of NCDΔKP equation is investigated.

Let us consider the difference [123, 124] analogue of a pseudo-differential operator,

\[
L = \Delta + \sum_{i=0}^{\infty} u_i \Delta^{-i},
\]

where \( u_i \triangleq u_i(n,y,t) \), \( t = (t_1, t_2, \cdots) \) and \( u \in A \). Here, \( \Delta \) denotes the forward difference operator defined by \( \Delta f(n) = (E - 1)f(n) = f(n + 1) - f(n) \). The operators \( \Delta \) and \( E \) are connected by \( \Delta \equiv E - 1 \) and \( \Delta\Delta^{-1} = \Delta^{-1}\Delta = 1 \). The difference operator on the product of two functions is defined by

\[
\Delta(fg) = (Ef)(\Delta g) + (\Delta f)g, \quad f, g \in A.
\]

It is important to observe that \( \Delta(fg) \neq \Delta(gf) \), when \( f \neq g \), \( \forall \, f, g \in A \). Using, difference operator, we give the Leibnitz rule [124] on the product of two functions \( f \) and \( g \) as

\[
\Delta^m(fg) = \sum_{r=0}^{\infty} \frac{m(m-1)\cdots(m-r+1)}{r!} \left( E^{m-r}\Delta^r f \right) \Delta^{m-r}g,
\]

for all positive integers \( m \).

From (6.1.2), we list the following formulae for different values \( m \) as

\[
\begin{align*}
\Delta(fg) &= Ef\Delta g + (\Delta f)g, \\
\Delta^2(fg) &= E^2f\Delta^2 g + 2(E\Delta f)g + (\Delta^2 f)g, \\
\Delta^3(fg) &= E^3f\Delta^3 g + 3(E^2\Delta f)\Delta^2 g + 3(E\Delta^2 f)\Delta g + (\Delta^3 f)g, \\
\vdots
\end{align*}
\]

\[
\begin{align*}
\Delta^{-1}(fg) &= (E^{-1}f)\Delta^{-1} g - (E^{-2}f)\Delta^{-2} g + (E^3\Delta^2 f)\Delta^{-3} g + \cdots, \\
\Delta^{-2}(fg) &= (E^{-2}f)\Delta^{-2} g - 2(E^{-3}f)\Delta^{-3} g + 3(E^4\Delta^2 f)\Delta^{-4} g + \cdots, \\
\Delta^{-3}(fg) &= (E^{-3}f)\Delta^{-3} g - 3(E^4\Delta f)\Delta^{-4} g + 6(E^5\Delta^2 f)\Delta^{-5} g + \cdots, \\
\vdots
\end{align*}
\]
In [127], Sun et al studied the underlying Lie algebraic structure of DΔKP equation by introducing new form of spectral condition. Here, we make use of the Lax triad approach to derive the flows associated with NCDΔKP equation.

### 6.2 Isospectral flows of NCDΔKP equation

Consider the isospectral problem of NCDΔKP equation with the eigenfunction \( \phi = \phi(t, \eta) \) and the spectral parameter \( \eta \) as

\[
L \phi = \eta \phi, \tag{6.2.1a}
\]

\[
\phi_y = B_1 \phi, \quad B_1 = \Delta + u_0, \tag{6.2.1b}
\]

\[
\phi_{t_m} = B_m \phi, \quad m = 1, 2, \ldots. \tag{6.2.1c}
\]

Here, the operators \( B_m \) are defined over the linear associative algebra \( \mathcal{A} \) with the form

\[
B_m = \Delta^m + \sum_{i=1}^m g_i^{(m)} \Delta^{m-i}, \quad B_m|_{u=0} = \Delta^m, \tag{6.2.2}
\]

where \( g_i^{(m)} \in \mathcal{A} \) and \( u = (u_0, u_1, u_2 \ldots) \). The compatibility conditions of (6.2.1) give the following equations

\[
L_y = [B_1, L], \tag{6.2.3a}
\]

\[
L_{t_m} = [B_m, L], \tag{6.2.3b}
\]

\[
B_{1,t_m} = B_{m,y} - [B_1, B_m]. \tag{6.2.3c}
\]

