CHAPTER- 3

Generalised Recurrent Kaehler Manifolds of Second Order

1. Introduction:

A non-flat riemannian manifold $V_n$ whose curvature tensor satisfies the relation

\begin{equation}
R_{kijh,lm} = R_{kijh,m} a_m + b_{ln} R_{kjh}
\end{equation}

where $a_m$ and $b_{lm}$ are not both zero, has been called a generalised recurrent manifold of second order or a generalised 2-recurrent manifold\cite{4}. If $a_m = 0$, the manifold reduces to what is known to be a 2-recurrent manifold named by Lichnerowicz\cite{3}. $a_m$ and $b_{lm}$ are respectively called vector of recurrence and tensor of recurrence of the manifold.

We consider generalised second order recurrent, Ricci-recurrent, H-conharmonic recurrent and Bochner-recurrent Kaehler
manifolds and study some relations existing between them.

The so-called H-coharmonic curvature tensor[6] is given by,

\[
(1.2) \quad H_{kjh} = R_{kjh} + \frac{1}{n + 4} \left\{ R_{ki}g_{jh} - R_{ji}g_{kh} + g_{ji}R_{kjh} - g_{jih}R_{kij} + S_{ki}J_{jh} + S_{ji}J_{kh} + J_{ki}S_{jh} - J_{ji}S_{kh} + 2S_{kj}J_{ih} + 2J_{ki}S_{jih} \right\}
\]

The covariant components of the Bochner curvature tensor and the H-concircular curvature tensor are respectively given by-

\[
(1.3) \quad B_{kjh} = R_{kjh} + \frac{1}{n + 4} \left\{ R_{ji}g_{kh} - R_{ki}g_{jh} + S_{ji}J_{kh} - S_{ki}J_{jh} - 2S_{kj}J_{ih} + R_{kh}g_{ji} - R_{jh}g_{ki} + S_{kh}J_{ji} - S_{jh}J_{ki} - 2S_{ih}J_{kj} \right\}
\]

\[
+ \frac{R}{(n + 2)(n + 4)} \left\{ g_{ji}g_{kh} - g_{ki}g_{jh} + J_{ji}J_{kh} - J_{ki}J_{jh} - 2J_{kj}J_{ih} \right\}
\]

and,

\[
(1.4) \quad U_{kjh} = R_{kjh} - \frac{R}{n(n + 2)} \left\{ g_{ji}g_{kh} - g_{ki}g_{jh} + J_{ji}J_{kh} - J_{ki}J_{jh} - 2J_{kj}J_{ih} \right\}
\]

It would be helpful to make use of the tensors \( T \) and \( V \) defined by

\[
(1.5) \quad T_{kjh} = g_{kh}g_{ji} - g_{ki}g_{jh} + J_{ki}J_{jh} - J_{kj}J_{ih} - 2J_{kj}J_{ih}
\]

and,
\[ V_{kib} = g_{ki} R_{ji} - g_{ki} R_{ji} + R_{ikb} g_{ji} - R_{ik} g_{ji} + J_{kib} S_{ji} - J_{ki} S_{ji} \]

\[ + S_{kb} J_{ji} - S_{ki} J_{jb} - 2 J_{ki} S_{ih} - 2 S_{kj} J_{ih} \]

respectively. Then if $H$, $B$ and $U$ stand for the $H$-conharmonic, Bochner and $H$-concircular tensors respectively, we've

\[ H = R - \frac{1}{n+4} V, \]

\[ B = R - \frac{1}{n+4} V + \frac{R}{(n+2)(n+4)} T \]

and,

\[ U = R - \frac{R}{n(n+2)} T, \]

where $R$ denotes the curvature tensor. It follows from (1.7) and (1.8) that

\[ B = H + \frac{R}{(n+2)(n+4)} T \]

and from (1.7) and (1.9) that

\[ H = U - \frac{1}{n+4} V + \frac{R}{n(n+2)} T \]

