CHAPTER - V

\textit{t-CONNECTEDNESS AND}
\textit{t-DISCONNECTEDNESS IN SMOOTH}
\textit{FUZZY BITOPOLOGICAL SPACES}
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**t-CONNECTEDNESS AND t-DISCONNECTEDNESS IN SMOOTH FUZZY BITOPOLOGICAL SPACES.**

The concept of $G_δ$-connectedness and $G_δ$-disconnectedness in fuzzy bitopological spaces was introduced by E.Roja, M.K.Uma and G.Balasubramanian[57]. Pairwise fuzzy connected and pairwise fuzzy extremely disconnected spaces were found in [21]. Pairwise fuzzy basically disconnected space was found in [12]. In this chapter, the concepts of pairwise smooth fuzzy $t$-connected spaces and pairwise smooth fuzzy extremely $t$-disconnected spaces and pairwise smooth fuzzy basically $t$-disconnected spaces are introduced. Characterizations of the above spaces are given besides giving several examples. Interrelations among the spaces introduced are studied and some relevant counter examples are given.
5.1 PAIRWISE SMOOTH FUZZY $t$-CONNECTED SPACES.

In this section, the concept of pairwise smooth fuzzy $t$-connected spaces are introduced with some interesting examples.

**Definition 5.1.1.**

A smooth fuzzy bitopological space $(X, T_1, T_2)$ is said to be pairwise smooth fuzzy $t$-connected iff $(X, T_1, T_2)$ has no $T_1$-$r$-fuzzy $t$-open set $\lambda_1$ and $T_2$-$r$-fuzzy $t$-open set $\lambda_2$ such that $\lambda_1, \lambda_2 \neq 0,1$ with $\lambda_1 + \lambda_2 = 1$, $\lambda_1, \lambda_2 \in \mathcal{I}^X$, $r \in I_0$. A smooth fuzzy bitopological space $(X, T_1, T_2)$ is pairwise smooth fuzzy $t$-disconnected if it is not pairwise smooth fuzzy $t$-connected.

**Definition 5.1.2.**

A smooth fuzzy topological space $(X, T)$ is smooth fuzzy $t$-connected iff it has no $r$-fuzzy $t$-open sets $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$ such that $\lambda_1 + \lambda_2 = \bar{1}$, $\lambda_1, \lambda_2 \in \mathcal{I}^X$, $r \in I_0$.

**Remark 5.1.1.**

A pairwise smooth fuzzy $t$-connectedness of $(X, T_1, T_2)$ is not governed by the smooth fuzzy $t$-connectedness of the spaces $(X, T_1)$ and $(X, T_2)$ as shown in the following Examples 5.1.1 and 5.1.2.

**Example 5.1.1.**

Let $X = \{a, b\}$. Let $\lambda_1, \mu_1 \in \mathcal{I}^X$ be defined as follows: $\lambda_1(a) = 0.7$, $\lambda_1(b) = 0.4$, $\mu_1(a) = 0.3$, $\mu_1(b) = 0.6$. Let $r = 0.01$. Define smooth
Fuzzy topologies $T_1, T_2: I^X \to I$ as follows:

$$T_1(\lambda) = \begin{cases} 
1 & \lambda = 0, 1 \\
0.5 & \lambda = \lambda_1 \\
0 & \text{otherwise},
\end{cases}$$

$$T_2(\lambda) = \begin{cases} 
1 & \lambda = 0, 1 \\
0.5 & \lambda = \mu_1 \\
0 & \text{otherwise}.
\end{cases}$$

Clearly, $(X, T_1, T_2)$ is not a pairwise smooth fuzzy $t$-connected space but $(X, T_1)$ and $(X, T_2)$ are both smooth fuzzy $t$-connected spaces.

**Example 5.1.2.**

Let $X = \{a, b\}$. Let $\lambda_1, \lambda_2, \mu_1, \mu_2 \in I^X$ be defined as follows:

$\lambda_1(a) = 0.4$, $\lambda_1(b) = 0.45$, $\mu_1(a) = 0.6$, $\mu_1(b) = 0.65$, $\lambda_2(a) = 0.45$, $\lambda_2(b) = 0.55$, $\mu_2(a) = 0.35$, $\mu_2(b) = 0.35$. Let $r = 0.02$. Define smooth fuzzy topologies $T_1, T_2: I^X \to I$ as follows:

$$T_1(\lambda) = \begin{cases} 
1 & \lambda = 0, 1 \\
0.5 & \lambda = \lambda_1 \\
0.5 & \lambda = \lambda_2 \\
0 & \text{otherwise},
\end{cases}$$

$$T_2(\lambda) = \begin{cases} 
1 & \lambda = 0, 1 \\
0.5 & \lambda = \mu_1 \\
0.5 & \lambda = \mu_2 \\
0 & \text{otherwise}.
\end{cases}$$
Clearly, \((X, T_1)\) and \((X, T_2)\) are not smooth fuzzy t-connected space but \((X, T_1, T_2)\) is a pairwise smooth fuzzy t-connected space.