In order to derive the isospectral flows of NCDΔKP equation, first we need to find the operators \( B_m \). It is known that the operators \( B_m \) can be obtained by taking \( B_m = (L^m)_{\geq 0} \), projection of positive powers of \( \Delta \). For this purpose, we derive the various powers of \( L \) as follows

\[
L = \Delta + u_0 + u_1 \Delta^{-1} + u_2 \Delta^{-2} + u_3 \Delta^{-3} + u_4 \Delta^{-4} + \cdots,
\]

\[
L^2 = \Delta^2 + a_{21} \Delta + a_{22} + a_{23} \Delta^{-1} + a_{24} \Delta^{-2} + a_{25} \Delta^{-3} \cdots,
\]

\[
L^3 = \Delta^3 + a_{31} \Delta^2 + a_{32} \Delta + a_{33} + a_{34} \Delta^{-1} + a_{35} \Delta^{-2} \cdots,
\]

\vdots
where
\[
a_{21} = \Delta u_0 + 2u_0, \\
a_{22} = \Delta u_0 + \Delta u_1 + u_0^2 + 2u_1, \\
a_{23} = \Delta u_1 + \Delta u_2 + 2u_2 + u_0u_1 + u_1E^{-1}u_0, \\
a_{24} = \Delta u_2 + \Delta u_3 + 2u_3 + u_0u_2 - u_1E^{-2}\Delta u_0 + u_2E^{-2}u_0 + u_1E^{-1}u_1, \\
a_{25} = \Delta u_3 + \Delta u_4 + 2u_4 + u_0u_3 + u_1E^{-3}\Delta^2u_0 - 2u_2E^{-3}\Delta u_0 + u_3E^{-3}u_0 - u_1E^{-2}\Delta u_1 \\
\quad - u_1E^{-1}u_2 + u_2E^{-2}u_1, \\
\vdots \\
a_{31} = \Delta^2u_0 + 3\Delta u_0 + 3u_0, \\
a_{32} = 2\Delta^2u_0 + 3\Delta u_0 + 3u_0^2 + 2u_0\Delta u_0 + \Delta u_0u_0 + (\Delta u_0)^2 + 3u_1 + 3\Delta u_1 + \Delta^2u_1, \\
a_{33} = \Delta^2u_0 + 3u_0u_1 + 2u_1u_0 + 2u_0\Delta u_0 + (\Delta u_0)u_0 + (\Delta u_0)^2 + (\Delta u_0)\Delta u_1 + 2u_0\Delta u_1 \\
\quad + (\Delta u_1)u_0 + (\Delta u_0)u_1 + u_1E^{-1}u_0 + 2\Delta^2u_1 + 3\Delta u_1 + 3\Delta u_2 + 3u_2 + \Delta^2u_2 + u_3^2, \\
a_{34} = \Delta a_{23} + Ea_{24} + u_0a_{23} + u_1E^{-1}a_{22} + u_2E^{-2}a_{21} - u_1E^{-2}\Delta a_{21} + u_3, \\
a_{35} = \Delta a_{24} + Ea_{25} + u_0a_{24} + u_1E^{-3}\Delta^2a_{21} - u_1E^{-2}\Delta a_{22} + u_1E^{-1}a_{23} - 2u_2E^{-3}\Delta a_{21} \\
\quad - 2u_2E^{-3}\Delta a_{21} + u_2E^{-2}a_{22} + u_3E^{-3}a_{21} + u_4, \\
\vdots
\]

From the fact, \( B_m = (L^m)_{\geq 0} \), the operators \( B_m \) can be determined from the above set of equations
\[
\begin{align*}
B_1 &= \Delta + u_0, \quad (6.2.4a) \\
B_2 &= \Delta^2 + a_{21}\Delta + a_{22}, \quad (6.2.4b) \\
B_3 &= \Delta^3 + a_{31}\Delta^2 + a_{32}\Delta + a_{33}, \quad (6.2.4c) \\
\vdots
\end{align*}
\]

