Chapter 4

 Bounds for Wiener index and RC-Wiener index of graphs
4.1 Introduction

The Wiener number $W(G)$ of a graph $G$ is defined as the sum of the distances between all unordered pairs of vertices of $G$ [139], that is,

$$W(G) = \sum_{1 \leq i < j \leq n} d(v_i, v_j).$$

(4.1)

In the family of Wiener-like molecular descriptors [134], reciprocal complementary Wiener index is the newest addition. It has been introduced by Ivanciuc [83] and debated by Ivanciuc et al. [83, 84, 85, 86].

The reciprocal complementary Wiener index of a graph $G$ is denoted by $RCW(G)$ and is defined as follows

$$RCW(G) = \sum_{1 \leq i < j \leq n} \frac{1}{1 + D - d(v_i, v_j)}.$$  

(4.2)

where $D$ is the diameter of $G$.

This graph invariant has been successfully applied in the structure-property modeling of the molar heat capacity, standard Gibbs energy of formation and vaporization enthalpy of 134 alkanes $C_6 - C_{10}$ [83, 155]. For some recent results on this index we refer the reader to see [23, 152] and for surveyed paper [145].

The distance number $d(v_i|G)$ and reciprocal complementary distance number $RCD(v_i|G)$ of a vertex $v_i$ of $G$, are defined as

$$d(v_i|G) = \sum_{v_j \in V(G)} d(v_i, v_j)$$

(4.3)
and

\[ RC\!D(v_i|G) = \sum_{v_j \in V(G)} \frac{1}{1 + D - d(v_i, v_j)}. \quad (4.4) \]

From the Eqs. (4.3) and (4.4) we can rewrite the definitions of \( W(G) \) and \( RCW(G) \), which are as follows

\[ W(G) = \frac{1}{2} \sum_{v_i \in V(G)} d(v_i|G), \quad RCW(G) = \frac{1}{2} \sum_{v_i \in V(G)} RC\!D(v_i|G). \quad (4.5) \]

In [152] Zhou et al. determined various lower and upper bounds for the \( RCW \) index and also given Nordhaus Gaddum type results for the same. Further more, Xu et al. [144] concluded that for any tree on \( n \) vertices, the path \( P_n \) and \( S_n = K_{1,n-1} \) are the least value and maximum value for the \( RCW \) index respectively. That is

\[ RCW(P_n) \leq RCW(T) \leq RCW(S_n). \quad (4.6) \]

In the continuation of the study on \( RCW \) index Qi and Zhou [116] characterize the trees of fixed number of vertices and matching number with the smallest \( RCW \) index, and the non-caterpillars on \( n \geq 7 \) vertices with the smallest, the second-smallest and the third-smallest \( RCW \) index.

The present chapter provides the correct version of the Theorem 7 of [137] related to the Wiener index of a graph interms of the chromatic number and lower and upper bounds for \( RCW \) in terms of some graph parameters like \( n, m, D, \) vertex connectivity \( (k) \), independence number \( (\beta_0) \), independence domination number \( (\gamma_0) \), and
chromatic number(\(\chi\)). Furthermore we obtained sharp lower bounds for the \(rCW\) index by means of \(\delta, \triangle, \triangle^*\).

**Example 4.1.1.**

![Graph diagram]

*Fig. 4.1*

From the Example 4.1.1, we get to know that the graph has diameter \(D = 2\) and the distance between each vertex is

\[
d(v_1, v_2) = 1, \quad d(v_1, v_3) = 2, \quad d(v_1, v_4) = 2, \quad d(v_1, v_5) = 2, \\
d(v_2, v_1) = 1, \quad d(v_2, v_3) = 1, \quad d(v_2, v_4) = 1, \quad d(v_2, v_5) = 1, \\
d(v_3, v_1) = 2, \quad d(v_3, v_2) = 1, \quad d(v_3, v_4) = 1, \quad d(v_3, v_5) = 2, \\
d(v_4, v_1) = 2, \quad d(v_4, v_2) = 1, \quad d(v_4, v_3) = 1, \quad d(v_4, v_5) = 1, \\
d(v_5, v_1) = 2, \quad d(v_5, v_2) = 1, \quad d(v_5, v_3) = 2, \quad d(v_5, v_4) = 1.
\]

Therefore the distance number \(d(v_i|G)\) and the reciprocal complimentary distance number \(RCD(v_i|G)\) of the vertices \(v_1, v_2, \ldots, v_5\) are as follows

\[
d(v_1|G) = 7, \quad d(v_2|G) = 4, \quad d(v_3|G) = 6, \\
d(v_4|G) = 5, \quad d(v_5|G) = 6.
\]
$RCD(v_1|G) = 3.5, \; RCD(v_2|G) = 2, \; RCD(v_3|G) = 3,$
$RCD(v_4|G) = 2.5, \; RCD(v_5|G) = 3.$

Then from Eq. (4.5) we get $W(G) = 14$ and $RCW(G) = 7$.