**Proposition 5.1.1.**

Let \((X, T_1, T_2)\) be a smooth fuzzy bitopological space. Then the following statements are equivalent:

(a) \((X, T_1, T_2)\) is a pairwise smooth fuzzy t-connected space.

(b) There exists no \(T_1\)-r-fuzzy t-open set \(\lambda_1 \neq 0\) and \(T_2\)-r-fuzzy t-open set \(\lambda_2 \neq 0\) such that \(\lambda_1 + \lambda_2 = \bar{1}, \lambda_1, \lambda_2 \in \Gamma^X, r \in I_0\).

(c) There exists no \(T_1\)-r-fuzzy t-closed set \(\lambda_1 \neq \bar{1}\) and \(T_2\)-r-fuzzy t-closed set \(\lambda_2 \neq \bar{1}\) such that \(\lambda_1 + \lambda_2 = \bar{1}, \lambda_1, \lambda_2 \in \Gamma^X, r \in I_0\).

(d) \((X, T_1, T_2)\) contains no \(\lambda \neq \bar{0}, \bar{1}\) and it is both \(T_1\)-r-fuzzy t-open set and \(T_2\)-r-fuzzy t-closed set or both \(T_2\)-r-fuzzy t-open set and \(T_1\)-r-fuzzy t-closed set, \(\lambda \in \Gamma^X, r \in I_0\).

**Proof:**

(a) \(\Rightarrow\) (b).

Assume that (a) is true. Then, (b) follows from the definition 5.1.1.
(b) $\Rightarrow$ (c).

Assume that (b) is true. Let us suppose that there exists a $T_1$-r-fuzzy t-closed set $\lambda_1 \neq \bar{1}$ and $T_2$-r-fuzzy t-closed set $\lambda_2 \neq \bar{1}$ such that $\lambda_1 + \lambda_2 = \bar{1}$, $\lambda_1, \lambda_2 \in I^X, r \in I_0$. Then, $\bar{1} - \lambda_1 \neq \bar{1} - \bar{1} \neq \bar{0}$ is a $T_1$-r-fuzzy t-open set. Similarly, $\bar{1} - \lambda_2 \neq \bar{0}$ is a $T_2$-r-fuzzy t-open set.

Now, $(\bar{1} - \lambda_1) + (\bar{1} - \lambda_2) = \bar{1}$. Contradiction. Hence, (c) is proved.

(c) $\Rightarrow$ (d).

Assume that (c) is true. Suppose that $\lambda \neq \bar{0}, \bar{1}$ be both $T_1$-r-fuzzy t-open and $T_2$-r-fuzzy t-closed set, $\lambda \in I^X, r \in I_0$. Then, $\bar{1} - \lambda \neq \bar{0}, \bar{1}$ is a $T_1$-r-fuzzy t-closed set. Also by assumption, $\lambda$ is a $T_2$-r-fuzzy t-closed set. Now, $(\bar{1} - \lambda) + \lambda = \bar{1}$. Contradiction. Hence, (d) is proved.

(d) $\Rightarrow$ (a).

Assume that (d) is true. Let us assume that $(X, T_1, T_2)$ is not a pairwise smooth fuzzy t-connected space. Then, $(X, T_1, T_2)$ has $T_1$-r-fuzzy t-open set $\lambda_1$ and $T_2$-r-fuzzy t-open set $\lambda_2, \lambda_1, \lambda_2 \neq \bar{0}, \bar{1}$ such that $\lambda_1 + \lambda_2 = \bar{1}, \lambda_1, \lambda_2 \in I^X, r \in I_0$. Now, $\lambda_1 + \lambda_2 = \bar{1}$ implies that, $\lambda_1 = \bar{1} - \lambda_2$. This implies that, $\lambda_1$ is a $T_2$-r-fuzzy t-closed set. Clearly, $\lambda_1$ is both $T_1$-r-fuzzy t-open and $T_2$-r-fuzzy t-closed set. Contradiction. Hence, (a) is proved.
5.2 PAIRWISE SMOOTH FUZZY STRONGLY $t$-CONNECTED SPACES.

In this section, the concept of pairwise smooth fuzzy strongly $t$-connected spaces and weakly $t$-disconnected spaces are discussed with examples.

**Definition 5.2.1.**

A smooth fuzzy bitopological space $(X, T_1, T_2)$ is said to be pairwise smooth fuzzy strongly $t$-connected space if it has no $T_1$-r-fuzzy $t$-closed sets (or) $T_2$-r-fuzzy $t$-closed sets $X_i$ and $X_2$, $aX_2 \neq 0, 1$, such that $X_i + aX_2 \subseteq Y$, $r \in I_0$. If $(X, T_1, T_2)$ is not a pairwise smooth fuzzy strongly $t$-connected space, then it is called a pairwise smooth fuzzy weakly $t$-connected space.