From (6.2.3c), we write the equation for the isospectral flows of NCD\(\Delta\)KP equation as
\[
u_{0,t_m} = B_{m,y} - [B_1, B_m], \quad m = 1, 2, \cdots. \quad (6.2.5)
\]

Now, by using (6.2.4) in (6.2.5), we arrive different cases:
Case 1
For $m = 1$ in (6.2.5), we have
\[ u_{0,t_1} = u_{0,y}, \]  
(6.2.6a)

Case 2
For $m = 2$ in (6.2.5), we get
\[ u_{0,t_2} = \Delta u_{0,y} + u_{0,y}u_0 + u_0u_{0,y} + \Delta u_{1,y} + 2u_{1,y} + [\Delta u_1, u_0] + 2[u_1, u_0] - \Delta^2 u_1 - 2\Delta u_1. \]  
(6.2.6b)

Case 3
For $m = 3$ in (6.2.5), we obtain
\[ u_{0,t_3} = a_{33,y} + \Delta^3 u_0 + a_{31} \Delta^2 u_0 + a_{32} \Delta u_0 + [a_{33}, u_0] - \Delta a_{33}. \]  
(6.2.6c)

From (6.2.3a), we can easily express $u_j (j > 0)$ in terms of $u_0$ as
\[ u_1 = \Delta^{-1} u_{0,y}, \]  
(6.2.7a)
\[ u_2 = \Delta^{-2} u_{0,yy} - \Delta^{-1} u_{0,y} - \Delta^{-1} (u_0 \Delta^{-1} u_{0,y}) + \Delta^{-1} (\Delta^{-1} u_{0,yy} E^{-1} u_0), \]  
(6.2.7b)
\[ u_3 = \Delta^{-3} u_{0,yyy} - 2\Delta^{-2} u_{0,yy} + \Delta^{-2} (\Delta^{-1} u_{0,yy} E^{-1} u_0) + \Delta^{-1} u_{0,y} \]
\[ + \Delta^{-1} (\Delta^{-2} u_{0,yy} E^{-2} u_0) - 2\Delta^{-1} (\Delta^{-1} u_{0,yy} E^{-1} u_0) \]
\[ + \Delta^{-2} (\Delta^{-1} u_0 E^{-1} u_0) - \Delta^{-2} (u_0 \Delta^{-1} u_{0,y}) \]
\[ + \Delta^{-1} (\Delta^{-1} u_{0,yy} E^{-1} u_0) E^{-2} u_0 - \Delta^{-2} (u_0 \Delta^{-1} u_{0,yy}) \]
\[ - \Delta^{-1} (u_0 \Delta^{-1} u_{0,yy}) E^{-2} u_0 - \Delta^{-1} (u_0 \Delta^{-2} u_{0,yy}) \]
\[ + 2\Delta^{-1} (u_0 \Delta^{-1} u_{0,y}) - \Delta^{-1} (u_0 \Delta^{-1} (\Delta^{-1} u_{0,yy} E^{-1} u_0)) \]
\[ + \Delta^{-1} (u_0 \Delta^{-1} (u_0 \Delta^{-1} u_{0,y})), \]  
(6.2.7c)
\[ \vdots \]

By taking $u_0 = u$ and substituting the above set of equations in (6.2.6a), (6.2.6b), (6.2.6c), $\cdots$, we obtain the isospectral flows of NCD$\Delta$KP equation as
\[ u_{t_1} = K_1 = u_y, \]  
(6.2.8a)
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\( u_{t_2} = K_2 = (1 + 2\Delta^{-1})u_{yy} - 2u_y + 2u_yu + 2[\Delta^{-1}u_y, u], \quad (6.2.8b) \)