4.2 Bounds for $W(G)$ and $RCW(G)$

**Theorem 4.2.1.** [137] Let $G$ be any connected graph of order $n$ with chromatic number $\chi(G) = t$. Then

$$W(G) \geq \frac{n(t + 1) - 2t}{2}. \quad (4.7)$$

Further the equality holds if and only if $G \cong K_{n_1,n_2,\ldots,n_t}$.

Unfortunately some mistakes are found in the proof of the Theorem 4.2.1 and hence this theorem is not correct. Among several examples, one of the counterexample to the equality of Eq. (4.7) is the complete multipartite graph $K_{1,1,2}$. For which $W(K_{1,1,2}) = 7$ and $(n(t + 1) - 2t)/2 = 5$.

The following theorem gives the correct statement of the Theorem 4.2.1

**Theorem 4.2.2.** Let $G$ be a connected graph of order $n$ with chromatic number $\chi(G) = t$. Let $C_1, C_2, \ldots, C_t$ be the color classes of $G$, where $|C_i| = n_i$, $i = 1, 2, \ldots, t$. Then

$$W(G) \geq \frac{n(n - 2)}{2} + \frac{1}{2} \sum_{i=1}^{t} n_i^2. \quad (4.8)$$
with equality holds if and only if $G \cong K_{n_1, n_2, \ldots, n_t}$.

Proof. Vertex set $V(G)$ can be partitioned into $t$ color classes $C_1, C_2, \ldots, C_t$ where $|C_i| = n_i$, $i = 1, 2, \ldots, t$ and $n = \sum_{i=1}^{t} n_i$. If the vertices $v_i, v_j \in C_i$, then $d(v_i, v_j) \geq 2$ and if $v_i \in C_i$ and $v_j \in V(G) - C_i$, then $d(v_i, v_j) \geq 1$.

Let $v_i \in C_i$, $i = 1, 2, \ldots, t$. Then

$$d(v_i | G) = \sum_{v_j \in V(G)} d(v_i, v_j)$$
$$= \sum_{v_j \in C_i} d(v_i, v_j) + \sum_{v_j \in V(G) - C_i} d(v_i, v_j)$$
$$\geq 2(n_i - 1) + (n - n_i)$$
$$= n + n_i - 2.$$ 

Therefore

$$W(G) = \frac{1}{2} \sum_{v_i \in V(G)} d(v_i | G)$$
$$= \frac{1}{2} \sum_{i=1}^{t} \sum_{v_i \in C_i} d(v_i | G)$$
$$\geq \frac{1}{2} \sum_{i=1}^{t} \sum_{v_i \in C_i} (n + n_i - 2)$$
$$= \frac{1}{2} \sum_{i=1}^{t} n_i(n + n_i - 2)$$
$$= \frac{1}{2} \left[ (n - 2) \sum_{i=1}^{t} n_i + \sum_{i=1}^{t} n_i^2 \right]$$
$$= \frac{n(n - 2)}{2} + \frac{1}{2} \sum_{i=1}^{t} n_i^2.$$
For equality:

If \( G = K_{n_1,n_2,\ldots,n_k} \), then \( W(G) = \frac{n(n-2)}{2} + \frac{1}{2} \sum_{i=1}^{t} n_i^2 \).