We can also express $B$ in terms of $U$ by
\begin{equation}
B = U \left(- \frac{1}{(n+4)} \right) V + \frac{2R}{n(n+4)} T
\end{equation}

and it is worth nothing that if we define \( X \) to be the tensor whose components \( X_{k\ell ni} \) are given by (1.6) but with the Ricci tensor replaced by the Einstein tensor, that is, the tensor having components

\[ R_{ji} = -\frac{R}{n} g_{ji} \]

then

\begin{equation}
B = U \left(- \frac{1}{(n+4)} \right) X
\end{equation}

From (1.6) we get

\begin{equation}
g^{kb} V_{k\ell ni} = (n+4)R_{ji} + R g_{ji}
\end{equation}

\begin{equation}
g^{kb} g^{ji} V_{k\ell ni} = 2(n+2)R
\end{equation}

and from (1.7) we then get

\begin{equation}
g^{kb} H_{k\ell ji} = -\frac{R}{(n+4)} g_{ji}
\end{equation}
2. Generalised Recurrent, Ricci-Reccurrent, H-Conharmonic-Recurrent and H-Concircular-Recurrent Kaehler Manifolds of Second Order:

Def (2.1) A Kaehler Manifold $K_n$ is said to be generalised Kaehler manifold of second order (or briefly G-2 Recurrent Kaehler Manifolds) if,

\begin{equation}
R_{kjh,lm} = R_{kjh,lm}a_m + b_{lm}R_{kjh}
\end{equation}

and to be generalised Ricci-recurrent Kaehler Manifold of second order (or briefly G-2 Ricci-recurrent Kaehler Manifold) if

\begin{equation}
R_{kk,h,l} = R_{kk,h,l}a_m + b_{lm}R_{kh,l}
\end{equation}

where $a_m$ and $b_{lm}$ are not both zero.

Def(2.2) A Kaehler Manifold $K_n$ is said to be generalised H-Conharmonic Recurrent Kaehler Manifold of second order (or briefly G-2H Conharmonic recurrent Kaehler Manifold).

\begin{equation}
H_{kjh,lm} = H_{kjh,lm}a_m + b_{lm}H_{kjh}
\end{equation}

and to be generalised H-Concircular recurrent Kaehler Manifold of second order (or briefly G-2 H-Concircular recurrent Kaehler Manifold) if
\[ U_{kib} = U_{kib} a_m + b_{im} U_{kib} \]

where \( a_m \) and \( b_{im} \) are not both zero.

For convenience we shall call two tensors co-recurrent if both satisfy a recurrence condition of the same type, with the same vector of recurrence and/or the same tensors of recurrence, as the case may be. Thus if \( R \) is recurrent then the Ricci tensor is co-recurrent with \( R \) likewise the scalar curvature is co-recurrent with both \( R \) and the Ricci tensor.

The tensor \( V \) is recurrent iff the Ricci tensor is recurrent, the two tensors then being co-recurrent.

**Theorem 2.1** If \( R \) is G-2 Recurrent, \( V, H, B, U \) and the scalar curvature \( R \) and G-2 co-recurrent with \( R \) and with one-another.

**Proof:** The statement follows from the equations (1.7), (1.8), and (1.9).

**Theorem 2.2** In a G2 H-Conharmonic recurrent Kaehler manifold, relation

\[ R_{im} - P_{ij} a_m - b_{im} R = 0 \]

is satisfied.

[68]
Proof: Equation (1.16) shows that if $H$ is $G_2$ recurrent, then $\tilde{R}$ is $G_2$ corecurrent with $\tilde{H}$, so that in particular if $\tilde{H}$ satisfies equation (2.3) then we've (2.5).

**Theorem 2.3** If $\tilde{H}$ is $G_2$ recurrent then $\tilde{R}$ is $G_2$ corecurrent with $\tilde{H}$ iff the Ricci tensor is $G_2$ corecurrent with $\tilde{H}$.

Proof: The statement follows from (1.7).

The above result can also be stated as,

**Theorem 2.4** If a Kaehler manifold $K_n$ satisfies any two of the properties-

(i) $K_n$ is $G_2$ H-conharmonic recurrent;

(ii) $K_n$ is $G_2$ recurrent;

(iii) $K_n$ is $G_2$ Ricci-recurrent;

then it must also satisfy the third, the tensor $H$, $R$ and the Ricci tensor then being $G_2$ corecurrent.
3. Generalised Bochner Recurrent Kaehler Manifolds of Second Order:

Def 3.1 A Kaehler manifold Kn is said to be generalised Bochner recurrent Kaehler manifold of second order (or briefly G2 Bochner recurrent Kaehler manifold) if

\[ B_{kijh,lm} = B_{kijh,1}a_m + B_{kijh}b_{lm}, \]  

(3.1)

where \( a_m \) and \( b_{lm} \) are not both zero.