**Proposition 5.2.1.**

A smooth fuzzy bitopological space $(X, T_1, T_2)$ is a pairwise smooth fuzzy strongly $t$-connected space iff it has no $T_1$-r-fuzzy $t$-open sets (or) $T_2$-r-fuzzy $t$-open sets $X_i$, $\mu \neq 0, 1$ such that $X_i + \mu \subseteq X$, $\mu \in I^X, r \in I_0$. 

**Proof:**

Suppose that $(X, T_1, T_2)$ is a pairwise smooth fuzzy weakly $t$-connected space iff it has $T_1$-r-fuzzy $t$-closed sets (or) $T_2$-r-fuzzy $t$-closed sets $X_i$, $\lambda \neq 0, 1$ such that $X_i + \lambda \subseteq X$, $\lambda \in I^X, r \in I_0$, iff it has $T_1$-r-fuzzy $t$-open sets (or) $T_2$-r-fuzzy $t$-open sets $\lambda$, $\mu \neq 0, 1$ where
\[ \lambda = \bar{1} - \lambda_1, \mu = \bar{1} - \lambda_2 \text{ such that, } \lambda + \mu = (\bar{1} - \lambda_1) + (\bar{1} - \lambda_2) \geq \bar{1}. \]

**Remark 5.2.1.**

Pairwise smooth fuzzy strongly t-connectedness implies pairwise smooth fuzzy t-connectedness. However, the converse is not true as shown in Example 5.2.1.

**Example 5.2.1.**

Let \( X = \{ a, b \} \) be a set. Let \( \lambda_1, \lambda_2, \mu_1, \mu_2 \in I^X \) be defined as follows:

\[
\begin{align*}
\lambda_1(a) &= 0.4, \quad \lambda_1(b) = 0.45, \\
\mu_1(a) &= 0.6, \quad \mu_1(b) = 0.65, \\
\lambda_2(a) &= 0.45, \quad \lambda_2(b) = 0.55, \\
\mu_2(a) &= 0.35, \quad \mu_2(b) = 0.35.
\end{align*}
\]

Let \( r = 0.01 \). Define smooth fuzzy topologies \( T_1, T_2 : I^X \to I \) as follows:

\[
T_1(\lambda) = \begin{cases} 
1 & \lambda = \bar{0}, \bar{1} \\
0.5 & \lambda = \lambda_1 \\
0.5 & \lambda = \lambda_2 \\
0 & \text{otherwise},
\end{cases}
\]

\[
T_2(\lambda) = \begin{cases} 
1 & \lambda = \bar{0}, \bar{1} \\
0.5 & \lambda = \mu_1 \\
0.5 & \lambda = \mu_2 \\
0 & \text{otherwise}.
\end{cases}
\]

Clearly, \((X, T_1, T_2)\) is a pairwise smooth fuzzy t-connected space but not a pairwise smooth fuzzy strongly t-connected space.
Proposition 5.2.2.

Let \((X, T_1, T_2)\) be any smooth fuzzy bitopological space and \(A \subseteq X\) be any subset. Then the following statements are equivalent:

(a) \((A, T_1/A, T_2/A)\) is a pairwise smooth fuzzy strongly t-connected subspace of \((X, T_1, T_2)\).

(b) For any \(T_1\)-r-fuzzy t-open sets (or) \(T_2\)-r-fuzzy t-open sets \(\lambda_1, \lambda_2 \neq 0, 1\) such that \(I_A \subseteq \lambda_1/A + \lambda_2/A\) implies either \(I_A = \lambda_1/A\) (or) \(I_A = \lambda_2/A\), \(\lambda_1, \lambda_2 \in I^X, r \in I_0\).

Proof:

(a) \(\Rightarrow\) (b).

Suppose that there exists \(T_1\)-r-fuzzy t-open sets (or) \(T_2\)-r-fuzzy t-open sets \(\lambda_1, \lambda_2 \neq 0, 1\) such that \(I_A \subseteq \lambda_1/A + \lambda_2/A\) but both \(I_A \neq \lambda_1/A\) and \(I_A \neq \lambda_2/A\), \(\lambda_1, \lambda_2 \in I^X, r \in I_0\). By Proposition 5.2.1., \(A\) is not a pairwise smooth fuzzy strongly t-connected subspace. Contradiction. Hence, (b) is proved.

(b) \(\Rightarrow\) (a).