On the other hand, if \( W(G) = \frac{n(n-2)}{2} + \frac{1}{2} \sum_{i=1}^{t} n_i^2 \), then the vertex set \( V(G) \) can be partitioned into \( t \) color classes \( C_1, C_2, \ldots, C_t \), where \( |C_i| = n_i, \ i = 1, 2, \ldots, t \). We claim that if \( v_i \in C_i \) and \( v_j \in C_j, \ i \neq j \) then \( v_i \) and \( v_j \) are adjacent. For, if \( v_i \) and \( v_j \) are not adjacent, where \( v_i \in C_i \) and \( v_j \in C_j, \ i \neq j \), then \( \sum_{v_j \in V(G)-C_i} d(v_i, v_j) > n - n_i \). This implies that

\[
\begin{align*}
\quad d(v_i | G) &= \sum_{v_j \in V(G)} d(v_i, v_j) \\
&= \sum_{v_j \in C_i} d(v_i, v_j) + \sum_{v_j \in V(G)-C_i} d(v_i, v_j) \\
&> 2(n_i - 1) + (n - n_i) \\
&= n + n_i - 2.
\end{align*}
\]

Therefore

\[
W(G) > \frac{n(n-2)}{2} + \frac{1}{2} \sum_{i=1}^{t} n_i^2,
\]

a contradiction.

Again, if \( v_i \) and \( v_j \) belongs to the same color class, then \( d(v_i, v_j) = 2 \). For, if \( v_i \) and \( v_j \) are adjacent in \( C_i \), then \( \sum_{v_j \in C_i} d(v_i, v_j) = n_i - 1 \).
Therefore

\[ d(v_i|G) = \sum_{v_j \in V(G)} d(v_i, v_j) = \sum_{v_j \in C_i} d(v_i, v_j) + \sum_{v_j \in V(G) - C_i} d(v_i, v_j) > (n_i - 1) + (n - n_i) = n - 1. \]

Therefore

\[ W(G) = \frac{1}{2} \sum_{v_i \in V(G)} d(v_i|G) > \frac{n(n - 1)}{2}, \]

again a contradiction.

Hence \( G \cong K_{n_1, n_2, \ldots, n_t} \). \( \square \)

We have given another proof for the following Theorem 4.2.3 and this assist to improve our succeeding results.

**Theorem 4.2.3.** [152] Let \( G \) be a non-complete connected graph of order \( n \) and size \( m \). Then

\[ RCW(G) = \frac{n(n - 1) - m}{2}. \]

only if \( D = 2 \).

**Proof.** Suppose \( G \) be a graph of order \( n \) and size \( m \) with \( D = 2 \). Define the sets \( A = \{ v_i \in V | e(v_i) = 1 \} \) and \( B = \{ v_i \in V | e(v_i) = 2 \} \). Then
$|A| + |B| = n$. If $v_i \in A$, then $\text{RCD}(v_i|G) = \frac{n-1}{D}$ and if $v_i \in B$, then define two sets $B_1$ and $B_2$ as

$$B_1 = \{v_j \in V | 1/1 + D - d(v_i, v_j) = 1/D\},$$

$$B_2 = \{v_j \in V | 1/1 + D - d(v_i, v_j) = 1/D - 1\}.$$

Then,

$$\text{RCD}(v_i|G) = \frac{|B_1|}{D} + \frac{|B_2|}{D - 1}$$

$$= |B_1| + |B_2| + \frac{|B_1|}{D} - |B_1| + \frac{B_2}{D - 1} - |B_2|$$

$$= n - 1 + \frac{1 - D}{D}|B_1| + \frac{(2 - D)(n - 1 - |B_1|)}{D - 1}$$

$$= \frac{Dn - D - d_G(v_i)}{D(D - 1)}. \quad (4.9)$$

Since $D = 2$. Therefore,

$$\text{RCD}(v_i|G) = \frac{2n - 2 - d_G(u)}{2}.$$
Then,

\[ RCW(G) = \frac{1}{2} \sum_{v_i \in V} RCD(v_i | G) \]

\[ = \frac{1}{2} \left[ \sum_{v_i \in A} RCD(v_i | G) + \sum_{v_i \in B} RCD(v_i | G) \right] \]

\[ = \frac{1}{2} \left[ \left( \frac{n-1}{2} \right) (|A| + |B|) \right. \]

\[ + \left. \left( \frac{n-1}{2} \right) |B| - \sum_{v_i \in B} \frac{d_G(v_i)}{2} \right] \]

\[ = \frac{1}{4} \left[ n(n-1) + (n-1)(n-|A|) - \sum_{v_i \in B} d_G(v_i) \right] \]

\[ = \frac{1}{4} \left[ 2n(n-1) - \sum_{v_i \in A} d_G(v_i) - \sum_{v_i \in B} d_G(v_i) \right] \]

\[ RCW(G) = \frac{n(n-1) - m}{2}. \]

\[ \square \]

**Theorem 4.2.4.** Let \( G \) be a connected graph of order \( n \), size \( m \) with \( D \geq 3 \),

\[ RCW(G) \geq \left[ \frac{D^2 n(n-1) - 2m(D-2) - 2Dn(n-1) + 2D}{2D(D-1)(D-2)} \right]. \]

Equality holds if \( G \) contains exactly two vertices of eccentricity three and rest are of eccentricity two.