\( u_{t_3} = K_3 = (3\Delta^{-2} + 3\Delta^{-1} + 1)u_{yyy} + 3\Delta^{-1}(u_y^2) + 3\Delta^{-1}(u_{yy})u - 6\Delta^{-1}u_{yy} + 3u_y \\
+ 3\Delta^{-1}(u_y)u_y - 3u_{yy}u - 3u_y + 3u_y^2 + 3u_yu^2 - 6u_yu \\
+ [Eu, u_y] + 2[\Delta u, u_y] + 3[\Delta^{-2} + \Delta^{-1})u_{yy}, u] + 3[\Delta^{-1}u_yE^{-1}u, u] \\
+ 3[\Delta^{-1}u_y, u]u + 6[u, \Delta^{-1}u_y] + 3\Delta^{-1}[\Delta^{-1}u_{yy}, u] + 3\Delta^{-1}[\Delta^{-1}u_y, u_y] \\
+ 3[\Delta^{-1}(\Delta^{-1}u_yE^{-1}u), u] + 3[u, \Delta^{-1}(\Delta^{-1}u_yu)], \quad (6.2.8c) \)

Note that the equation (6.2.8b) is the isospectral NCD\( D\Delta KP \) equation together with (6.2.8a), (6.2.8c), \( \cdots \) form the isospectral flows of NCD\( D\Delta KP \) equation [60].

### 6.3 Nonisospectral flows of NCD\( D\Delta KP \) equation

By considering the spectral parameter \( \eta(t_1, t_2, t_3, \cdots) \), Sun et al [127], presented the condition on the spectral parameter as \( \eta_{t_m} = \eta^m + \eta^{m-1} \) for the nonisospectral flows of \( D\Delta KP \) equation. In the case of NCD\( D\Delta KP \) equation, we derive the same condition on the spectral parameter. Here, it is important to note that the condition on the nonisospectral operator \( A_m \) for NCD\( D\Delta KP \) equation at \( u = 0 \) is slightly different. More details can be found in the Appendix B. Now, we make use of Lax triad approach to consider the spectral problem associated with the nonisospectral flows of NCD\( D\Delta KP \) equation as follows

\[ L\phi = \eta\phi, \quad (6.3.1a) \]

\[ \phi_y = B_1\phi, \quad (6.3.1b) \]

\[ \phi_{t_m} = A_m\phi, \quad m = 1, 2, \cdots, \quad (6.3.1c) \]

where the operators \( A_m, \ m \in \mathbb{Z}^+ \) are of the form

\[ A_m = \sum_{i=0}^{m} h_i^{(m)} \Delta^{m-i}, \quad h_i^{(m)} \in \mathcal{A}, \quad (6.3.2) \]

with

\[ A_m|_{u=0} = y\Delta^m + (y + n)\Delta^{m-1}, \quad u = (u_0, u_1, u_2, \cdots). \quad (6.3.3) \]
From the fact $\eta_m = \eta^m + \eta^{m-1}$, the compatibility condition of (6.3.1) reads the following equations

\begin{align*}
L_y &= [B_1, L], \quad (6.3.4a) \\
L_{t_m} &= [A_m, L] + L^m + L^{m-1}, \quad (6.3.4b) \\
B_{1,t_m} &= A_{m,y} - [B_1, A_m]. \quad (6.3.4c)
\end{align*}

In order to derive the nonisospectral flows, we need to construct the operators $A_m$ in (6.3.2) explicitly. This is achieved by substituting (6.3.2) in (6.3.4b) along with (6.3.3), we can find the functions $h_i^{(m)}$ in $A_m$. As we discussed in the previous Chapter 5, let us illustrate this construction by taking $m = 2$ in (6.3.2). Thus, we write

$$A_2 = h_0^{(2)} \Delta^2 + h_1^{(2)} \Delta + h_2^{(2)}.$$  

Substituting (6.3.5) in (6.3.4b) and collecting the various coefficients of $\Delta^i$, we get

\begin{align*}
\Delta^3 : \quad &h_0^{(2)} - Eh_0^{(2)} = 0, \quad (6.3.6a) \\
\Delta^2 : \quad &h_0^{(2)} E^2 u_0 + h_1^{(2)} - \Delta h_0^{(2)} - Eh_1^{(2)} - u_0 h_0^{(2)} + 1 = 0, \quad (6.3.6b) \\
\Delta^1 : \quad &2h_0^{(2)} E \Delta u_0 + h_0^{(2)} E^2 u_1 + h_1^{(2)} E u_0 + h_2^{(2)} - \Delta h_1^{(2)} \\
&- Eh_2^{(2)} - u_0 h_1^{(2)} - u_1 E^{-1} h_0^{(2)} + \Delta u_0 + 2u_0 + 1 = 0, \quad (6.3.6c) \\
&\vdots
\end{align*}