Suppose that \(A\) is not a pairwise smooth fuzzy strongly t-connected subset of \(X\). Then, there exist \(T_1/A\)-r-fuzzy t-closed sets (or) \(T_2/A\)-r-fuzzy t-closed sets \(\lambda, \mu \neq 0, 1\) such that \(\lambda + \mu \leq I_A\), \(\lambda, \mu \in I^X, r \in I_0\). This implies that, \(T_1\)-r-fuzzy t-open sets (or) \(T_2\)-r-fuzzy t-open sets \(\lambda_1, \lambda_2 \neq 0, 1\) such that \(\lambda_1/A = I_A - \lambda, \lambda_2/A = I_A - \mu\). Then,
\[
\lambda_{1}/A + \lambda_{2}/A = (\overline{1}_{A} - \lambda) + (\overline{1}_{A} - \mu) \geq \overline{1}_{A}.
\]

Since, \(0 < \lambda_{1}/A < \overline{1}_{A}, \ 0 < \lambda_{2}/A < \overline{1}_{A}\) and \(\lambda_{1}/A + \lambda_{2}/A \geq \overline{1}_{A}\). It follows that, \(\overline{1}_{A} \neq \lambda_{1}/A\) and \(\overline{1}_{A} \neq \lambda_{2}/A\). Contradiction. Hence, (a) is proved.

**Proposition 5.2.3.**

Let \((X, T_{1}, T_{2})\) be any smooth fuzzy bitopological space. Let \(F \subseteq X\) be such that \(\overline{1}_{F}\) is a \(T_{1}\)-r-fuzzy \(t\)-closed set (or) \(T_{2}\)-r-fuzzy \(t\)-closed set. Then, \((X, T_{1}, T_{2})\) is a pairwise smooth fuzzy strongly \(t\)-connected space implies that \((F, T_{1}/F, T_{2}/F)\) is a pairwise smooth fuzzy strongly \(t\)-connected subspace.

**Proof:**

Let \(F \subseteq X\) be such that \(\overline{1}_{F}\) is a \(T_{1}\)-r-fuzzy \(t\)-closed set (or) \(T_{2}\)-r-fuzzy \(t\)-closed set. Suppose that \((F, T_{1}/F, T_{2}/F)\) is not a pairwise smooth fuzzy strongly \(t\)-connected subspace. Now, there exist \(T_{1}/F\)-r-fuzzy \(t\)-closed sets (or) \(T_{2}/F\)-r-fuzzy \(t\)-closed sets \(\lambda, \mu \neq 0, 1\) such that \(\lambda + \mu \leq \overline{1}, \lambda, \mu \in I_{X}, r \in I_{0}\). Hence, \(T_{1}\)-r-fuzzy \(t\)-closed sets (or) \(T_{2}\)-r-fuzzy \(t\)-closed sets \(\lambda_{1}, \lambda_{2} \neq 0, 1\) such that \(\lambda = \lambda_{1}/F, \mu = \lambda_{2}/F\). Since \(\overline{1}_{F}\) is a \(T_{1}\)-r-fuzzy \(t\)-closed set (or) \(T_{2}\)-r-fuzzy \(t\)-closed set, \(\lambda_{1} \wedge \overline{1}_{F}\) and \(\lambda_{2} \wedge \overline{1}_{F}\) are \(T_{1}\)-r-fuzzy \(t\)-closed set (or) \(T_{2}\)-r-fuzzy \(t\)-closed set. Therefore, \((\lambda_{1} \wedge \overline{1}_{F}) + (\lambda_{2} \wedge \overline{1}_{F}) \leq \overline{1}\). Thus, \((X, T_{1}, T_{2})\) is not a pairwise smooth fuzzy strongly \(t\)-connected space. Contradiction. Hence proved.
5.3 PAIRWISE SMOOTH FUZZY EXTREMALLY t-DISCONNECTED SPACES.

In this section, the concept of pairwise smooth fuzzy extremely t-disconnected spaces are studied with suitable examples.

Definition 5.3.1.

A smooth fuzzy bitopological space \(( X, T_1, T_2 )\) is said to be pairwise smooth fuzzy extremally t-disconnected if \( t-C_{T_1} ( \lambda, r ) \) is a \( T_2 - r \)-fuzzy t-open set for every \( T_2 - r \)-fuzzy t-open set \( \lambda \) and \( t-C_{T_2} ( \lambda, r ) \) is a \( T_1 - r \)-fuzzy t-open set for every \( T_1 - r \)-fuzzy t-open set \( \lambda, \lambda \in I^X, r \in I_0 \).

Example 5.3.1.

In Example 5.1.1., we can easily see that the smooth fuzzy bitopological space \(( X, T_1, T_2 )\) is a pairwise smooth fuzzy extremally t-disconnected space but \(( X, T_1 )\) and \(( X, T_2 )\) are both smooth fuzzy t-connected spaces.

Proposition 5.3.1.

Let \(( X, T_1, T_2 )\) be smooth fuzzy bitopological space then the following statements are equivalent:

(a) \(( X, T_1, T_2 )\) is a pairwise smooth fuzzy extremally t-disconnected space.