Theorem 3.1 If \( \tilde{H} \) is G2 recurrent, then \( B \) is G2 corecurrent with \( \tilde{H} \).

Proof: The statement follows from (2.5) and (1.10).

Theorem 3.2 If \( \tilde{H} \) is G2 recurrent, then \( B \) is G2 corecurrent with \( \tilde{H} \) iff

\[ R_{jlm} - R_{j}a_m - b_{lm}R = 0. \]  

(3.2)

Proof: The statement follows from (1.10).
4. Generalised Ricci-Recurrent Kaehler Manifolds of Second Order:

**Theorem 4.1** In a G2 Ricci-recurrent Kaehler manifold with \( a_l \) and \( b_{lm} \) as vector of recurrence and tensor of recurrence respectively, relation

\[
(4.1) \quad H_{kij,lm} - H_{kij,lm} \tilde{a}_m - b_{lm} H_{kij} = R_{kij,lm} - R_{kij,lm} \tilde{a}_m - b_{lm} R_{kij},
\]

is satisfied.

**Proof:** The result follows from (1.6) and (1.7).

**Theorem 4.2** In a G2 Ricci-recurrent Kaehler manifold with \( a_m \) and \( b_{lm} \) as vector and tensor of recurrence, relation

\[
(4.2) \quad B_{kij,lm} - B_{kij,lm} \tilde{a}_m - b_{lm} B_{kij} = R_{kij,lm} - R_{kij,lm} \tilde{a}_m - b_{lm} R_{kij},
\]

is satisfied.

**Proof:** The statement follows from (1.8).

**Theorem 4.3** In a G2 Ricci-recurrent Kaehler manifold, with \( a_i \) and \( b_{im} \) as vector of recurrence and tensor of recurrence respectively, relation

[71]
\[(4.3) \quad B_{k\bar{m},lm} - B_{k\bar{m},l} a_m - b_{\bar{m} l} B_{k\bar{m}} = H_{k\bar{m},lm} - H_{k\bar{m},l} a_m - b_{\bar{m} l} H_{k\bar{m}},\]

is satisfied.

**Proof:** The statement follows from (1.10).

**Theorem 4.4** If the Ricci tensor is G2 recurrent, then \( \mathcal{R} \) is G2 corecurrent if and only if \( \mathcal{H} \) is G2 corecurrent, the three tensors then being G2 corecurrent.

**Proof:** The statement follows from (1.7).

The statement of the following theorem can easily be verified.

**Theorem 4.5** If a Kaehler manifold satisfies any two of the following properties:

(i) it is G2 Bochner recurrent

(ii) it is G2 recurrent

(iii) it is G2 Ricci-recurrent

then it must also satisfy third, the tensors \( \mathcal{B}, \mathcal{R} \) and the Ricci tensor then being G2 corecurrent.
Remark: G2 recurrence can also be considered as a mapping $F$ which associates with each tensor $\mathcal{X}$ of type $(r,s)$ a tensor $F(\mathcal{X})$ of type $(r,s+2)$ satisfying

$$ (4.4) \quad F(c_1 X_{\mathcal{X}_1} + c_2 X_{\mathcal{X}_2}) = c_1 F(\mathcal{X}_{\mathcal{X}_1}) + c_2 F(\mathcal{X}_{\mathcal{X}_2}) $$

for any constants $c_1, c_2$; for example when $\mathcal{X}$ is the Ricci tensor, $F(\mathcal{X})$ is the tensor having components

$$ R_{j_1 k_1} - R_{j_1 k_2} a_{j_2} - R_{j_2 k_1}. $$

The tensor $\mathcal{X}$ is then G2 recurrent with vector of recurrence $a_1$ and tensor of recurrence $b_{kl}$ if and only if $F(\mathcal{X}) = 0$. Suppose that $\mathcal{Y}$ is a tensor obtained from $\mathcal{X}$ by transvection with the components of the metric tensor or the tensor $J$. Then if $F(\mathcal{X}) = 0$ we have $F(\mathcal{Y}) = 0$. It can thus be seen a generalisation can be achieved.
References

[1.] Bochner, S. : Curvature and Betti numbers, Ann. of Math. 50(1949), 77-93.