Let us take $h_0^{(2)} = y$ as in the case of NCKP equation and from (6.3.6b), we arrive

$$\Delta h_1^{(2)} = y E^2 u_0 - y u_0 + 1.$$  

Operating $\Delta^{-1}$ on both sides of the above equation, and using (6.3.3), we have

$$h_1^{(2)} = y \Delta^{-1} E^2 u_0 - y \Delta^{-1} u_0 + n + y.$$  

Substituting $h_0^{(2)}$ and $h_1^{(2)}$ in (6.3.6c), we obtain

$$\Delta h_2^{(2)} = y \left( \Delta^2 u_0 + \Delta^2 u_1 + 2 \Delta u_1 + \Delta u_0 + (\Delta u_0)^2 + u_0 \Delta u_0 + (\Delta u_0) u_0 \right) + n \Delta u_0 + 2u_0 + \Delta u_0.$$  

Taking $\Delta^{-1}$ on both sides of the above equation, and using (6.3.3), we arrive

$$h_2^{(2)} = y \left( \Delta u_0 + \Delta u_1 + 2u_1 + u_0 + u_0^2 \right) + nu_0 + \Delta^{-1} u_0.$$
Now, by using $h_0^{(2)}$, $h_1^{(2)}$, and $h_2^{(2)}$ in (6.3.5), we get

$$A_2 = y\Delta^2 + (y\Delta^{-1}E^2u_0 - y\Delta^{-1}u_0 + n + y) \Delta + y(\Delta u_0 + \Delta u_1 + 2u_1 + u_0 + u_0^2) + nu_0 + \Delta^{-1}u_0.$$ 

Repeating the same procedure, we list $A_m$ in terms of $B_k$ ($k = 1, 2, \cdots$) as

$$A_1 = yB_1 + y + n,$$

$$A_2 = yB_2 + yB_1 + nB_1 + \Delta^{-1}u_0,$$

$$A_3 = yB_3 + yB_2 + nB_2 + \Delta^{-1}u_0\Delta + u_0\Delta^{-1}u_0 + \Delta^{-1}(2u_1 - u_0 + u_0^2),$$

$$\vdots.$$ 

From (6.3.4c), we write the equation for the nonisospectral flows of NCD$\Delta$KP equation as

$$u_{0,t_m} = A_{m,y} - [B_1, A_m], \quad m = 1, 2, 3, \cdots.$$ 

Next, consider the various values of $m$ in (6.3.7), we have the following cases:

**Case 1**

Substituting $m = 1$ in (6.3.7), we obtain

$$u_{0,t_1} = yu_{0,y} + u_0.$$ 

**Case 2**

Substituting $m = 2$ in (6.3.7), we get

$$u_{0,t_2} = y(\Delta u_{0,y} + u_{0,y}u_0 + u_0u_{0,y} + \Delta u_{1,y} + 2u_{1,y} + [\Delta u_1, u_0] + 2[u_1, u_0]$$

$$-\Delta^2 u_1 - 2\Delta u_1) + yu_{0,y} + nu_{0,y} + u_0^2 - u_0 + \Delta u_1 + 2u_1 + \Delta^{-1}u_{0,y}$$

$$+ [\Delta^{-1}u_0, u_0].$$ 

**Case 3**

Substituting $m = 3$ in (6.3.7), we arrive

$$u_{0,t_3} = y(a_{33,y} + \Delta^3 u_0 + a_{31}\Delta^2 u_0 + a_{32}\Delta u_0 + [a_{33}, u_0] - \Delta a_{33})$$

$$+ (y + n)(a_{22,y} + \Delta^2 u_0 + a_{21}\Delta u_0 + [a_{22}, u_0] - \Delta a_{22}) - \Delta^2 u_0 - \Delta(u_0^2)$$