(b) If \( \lambda \) is a \( T_1 - r \)-fuzzy t-closed set, then \( t-I_{T_2} ( \lambda, r ) \) is a \( T_1 - r \)-fuzzy t-closed set and if \( \mu \) is a \( T_2 - r \)-fuzzy t-closed set then \( t-I_{T_1} ( \mu, r ) \)
is a $T_2$-r-fuzzy t-closed set $\lambda, \mu \in I^X, r \in I_0$.

(c) If $\lambda$ is a $T_1$-r-fuzzy t-open set, then

$$t-C_{T_1} (\bar{1} - t-C_{T_2} (\lambda, r), r) = \bar{1} - t-C_{T_2} (\lambda, r).$$

Similarly, if $\lambda$ is a $T_2$-r-fuzzy t-open set, then

$$t-C_{T_2} (\bar{1} - t-C_{T_1} (\lambda, r), r) = \bar{1} - t-C_{T_1} (\lambda, r), \lambda \in I^X, r \in I_0.$$

(d) Every pair of $T_1$-r-fuzzy t-open set $\lambda$ and $\mu$, with $t-C_{T_2} (\lambda, r) + \mu = \bar{1}$ implies that, $t-C_{T_2} (\lambda, r) + t-C_{T_1} (\mu, r) = \bar{1}, \lambda, \mu \in I^X, r \in I_0$. Similarly, every pair of $T_2$-r-fuzzy t-open set $\lambda$ and $\mu, \lambda, \mu \in I^X, r \in I_0$ with $t-C_{T_1} (\lambda, r) + \mu = \bar{1}$ implies that, $t-C_{T_2} (\lambda, r) + t-C_{T_1} (\mu, r) = \bar{1}$.

**Proof:**

(a) $\Rightarrow$ (b).

Suppose that (a) is true. Let $\lambda$ be any $T_1$-r-fuzzy t-closed set, $\lambda \in I^X, r \in I_0$. Then $\bar{1} - \lambda$ is a $T_1$-r-fuzzy t-open set. Then from (a), $t-C_{T_2} (\bar{1} - \lambda, r)$ is a $T_1$-r-fuzzy t-open set. Now, $\bar{1} - t-C_{T_2} (\bar{1} - \lambda, r)$ is a $T_1$-r-fuzzy t-closed set. But, $\bar{1} - t-C_{T_2} (\bar{1} - \lambda, r) = t-I_{T_2} (\lambda, r)$ and therefore, $t-I_{T_2} (\lambda, r)$ is a $T_1$-r-fuzzy t-closed set. Similarly, $t-I_{T_1} (\mu, r)$ is a $T_2$-r-fuzzy t-closed set for $T_2$-r-fuzzy t-closed set $\mu \in I^X, r \in I_0$. Thus, (b) is proved.

(b) $\Rightarrow$ (c).

Assume (b) is true. Suppose that $\lambda$ is a $T_1$-r-fuzzy t-open set
\( \lambda \in \mathcal{I}^X, r \in I_0 \). Then \( \bar{1} - \lambda \) is a \( T_1 \)-fuzzy t-closed set. Now, \( t-I_{T_2}(\bar{1} - \lambda, r) \) is a \( T_1 \)-fuzzy t-closed set. But, \( t-I_{T_2}(\bar{1} - \lambda, r) = \bar{1} - t-C_{T_2}(\lambda, r) \) implies that, \( \bar{1} - t-C_{T_2}(\lambda, r) \) is a \( T_1 \)-fuzzy t-closed set. Therefore,

\[
t-C_{T_1}(\bar{1} - t-C_{T_2}(\lambda, r), r) = \bar{1} - t-C_{T_2}(\lambda, r).
\]

Similarly, \( t-C_{T_2}(\bar{1} - t-C_{T_1}(\lambda, r), r) = \bar{1} - t-C_{T_1}(\lambda, r) \) where \( \lambda \) is a \( T_2 \)-fuzzy t-open set. Hence, (c) is proved.

(c) \( \Rightarrow \) (d).

Assume (c) is true. Suppose that \( \lambda \) is a \( T_1 \)-fuzzy t-open set and \( \mu \) with \( t-C_{T_2}(\lambda, r) + \mu = \bar{1}, \lambda, \mu \in \mathcal{I}^X, r \in I_0 \) Then by (c),

\[
t-C_{T_2}(\lambda, r) + t-C_{T_1}(\bar{1} - t-C_{T_2}(\lambda, r), r) = \bar{1}
\]

implies that, \( \mu = t-C_{T_1}(\bar{1} - t-C_{T_2}(\lambda, r), r) \). Now,

\[
\bar{1} - t-C_{T_2}(\lambda, r) = t-C_{T_1}(\bar{1} - t-C_{T_2}(\lambda, r), r).
\]

Therefore, \( \bar{1} - t-C_{T_2}(\lambda, r) \) is a \( T_1 \)-fuzzy t-closed set and hence,

\[
t-C_{T_1}(\mu, r) = \bar{1} - t-C_{T_2}(\lambda, r).
\]

Thus, \( t-C_{T_2}(\lambda, r) + t-C_{T_1}(\mu, r) = \bar{1} \). Similarly,

\[
t-C_{T_2}(\lambda, r) + t-C_{T_1}(\mu, r) = \bar{1}.
\]

for every pair of \( T_1 \)-fuzzy t-open set \( \lambda \) and \( T_1 \)-fuzzy t-open set \( \mu \).