**Proof.** For, if \( D = k \geq 3 \) and let \( v_i \in V \) for \( i = 1, 2, \ldots, n \) be an arbitrary vertex in \( G \), then define the sets \( A \) and \( B \) as \( A = \{ v_i \in V | e(v_i) = 2 \} \), and \( B = \{ v_i \in V | e(v_i) \geq 3 \} \), where \( |A| + |B| = n \).

If \( v_i \in A \), then we will get Eq. (4.9) and If \( v_i \in B \), define three sets
$B_1$, $B_2$, $B_3$ as follows:

\[
B_1 = \{v_j \in V | 1/1 + D - d(v_i, v_j) = 1/D \},
\]

\[
B_2 = \{v_j \in V | 1/1 + D - d(v_i, v_j) = 1/D - 1 \},
\]

\[
B_3 = \{v_j \in V | 1/1 + D - d(v_i, v_j) \geq 1/D - 2 \}.
\]

Clearly $|B_1| + |B_2| + |B_3| = n - 1$.

\[
RCD(v_i | G) \geq \frac{|B_1|}{D} + \frac{|B_2|}{D-1} + \frac{|B_3|}{D-2}
\]

\[
= |B_1| + |B_2| + |B_3| + \frac{|B_1|}{D} - |B_1|
\]

\[
+ \frac{|B_2|}{D-1} - |B_2| + \frac{|B_3|}{D-2} - |B_3|
\]

\[
= n - 1 + \frac{d_G(v_i)}{D} - d_G(v_i) + \left( \frac{2 - D}{D-1} \right) |B_2|
\]

\[
+ \left( \frac{3 - D}{D-2} \right) |B_3|
\]

\[
= \frac{D^2 n - D^2 - 2Dn + 3D - Dd_G(v_i) + 2d_G(v_i)}{D(D-1)(D-2)}.
\]
Therefore

\[
RCW(G) = \frac{1}{2} \left\{ \frac{(Dn-D_dG(v_i))|A|}{D(D-1)} + \frac{D^2n^2-2Dn+3D-D_dG(v_i)+2d_G(v_i)}{D(D-1)(D-2)}|B| \right\}
\]

\[
= \frac{(D^2n - D^2)(|A| + |B|) - D \sum_{v_i \in V} d_G(v_i) - 2Dn(|A| + |B|) + 2D|A| + 3D|B| + 2 \sum_{v_i \in V} d_G(v_i)}{2D(D-1)(D-2)}
\]

\[
= \frac{n(D^2n - D^2) - 2mD - 2Dn^2 + 2D(|A| + |B|) + D|B| + 4m}{2D(D-1)(D-2)}
\]

\[
= \frac{D^2n(n-1) - 2m(D-2) - 2Dn(n-1) + 2D}{2D(D-1)(D-2)}.
\]

\[ \square \]

**Theorem 4.2.5.** Let \( G \) be a connected graph of order \( n \), size \( m \) with \( D = r = 3 \), then

\[
RCW(G) \leq \frac{n(n-2)}{2} - \frac{2m}{3}.
\]

Equality holds when \( G \cong C_6 \).

**Proof.** Suppose \( G \) be a graph with \( D = r = 3 \), then \( e(v_i) = 3 \), for every vertex \( v_i \) for \( i = 1, 2, \ldots, n \) in \( G \). Define the sets \( A_k(v_i) = \{ v_j \in V \mid \frac{1}{1+D_d(v_i,v_j)} = \frac{1}{4-k} \} \) for \( k = 0, 1, 2, 3 \) and \( i, j = 1, 2, \ldots, n \).

Clearly \( \bigcup_{k=0}^3 A_k(v_i) = n \).

\[
|A_2(v_i)| + |A_3(v_i)| = n - 1 - d_G(v_i).
\] (4.10)
Since $|A_0(v_i)| = 1$ and $|A_1(v_i)| = d_G(v_i)$.