$$- \Delta^2 u_1 - 3\Delta u_1 - \Delta u_0 - 3u_0^2 - 4u_1 + u_0[\Delta^{-1}u_0, u_0] - \Delta u_0u_0.$$
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\[ + [\Delta^{-1}u_0, \Delta u_0] + 2[\Delta^{-1}u_1, u_0] + [u_0, \Delta^{-1}u_0] + u_0 + [\Delta^{-1}u_0^2, u_0] + a_{33} + a_{22} \\
+ u_{0,y}[\Delta^{-1}u_0 + u_0[\Delta^{-1}u_{0,y} + 2\Delta^{-1}u_{1,y} - \Delta^{-1}u_{0,y}] + \Delta^{-1}(u_{0,y}u_0) + \Delta^{-1}(u_0u_{0,y})]. \]

(6.3.8c)

Now, by using the relations (6.2.7) in (6.3.8a), (6.3.8b), \ldots, we have ($u_0 = u$)

\[
\begin{align*}
    u_{t_1} &= \sigma_1 = yK_1 + u, \\
    u_{t_2} &= \sigma_2 = yK_2 + (y + n)K_1 + u_y + 3\Delta^{-1}u_y + u^2 - u + [\Delta^{-1}u, u], \\
    u_{t_3} &= \sigma_3 = yK_3 + (y + n)K_2 + [u, \Delta u] - 2u_y - 2u^2 - 6\Delta^{-1}u_y + u[\Delta^{-1}u, u] \\
    &+ [\Delta^{-1}u, \Delta u] + 2[\Delta^{-2}u_{y,y}, u] + [u, \Delta^{-1}u] + u + [\Delta^{-1}u^2, u] \\
    &+ u_y[\Delta^{-1}u + u\Delta^{-1}u_y + 5\Delta^{-2}u_{y,y} + \Delta^{-1}(u_yu) + \Delta^{-1}(uu_y)] \\
    &+ 3\Delta^{-1}u_yu + uu_y + 2u_yu + 3\Delta^{-1}u_{y,y} + 3\Delta^{-1}u_yE^{-1}u \\
    &- 3\Delta^{-1}(u\Delta^{-1}u_y) + 3\Delta^{-1}(\Delta^{-1}u_yE^{-1}u) + uu_y + u^3, \\
    \vdots
\end{align*}
\]

Equation (6.3.9b) is the nonisospectral NCD\$\Delta\$KP equation together with (6.3.9a), (6.3.9c), \ldots, form the nonisospectral flows of NCD\$\Delta\$KP equation [60].

6.4 Algebraic structure of NCD\$\Delta\$KP equation

Consider the differential-difference polynomial $D[n, x, u(n, x)]$ over $A$. Here, $x = (y, t)$. The $A$ valued evolution equation is of the form

\[ u_t = K(u), \quad K \in D. \] (6.4.1)

For a differential-difference operator,

\[ P(u) = \sum_{j \leq s} p_j(u)\Delta^j, \] (6.4.2)

its Gâteaux derivative in the direction of $h$ with respect to $u$ is defined by

\[ P'[h] = \sum_{j \leq s} p'_j[h]\Delta^j. \] (6.4.3)
The obtained isospectral and nonisospectral flows can be expressed through the following form of implicit flow representations with \( u_0 = u \) as

**Isospectral case:**

\[
K_s = B_{s,y} - [B_1, B_s],
\]

\[
B_s|_{u=0} = \Delta^s.
\]

**Nonisospectral case:**

\[
\sigma_r = A_{r,y} - [B_1, A_r],
\]

\[
A_r|_{u=0} = y\Delta^r + (y + n)\Delta^{r-1}.
\]

Next, we consider the following lemma.

**Lemma 12.** [60] Suppose the difference operator \( R \) is in the form of \( \sum_{i=0}^{k} a_i \Delta^{m-i} \) and \( X = X(u) \in D \), then

\[
X = R_y - [B_1, R], \quad R|_{u=0} = 0,
\]

admits only zero solution \( X = 0, \ R = 0 \). Here, \( B_1 = \Delta + u \), where we have taken \( u_0 = u \).