Hence, (d) is proved.
(d) $\Rightarrow$ (a).

Assume that (d) is true. Let $\lambda$ be any $T_1$-r-fuzzy t-open set $\lambda$, $\mu \in I^X$, $r \in I_0$. Put $t-C_{T_2}(\lambda, r) + \mu = \overline{1}$. That is, $\mu = \overline{1} - t-C_{T_2}(\lambda, r)$. By (d),

$$t-C_{T_2}(\lambda, r) + t-C_{T_1}(\mu, r) = \overline{1}$$

and hence, $t-C_{T_2}(\lambda, r)$ is a $T_1$-r-fuzzy t-open set. Similarly, $t-C_{T_1}(\lambda, r)$ is a $T_2$-r-fuzzy t-open set. Therefore, $(X, T_1, T_2)$ is a pairwise smooth fuzzy extremally t-disconnected space.

**Proposition 5.3.2.**

Let $(X, T_1, T_2)$ be a pairwise smooth fuzzy extremally t-disconnected space. If $A \subset X$ is such that $\overline{1}$ is a $T_1$-r-fuzzy t-open set and $T_2$-r-fuzzy t-open set, then $(A, T_1/A, T_2/A)$ is a pairwise smooth fuzzy extremally t-disconnected subspace.

**Proof:**

Let $A \subset X$ be such that $\overline{1}_A$ is a $T_1$-r-fuzzy t-open set and $T_2$-r-fuzzy t-open set. Let $\lambda_1$ be $T_1/A$-r-fuzzy t-open set and let $\lambda_2$ be $T_2/A$-r-fuzzy t-open set such that $t-C_{T_2/A}(\lambda_1, r) + \lambda_2 = \overline{1}$, $\lambda_1, \lambda_2 \in I^X$, $r \in I_0$. Then, there exist $T_1$-r-fuzzy t-open set $\mu_1$ and $T_1$-r-fuzzy t-open set $\mu_2$ such that $\mu_1/A = \lambda_1$ and $\mu_2/A = \lambda_2$, $\mu_1, \mu_2 \in I^X$, $r \in I_0$. That is, $\mu_1 \wedge \overline{1}_A = \lambda_1$ and $\mu_2 \wedge \overline{1}_A = \lambda_2$. Since $\overline{1}_A$ is a $T_1$-r-fuzzy t-open set and $T_2$-r-fuzzy t-open set, $\lambda_1 \wedge \overline{1}_A$ is a $T_1$-r-fuzzy t-open set and $\lambda_2 \wedge \overline{1}_A$ is a $T_2$-r-fuzzy t-open set.
set. That is, $\lambda_1$ is a $T_1$-r-fuzzy t-open set and $\lambda_2$ is a $T_2$-r-fuzzy t-open $\lambda_1$, $\lambda_2 \in I^X$, $r \in I_0$. Since $(X, T_1, T_2)$ is a pairwise smooth fuzzy extremally t-disconnected space, $t-C_{T_2}(\lambda_1, r) + t-C_{T_1}(\lambda_2, r) = \overline{1}$ in $(X, T_1, T_2)$ and therefore in $(A, T_1/A, T_2/A)$. Thus, $(A, T_1/A, T_2/A)$ is a pairwise smooth fuzzy extremally t-disconnected subspace.

5.4 PAIRWISE SMOOTH FUZZY BASICALLY t-DISCONNECTED SPACE.

In this section, the concept of pairwise smooth fuzzy basically t-disconnected spaces are discussed with relevant examples.

Definition 5.4.1.

A smooth fuzzy bitopological space $(X, T_1, T_2)$ is said to be pairwise smooth fuzzy basically t-disconnected if $t-C_{T_1}(\lambda, r)$ is a $T_2$-r-fuzzy t-open set for every $T_2$-r-fuzzy t-open, $T_2$-r-fuzzy $F_\sigma$-set $\lambda$ and $t-C_{T_2}(\lambda, r)$ is $T_1$-r-fuzzy t-open set for every $T_1$-r-fuzzy t-open, $T_1$-r-fuzzy $F_\sigma$-set, $\lambda \in I^X$, $r \in I_0$.

Proposition 5.4.1.