Also, $|A_2(v_i)| \geq 2$, for, otherwise there is a vertex $w \in A_2(v_i)$ such that $c(w) \leq 2$, a contradiction. Thus

$$RCD(v_i|G) = \sum_{v_j \in V} \frac{1}{1 + D - d(v_i, v_j)}$$
$$= \frac{|A_1(v_i)|}{D} + \frac{|A_2(v_i)|}{D - 1} + \frac{|A_3(v_i)|}{D - 2}$$
$$= \frac{1}{3} |A_1(v_i)| + \frac{1}{2} |A_2(v_i)| + |A_3(v_i)|$$
$$= \frac{d_G(v_i)}{3} + \frac{1}{2} [2|A_2(v_i)| + |A_3(v_i)|] + \frac{1}{2} |A_3(v_i)|. \quad (4.11)$$

Now from the Eq. (4.10) we have the following

$$|A_3(v_i)| = n - 1 - d_G(v_i) - |A_2(v_i)|$$
$$\leq n - 1 - d_G(v_i) - 2$$
$$\therefore |A_2(v_i)| \geq 2$$
$$= n - 3 - d_G(v_i).$$

Therefore the Eq. (4.11) becomes

$$RCD(v_i|G) \leq \frac{d_G(v_i)}{3} + \frac{1}{2} [n - 1 - d_G(v_i)] + \frac{1}{2} [n - 3 - d_G(v_i)]$$
$$= \frac{3n - 6 - d_G(v_i)}{3}.$$
\[ RCW(G) = \frac{1}{2} \sum_{v_i \in V} RC_D(v_i | G) \]

\[ \leq \frac{1}{2} \left[ \sum_{v_i \in V} \frac{(3n - 6 - 2d_G(v_i))}{3} \right] \]

\[ = \frac{1}{2} \left[ \frac{n(3n - 6) - 2 \sum_{v_i \in V} d_G(v_i)}{3} \right] \]

\[ = \frac{n(n - 2)}{2} - \frac{2m}{3} \cdot \sum_{v_i \in V} d_G(v_i) = 2m \]

\[ \square \]

**Theorem 4.2.6.** Let \( G \) be a non-complete connected graph of order \( n \) connectivity \( k(G) \) and \( H_1, H_2, \ldots, H_l \) be the connected components of \( G - S \), where \( |S| = k \). Then

\[ RCW(G) \geq \frac{n^2(D - 1) - 2l(l + k) + n(2l - D + 1)}{2D(D - 1)} , \]

where \( l = \min_{1 \leq i \leq l} \{|V(H_i)| \} \).

Further, the equality holds if and only if \( G = K_l + K_k + K_{n-l-k} \).

**Proof.** Let \( G \) be a graph with \( n \) vertices and \( |S| = k \), where \( S \) be any cut-set of a \( G \). Let \( H_i \) for \( i = 1, 2, \ldots, l \) are the connected components of \( G - S \), with \( l = \min_{1 \leq i \leq l} \{|V(H_i)| \} \) without loss of generality, assume that \( |V(H_1)| = l \), \( G_1 = H_1 \) and \( G_2 = \bigcup_{i=2}^{l} H_i \).
Then, \(|V(G_1)| = l\) and \(|V(G_2)| = n - k - l\). Now we have

\[
RCW(G) = \frac{1}{2} \sum_{v_i \in V} RCD(v_i|G)
= \frac{1}{2} \left[ \sum_{v_i \in V(G)} RCD(v_i|G) + \sum_{v_i \in S} RCD(v_i|G) + \sum_{v_i \in V(G_2)} RCD(v_i|G) \right]. \quad (4.12)
\]

Now, result follow in three cases

Case (i): Let \(v_i \in V(G_1)\).

\[
RCD(v_i|G) = \sum_{v_j \in V(G)} \frac{1}{1 + D - d(v_i, v_j)}
= \sum_{v_j \in V(G_1)} \frac{1}{1 + D - d(v_i, v_j)} + \sum_{v_j \in S} \frac{1}{1 + D - d(v_i, v_j)}
+ \sum_{v_j \in V(G_2)} \frac{1}{1 + D - d(v_i, v_j)}
\geq \frac{l - 1}{D} + \frac{k(G)}{D} + \frac{n - l - k(G)}{D - 1}
= \frac{D(n - D) - (l + k(G) - 1)}{D(D - 1)}.
\]

Since \(\frac{1}{1 + D - d(v_i, v_j)} \geq \frac{1}{D}\), if \(v_j \in V(G_1), v_j \in S\) and \(\frac{1}{1 + D - d(v_i, v_j)} \geq \frac{1}{2}\), if \(v_j \in V(G_2)\).