**Proof**

Comparing the coefficient of highest power of \( \Delta \) in (6.4.6), we get \( a_0 = 0 \). Proceeding this way, we obtain \( a_i = 0 \), for \( i = 1, 2, \cdots, k \). This yields \( R = 0 \) and consequently \( X = 0 \).

**Theorem 10.** [60] If we define the following kind of products

\[
\langle B_s, B_r \rangle = B'_s[K_r] - B'_r[K_s] + [B_s, B_r],
\]

\[
\langle B_s, A_r \rangle = B'_s[\sigma_r] - A'_r[K_s] + [B_s, A_r],
\]

\[
\langle A_s, A_r \rangle = A'_s[\sigma_r] - A'_r[\sigma_s] + [A_s, A_r],
\]

then we have

\[
[K_s, K_r] = \langle B_s, B_r \rangle y - [B_1, \langle B_s, B_r \rangle],
\]

\[
[K_s, \sigma_r] = \langle B_s, A_r \rangle y - [B_1, \langle B_s, A_r \rangle],
\]

\[
[\sigma_s, \sigma_r] = \langle A_s, A_r \rangle y - [B_1, \langle A_s, A_r \rangle],
\]
and

\[ \langle B_s, B_r \rangle |_{u=0} = 0, \quad (6.4.9a) \]
\[ \langle B_s, A_r \rangle |_{u=0} = s(\Delta^{s+r-1} + \Delta^{s+r-2}), \quad (6.4.9b) \]
\[ \langle A_s, A_r \rangle |_{u=0} = (s-r)[y\Delta^{s+r-1} + (2y+n)\Delta^{s+r-2} + (y+n)\Delta^{s+r-3}], \quad (6.4.9c) \]

**Proof.** We only prove the equations (6.4.8c) and (6.4.9c). The others can be obtained in a similar manner. Let us take the L.H.S of (6.4.8c) and establish its R.H.S. For this purpose, L.H.S of (6.4.8c) can be expressed in the form

\[ \left[ \sigma_s, \sigma_r \right] = (\sigma_s)'[\sigma_r] - (\sigma_r)'[\sigma_s]. \quad (6.4.10) \]

First, let us take the Gateaux derivative of \( \sigma_s \) in the direction of \( \sigma_r \) and it can be expressed as

\[
(\sigma_s)'[\sigma_r] = (A_{s,y} - [B_1, A_s])'[\sigma_r],
\]
\[
= (A_{s,y})'[\sigma_r] - (B_1)'[\sigma_r]A_s - B_1(A_s)'[\sigma_r] + (A_s)'[\sigma_r]B_1 + A_s(B_1)'[\sigma_r],
\]
\[
= (A_{s,y})'[\sigma_r] - \sigma_rA_s - B_1(A_s)'[\sigma_r] + (A_s)'[\sigma_r]B_1 + A_s[\sigma_r],
\]
\[
= (A_{s,y})'[\sigma_r] - (A_{r,y} - [B_1, A_r])A_s - B_1(A_s)'[\sigma_r] + (A_s)'[\sigma_r]B_1 + A_s[A_{r,y} - [B_1, A_r]],
\]
\[
= (A_{s,y})'[\sigma_r] - A_{r,y}A_s - B_1(A_s)'[\sigma_r] + (A_s)'[\sigma_r]B_1 + A_sA_{r,y}
\]
\[
+ [B_1, A_r, A_s].
\]

Similarly, we have

\[
(\sigma_r)'[\sigma_s] = (A_{r,y})'[\sigma_s] - A_{s,y}A_r - B_1(A_r)'[\sigma_s] + (A_r)'[\sigma_s]B_1 + A_r[A_{s,y}]
\]
\[
+ [B_1, A_s, A_r].
\]