Let $(X, T_1, T_2)$ be smooth fuzzy bitopological space. Then the following statements are equivalent:

(a) $(X, T_1, T_2)$ is a pairwise fuzzy basically t-disconnected.

(b) If $\lambda$ is a $T_1$-r-fuzzy t-closed, $T_1$-r-fuzzy $G_\delta$-set, then $t-I_{T_2}(\lambda, r)$ is a $T_1$-r-fuzzy t-closed set and if $\lambda$ is a $T_2$-r-fuzzy t-closed, $T_2$-r-fuzzy
$G_\delta$-set, then $t-I_{T_1}(\lambda, r)$ is a $T_2$-r-fuzzy t-closed set, $\lambda \in I^X$, $r \in I_0$.

(c) If $\lambda$ is a $T_1$-r-fuzzy t-open set, $T_1$-r-fuzzy $F_\sigma$-set, then, $\lambda \in I^X$, $r \in I_0$

$$t-C_{T_1}(t-I_{T_2}(\bar{1} - \lambda, r), r) = \bar{1} - t-C_{T_2}(\lambda, r).$$

Similarly, if $\lambda$ is a $T_2$-r-fuzzy t-open set, $T_2$-r-fuzzy $F_\sigma$-set, then

$$t-C_{T_2}(t-I_{T_1}(\bar{1} - \lambda, r), r) = \bar{1} - t-C_{T_1}(\lambda, r).$$

(d) If $\lambda$ is a $T_1$-r-fuzzy t-open set, $T_1$-r-fuzzy $F_\sigma$-set and $\mu, \lambda, \mu \in I^X$, $r \in I_0$ such that $t-C_{T_2}(\lambda, r) + \mu = \bar{1}$ then,

$$t-C_{T_2}(\lambda, r) + t-C_{T_1}(\mu, r) = \bar{1}.$$

Similarly, if $\lambda$ is a $T_2$-r-fuzzy t-open set, $T_2$-r-fuzzy $F_\sigma$-set and $\mu, \lambda, \mu \in I^X$, $r \in I_0$ such that $t-C_{T_2}(\lambda, r) + \mu = \bar{1}$, then,

$$t-C_{T_1}(\lambda, r) + t-C_{T_2}(\mu, r) = \bar{1}.$$  

**Proof:**

(a) $\Rightarrow$ (b).

Let $\lambda$ be any $T_1$-r-fuzzy t-closed set, $T_1$-r-fuzzy $G_\delta$-set, $\lambda \in I^X$, $r \in I_0$. Now, $\bar{1} - \lambda$ is a $T_1$-r-fuzzy t-open set, $T_1$-r-fuzzy $F_\sigma$-set. By (a),

$$t-C_{T_2}(\bar{1} - \lambda, r)$$

is a $T_1$-r-fuzzy t-open set. Therefore, $\bar{1} - t-C_{T_2}(\bar{1} - \lambda, r)$ is a $T_1$-r-fuzzy t-closed set. But, $\bar{1} - t-C_{T_2}(\bar{1} - \lambda, r) = t-I_{T_2}(\lambda, r)$.

Hence, $t-I_{T_2}(\lambda, r)$ is a $T_1$-r-fuzzy t-closed set. Similarly, $t-I_{T_1}(\lambda, r)$ is a $T_2$-r-fuzzy t-closed set for $T_2$-r-fuzzy t-closed, $T_2$-r-fuzzy $G_\delta$-set $\lambda$.  

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Hence, (b) is proved.

\[(b) \Rightarrow (c).\]

Let \( \lambda \) be a \( T_1 \)-r-fuzzy t-open set, \( T_1 \)-r-fuzzy \( F_\sigma \)-set, \( \lambda \in I^X, r \in I_0. \)

Now, \( \overline{1} - \lambda \) is a \( T_1 \)-r-fuzzy t-closed set, \( T_1 \)-r-fuzzy \( G_\delta \)-set. By (b),

\[t-I_{T_2} (\overline{1} - \lambda, r) \]

is a \( T_1 \)-r-fuzzy t-closed set. But, \( t-I_{T_2} (\overline{1} - \lambda, r) = \overline{1} - t-C_{T_2} (\lambda, r) \)

implies that, \( \overline{1} - t-C_{T_2} (\lambda, r) \) is a \( T_1 \)-r-fuzzy t-closed set. Therefore,

\[
t-C_{T_1} (t-I_{T_2} (\overline{1} - \lambda, r), r) = t-C_{T_1} (\overline{1} - t-C_{T_2} (\lambda, r), r).
\]

\[
t-C_{T_1} (t-I_{T_2} (\overline{1} - \lambda, r), r) = \overline{1} - t-C_{T_2} (\lambda, r).
\]