Case(ii) : Let \(v_i \in S\).
$$RCD(v_i|G) = \sum_{v_j \in V(G)} \frac{1}{1 + D - d(v_i, v_j)}$$

$$= \sum_{v_j \in V(G_1)} \frac{1}{1 + D - d(v_i, v_j)} + \sum_{v_j \in S} \frac{1}{1 + D - d(v_i, v_j)}$$

$$+ \sum_{v_j \in V(G_2)} \frac{1}{1 + D - d(v_i, v_j)}$$

$$\geq \frac{l}{D} + \frac{k - 1}{D} + \frac{n - l - k}{D}$$

$$= \frac{n - 1}{D}.$$  

Since $\frac{1}{1 + D - d(v_i, v_j)} \geq \frac{1}{D}$, if $v_j$ is in either sets $V(G_1)$ or $V(G_2)$

**Case (iii):** Let $v_i \in V(G_2)$, then we can prove that

$$RCD(v_i|G) \geq \frac{l}{D - 1} + \frac{k}{D} + \frac{n - l - k - 1}{D}$$

$$\geq \frac{D(n - 1) - (n - l - 1)}{D(D - 1)}.$$

Thus by Eq. (4.12) we have

$$RCW(G) \geq \frac{n^2(D - 1) - 2l(l + k) + n(2l - D + 1)}{2D(D - 1)}.$$  

Proof of the second part of this theorem holds from the proof of the inequality itself.

\[\square\]

**Theorem 4.2.7.** Let $G$ be a non-complete connected graph of order $n$. Then

$$RCW(G) \geq \frac{1}{2} \left[ \frac{(Dn - D - n + 1)n + \beta_0(\beta_0 - 1)}{D(D - 1)} \right]$$
Equality holds if and only if $G = K_{\beta_0} + K_{n-\beta_0}$.

Proof. Let $S$ be the maximum independent set with $|S| = \beta_0$ and $v_i$ be any vertex in $S$ for $i = 1, 2, \ldots, n$. Then

$$RCD(v_i|G) = \sum_{v_j \in V(G)} \frac{1}{1 + D - d(v_i, v_j)}$$

$$= \sum_{v_j \in S} \frac{1}{1 + D - d(v_i, v_j)} + \sum_{v_j \in V - S} \frac{1}{1 + D - d(v_i, v_j)}$$

$$\geq \frac{\beta_0 - 1}{D - 1} + \frac{n - \beta_0}{D}$$

$$= \frac{D n - D - n + \beta_0}{D(D - 1)}.$$  \hspace{1cm} (4.13)

Since $v_i \neq v_j$ and $v_i \in S$, so that there are $(\beta_0 - 1)$ vertices in $S$ which are at distance atleast two from $v_i$ and $\frac{1}{1 + D - d(v_i, v_j)} \leq 1$, for any $v_j \in V - S$.

Next, let $v_i \in V - S$. Then

$$RCD(v_i|G) = \sum_{v_j \in V(G)} \frac{1}{1 + D - d(v_i, v_j)}$$

$$\geq \frac{n - 1}{D}.$$  \hspace{1cm} (4.14)
Therefore,

\[
RCW(G) = \frac{1}{2} \sum_{v_i \in V(G)} RCD(v_i | G)
\]

\[
\geq \frac{1}{2} \left[ \left( \frac{Dn - D - n + \beta}{D(D - 1)} \right) \beta_0 \right] + (n - \beta_0) \left( \frac{n - 1}{D} \right)
\]

from Eqs. (4.13) and (4.14)

\[
= \frac{1}{2} \left[ \frac{n(Dn - D - n + 1) + \beta_0(\beta_0 - 1)}{D(D - 1)} \right]
\]

Further, if \( G = \overline{K}_{\beta_0} + K_{n-\beta_0} \). It is easy to see that

\[
RCW(G) = \frac{1}{2} \left[ \frac{n(Dn - D - n + 1) + \beta_0(\beta_0 - 1)}{D(D - 1)} \right]
\]

Conversely, suppose

\[
RCW(G) = \frac{1}{2} \left[ \frac{n(Dn - D - n + 1) + \beta_0(\beta_0 - 1)}{D(D - 1)} \right]
\]