Thus, (6.4.10) becomes

\[
\left[ \sigma_s, \sigma_r \right] = (A_{s,y})'[\sigma_r] - (A_{r,y})'[\sigma_s] + A_{s,y}A_r + A_s[A_{r,y} - A_rA_s] - A_r[A_{s,y}]
\]
\[
+ B_1(A_r)'[\sigma_s] - B_1(A_s)'[\sigma_r] + (A_s)'[\sigma_r]B_1 - (A_r)'[\sigma_s]B_1
\]
\[
+ [B_1, A_r, A_s] - [B_1, A_s, A_r]. \quad (6.4.11)
\]
Observe that

\[(A_m)'[\sigma_n])_y = (A_{m,y})'[\sigma_n], \quad \forall \ m, n \in \mathbb{Z}^+ \text{ is true,}\]

and by making use of the Jacobi identity, we have

\[[[A_s, A_r], B_1] = [[[B_1, A_r], A_s] - [[B_1, A_s], A_r]].\]

Using the above identities in (6.4.11), we get

\[J_{\sigma_s, \sigma_r}K_s = (A_s)'[\sigma_r] - (A_r)'[\sigma_s] + [A_s, A_r]_y - B_1(A_s)'[\sigma_r] + B_1(A_r)'[\sigma_s] + (A_s)'[\sigma_r]B_1 - (A_r)'[\sigma_s]B_1 - [B_1, [A_s, A_r]],\]

Thus, (6.4.8c) holds good. Next, note that \(\sigma_r|_{u=0} = 0, \forall r \geq 1.\) Using this fact in (6.4.7c) along with (6.4.5b), we obtain (6.4.9c) immediately. This completes the proof. \(\square\)

With the above theorem in hand, the algebraic relations among the flows \(K_s\) and \(\sigma_r\) can be derived from the following theorem.

**Theorem 11.** [60] The flows \{\(K_s\)\} and \{\(\sigma_r\)\} form a Lie algebra with structure

\[
\begin{align*}
[K_s, K_r] &= 0, \quad (6.4.12a) \\
[K_s, \sigma_r] &= sK_{s+r-1} + sK_{s+r-2}, \quad (6.4.12b) \\
[\sigma_s, \sigma_r] &= (s - r)(\sigma_{s+r-1} + \sigma_{s+r-2}), \quad (6.4.12c)
\end{align*}
\]

where, \(s, r \geq 1\) and we set \(K_0 = \sigma_0 = 0.\)

**Proof.** Using (6.4.6), it is easy to prove (6.4.12a). By considering (6.4.8a) with (6.4.9a) and using the Lemma 12, we can show that (6.4.12a) is true. Next, let us take

\[
\begin{align*}
\theta &= [K_s, \sigma_r] - sK_{s+r-1} - sK_{s+r-2}, \quad (6.4.13a) \\
\tilde{B} &= \langle B_s, A_r \rangle - sB_{s+r-1} - sB_{s+r-2}. \quad (6.4.13b)
\end{align*}
\]

From (6.4.8b) and (6.4.9b) and the isospectral implicit representation (6.4.4a) and (6.4.4b), one can show that

\[
\theta = \tilde{B}_y - [B_1, \tilde{B}], \quad \tilde{B}|_{u=0}, \quad (6.4.14)
\]
holds. From Lemma 12, the above equation admits only zero solution $\theta = 0$ and $\tilde{B} = 0$. Therefore, (6.4.12b) is true.

Similarly, take

$$\omega = [\sigma_s, \sigma_r] - (s - r)(\sigma_{s+r-1} + \sigma_{s+r-2}),$$

$$\tilde{A} = \langle A_s, A_r \rangle - (s - r)(A_{s+r-1} + A_{s+r-2}),$$

(6.4.15)  

(6.4.16)

together with (6.4.8c) and (6.4.9c) and the implicit flow representation (6.4.5a) and (6.4.5b), we arrive

$$\omega = \tilde{A}_y - [B_1, \tilde{A}], \quad \tilde{A}|_{u=0} = 0.$$  

(6.4.17)

Now, from Lemma 12, we get $\omega = 0$ and $\tilde{A} = 0$, which implies that (6.4.12c) is also correct. Thus, we complete the proof. \qed