Similarly, \( t-C_{T_2} (t-I_{T_1} (\overline{1} - \lambda, r), r) = \overline{1} - t-C_{T_1} (\lambda, r) \) for \( T_2 \)-r-fuzzy t-open, \( T_2 \)-r-fuzzy \( F_\sigma \)-set \( \lambda \). Hence, (c) is proved.

\[(c) \Rightarrow (d).\]

Let \( \lambda \) be any \( T_1 \)-r-fuzzy t-open set, \( T_1 \)-r-fuzzy \( F_\sigma \)-set and \( \mu \), such that \( t-C_{T_2} (\lambda, r) + \mu = \overline{1}, \lambda, \mu \in I^X, r \in I_0. \) Now, \( \mu = \overline{1} - t-C_{T_2} (\lambda, r). \)

By (c), \( t-C_{T_2} (\lambda, r) + t-C_{T_1} (\overline{1} - t-C_{T_2} (\lambda, r), r) = \overline{1}. \) This implies that,

\[
\mu = t-C_{T_1} (\overline{1} - t-C_{T_2} (\lambda, r), r).
\]

That is,

\[
\overline{1} - t-C_{T_2} (\lambda, r) = t-C_{T_1} (\overline{1} - t-C_{T_2} (\lambda, r), r) = t-C_{T_1} (\mu, r).
\]

\[t-C_{T_2} (\lambda, r) + t-C_{T_1} (\mu, r) = \overline{1}. \]

Similarly, \( t-C_{T_1} (\lambda, r) + t-C_{T_2} (\mu, r) = \overline{1} \)

for \( T_2 \)-r-fuzzy t-open set, \( T_2 \)-r-fuzzy \( F_\sigma \)-set \( \lambda \) and \( \mu \) such that \( t-C_{T_1} (\lambda, r) + \mu = \overline{1}, \lambda, \mu \in I^X, r \in I_0. \)
(d) ⇒ (a).

Let λ be any $T_1$-r-fuzzy t-open set, $T_1$-r-fuzzy $F_\sigma$-set and $\mu$ such that $t-C_{T_2} (\lambda, r) + \mu = \bar{1}$, $\mu \in I^X$, $r \in I_0$. Now, $\mu = \bar{1} - t-C_{T_2} (\lambda, r)$. Hence by (d), $t-C_{T_2} (\lambda, r) + t-C_{T_1} (\mu, r) = \bar{1}$. $t-C_{T_1} (\mu, r) = \bar{1} - t-C_{T_2} (\lambda, r)$. $1 - t-C_{T_1} (\mu, r) = t-C_{T_2} (\lambda, r)$. $\bar{1} - t-C_{T_1} (\bar{1} - t-C_{T_2} (\lambda, r), r) = t-C_{T_2} (\lambda, r)$. Therefore, $t-C_{T_2} (\lambda, r)$ is a $T_1$-r-fuzzy t-open set. Similarly, $t-C_{T_1} (\lambda, r)$ is a $T_2$-r-fuzzy t-open set for $T_2$-r-fuzzy t-open set, $T_2$-r-fuzzy $F_\sigma$-set $\lambda$. Hence, (a) is proved.

**Proposition 5.4.2.**

Let $(X, T_1, T_2)$ be a pairwise smooth fuzzy basically t-disconnected space and let $(Y, T_1/Y, T_2/Y)$ be any pairwise smooth fuzzy subspace of $(X, T_1, T_2)$. Then, $(Y, T_1/Y, T_2/Y)$ is a pairwise fuzzy basically t-disconnected subspace.

**Proof**

Let $\lambda_1$ be a $T_1/Y$-r-fuzzy t-open set, $T_1/Y$-r-fuzzy $F_\sigma$-set, such that $t-C_{T_2} (\lambda_1, r) + \lambda_2 = \bar{1}$, $\lambda_1, \lambda_2 \in I^Y$, $r \in I_0$. Define $\lambda_1^1, \lambda_2^2 \in I^X$ as follows:

$$
\lambda_1^1 (x) = \begin{cases} 
\lambda_1 (x) & \text{if } x \in X \\
0 & \text{otherwise},
\end{cases}
$$

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\[ \lambda_2^2(x) = \begin{cases} 
\lambda_2(x) & \text{if } x \in X \\
0 & \text{otherwise.} 
\end{cases} \]

Hence, \( \lambda_1 \) is a \( T_1 \)-r-fuzzy t-open set, \( T_1 \)-r-fuzzy \( F_0 \)-set such that,

\[ t-C_{T_2}(\lambda_1, r) + \lambda_2^2 = 1. \]

Since \( (X, T_1, T_2) \) is a pairwise smooth fuzzy basically t-disconnected space, \( t-C_{T_2}(\lambda_1, r) + t-C_{T_1}(\lambda_2^2, r) = 1. \) This implies that, \( t-C_{T_2/Y}(\lambda_1, r) + t-C_{T_1/Y}(\lambda_2^2, r) = 1. \)