We prove that \( G = \overline{K}_{\beta_0} + K_{n-\beta_0} \). If possible assume that \( G \neq \overline{K}_{\beta_0} + K_{n-\beta_0} \). Let \( S \) be the maximum independent set with \( |S| = \beta_0 \) in \( G \). For any two vertices \( v_i \) and \( v_j \) in \( G \), \( d(v_i, v_j) = 2 \) if both \( v_i \) and \( v_j \) are in \( V - S \) otherwise it will lead to

\[
RCW(G) = \frac{1}{2} \left[ \frac{n(Dn - D - n + 1) + \beta_0(\beta_0 - 1)}{D(D - 1)} \right]
\]

a contradiction. Thus \( \langle S \rangle = \overline{K}_{\beta_0} \) and \( \langle V - S \rangle = K_{n-\beta_0} \). Further if \( v_i \in S \) and \( v_j \in V - S \), we claim that \( \frac{1}{1 + D - d(v_i, v_j)} = \frac{1}{D} \), for, otherwise \( RCD(u | G) > \frac{n - \beta_0}{D} \) and there by

\[
RCW(G) = \frac{1}{2} \left[ \frac{n(Dn - D - n + 1) + \beta_0(\beta_0 - 1)}{D(D - 1)} \right]
\]
holds, a contradiction. Thus $G' = \overline{K_{\chi_0}} + \overline{K}_{n-\chi_0}$ holds.

\[ \square \]

**Theorem 4.2.8.** Let $G$ be any connected graph of order $n$. Then

\[
RCW(G) \geq \frac{1}{2} \left[ \frac{n(D_n - D - n + 1) + \gamma_0(\gamma_0 - 1)}{D(D - 1)} \right].
\]

**Proof.** The proof of this theorem follows from the Theorem 4.2.7.

\[ \square \]

**Theorem 4.2.9.** Let $G$ be a non-complete connected graph of order $n$ with chromatic number $\chi(G) = t$. Then

\[
RCW(G) \geq \frac{1}{2} \left[ \frac{D_n(n - 1) - n^2 + \sum_{i=1}^{t} n_i^2}{D(D - 1)} \right].
\]

Equality holds if and only if $G = K_{n_1, n_2, \ldots, n_t}$.

**Proof.** Suppose $\chi(G) = t$, then the vertex set $V(G)$ of $G$ can be partitioned into $t$ color classes $C_1, C_2, \ldots, C_t$ such that no two vertices in any $C_i$ adjacent and let $|C_i| = n_i$, for $i = 1, 2, \ldots, t$. Thus, $n = \sum_{i=1}^{t} n_i$.

Let $v_i \in C_i$. Therefore

\[
RCW(v_i|G) = \sum_{v_j \in V(G)} \frac{1}{1 + D - d(v_i, v_j)} = \sum_{v_j \in C_i} \frac{1}{1 + D - d(v_i, v_j)} + \sum_{v_j \in V - C_i} \frac{1}{1 + D - d(v_i, v_j)}
\]

\[
\geq \frac{n_i - 1}{D - 1} + \frac{n - n_i}{D}
\]

\[
= \frac{Dn - D - n + n_i}{D(D - 1)}.
\]
Since \( \frac{1}{1+D-d(v_i,v_j)} \geq \frac{1}{D-1} \), if \( v_j \in C_i \) and \( \frac{1}{1+D-d(v_i,v_j)} \geq \frac{1}{D} \), if \( v_j \in V - C_i \)

Therefore,

\[
RCW(G) = \frac{1}{2} \sum_{v_i \in V(G)} RCD(v_i|G)
\]

\[
= \frac{1}{2} \left[ \sum_{i=1}^{t} \sum_{v_i \in C_i} \frac{1}{1 + D - d(v_i, v_j)} \right]
\]

\[
\geq \frac{1}{2} \left[ \sum_{i=1}^{t} \frac{Dn_i^2 - Dn_i - n_i^2 + mn_i}{D(D-1)} \right]
\]

\[
= \frac{1}{2} \left[ \frac{Dn(n-1) - n^2 + \sum_{i=1}^{t} n_i^2}{D(D-1)} \right].
\]

Further if \( G = K_{n_1,n_2,\ldots,n_t} \), then it is not difficult to see that

\[
RCW(G) = \frac{1}{2} \left[ \frac{Dn(n-1) - n^2 + \sum_{i=1}^{t} n_i^2}{D(D-1)} \right].
\]

On the other hand, if \( RCW(G) = \frac{1}{2} \left[ \frac{Dn(n-1) - n^2 + \sum_{i=1}^{t} n_i^2}{D(D-1)} \right] \) and \( \chi(G) = t \), then the vertex set \( V(G) \) can be partitioned into the color classes \( C_1, C_2, \ldots, C_t \) such that \( |C_i| = n_i \), for \( i = 1, 2, \ldots, t \). Now, we claim that any two vertices \( v_i \) and \( v_j \) belonging to two different color classes are adjacent. For if \( v_i \in C_i \), and \( v_j \in C_j \) for \( i \neq j \) are not adjacent then

\[
\sum_{v_j \in V - C_i} \frac{1}{1+D-d(v_i,v_j)} \geq \frac{n-n_i}{D},
\]

which in turn implies that, \( RCD(v_i|G) > \frac{Dn-D-n+n_i}{D(D-1)} \) and there by it will lead to

\[
RCW(G) > \frac{1}{2} \left[ \frac{Dn(n-1) - n^2 + \sum_{i=1}^{t} n_i^2}{D(D-1)} \right],
\]

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a contradiction. Again, if both \( u \) and \( v \) belongs to the same color class then 
\[
\frac{1}{1 + D - d(u, v)} = \frac{1}{D - 1},
\]
otherwise it leads to the same contradiction. Hence \( G = K_{n_1, n_2, \ldots, n_t} \) holds.

\[ \square \]

**Note:** For any graph \( G \), we know that diameter will be in terms of \( n \) and \( \Delta \) [59]. i.e \( D(G) \leq n - \Delta + 1 \). Suppose the maximum degree is \( \Delta \) and second maximum degree is \( \Delta^* \) then the diameter is of the form \( D(G) \leq n - (\Delta + \Delta^*) + 2 \).

The following results gives the sharp bounds for the reciprocal complementary Wiener index in terms of \( \Delta, \delta \) and \( \Delta^* \).

**Theorem 4.2.10.** Let \( G \) be a graph with \( n \) vertices, \( m \) edges and \( \Delta \) is the maximum degree. Then

\[
RCW(G) \geq \left[ \frac{1}{n - \Delta} \right] \left[ \frac{n(n - 1)}{2} - \frac{m}{n - \Delta + 1} \right].
\]

Equality when \( D = n - \Delta + 1 \).

**Proof.** Since \( D(G) \leq n - \Delta + 1 \). Now we have \( m \) – pairs are at distance 1 and \( \left( \binom{n}{2} - m \right) \) – pairs are at distance 2. So that

\[
RCW(G) = \sum_{1 \leq i < j \leq n} \frac{1}{1 + D - d(v_i, v_j)} \\
\geq m \left[ \frac{1}{n - \Delta + 1} - \frac{1}{n - \Delta} \right] + \left[ \frac{n(n - 1)}{2} \right] \left[ \frac{1}{n - \Delta} \right] \\
= \left[ \frac{n(n - 1)}{2} - \frac{m}{n - \Delta + 1} \right] \left[ \frac{1}{n - \Delta} \right].
\]
Theorem 4.2.11. Let \( G \) be a graph with order \( n \) and size \( m \), \( \Delta \) and \( \delta \) are the maximum degree and minimum degree of a graph \( G \). Then

\[
RCW(G) \geq \frac{1}{n - (\Delta + \delta) + 1} \left[ \frac{n(n - 1)}{2} - \frac{m}{n - (\Delta + \delta) + 2} \right].
\]

Equality when \( D = n - (\Delta + \delta) + 2 \).

Proof. The proof of this theorem is same as the Theorem 4.2.10.

Theorem 4.2.12. Let \( G \) be a graph of order \( n \) and size \( m \) with \( \Delta + \Delta^* \leq n \). Then

\[
RCW(G) \geq \frac{1}{2} \left[ \frac{n(n - 1)(n - (\Delta + \Delta^*) + 2) - 2m}{(n - (\Delta + \Delta^*) + 2)(n - (\Delta + \Delta^*) + 1)} \right].
\]

Equality when \( D = n - (\Delta + \Delta^*) + 2 \).

Proof. The proof of this theorem is same as the Theorem 4.2.10.

4.3 Conclusion

interms of the chromatic number. Further we have given lower and upper bounds for the reciprocal complementary Wiener index interms of some graph parameters.