CHAPTER - VI

OPERATION APPROACHES ON SEMI OPEN (resp. $G_\delta$) SETS IN SMOOTH FUZZY TOPOLOGICAL SPACES
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OPERATION APPROACHES ON SEMI OPEN
(resp. G₆) SETS IN SMOOTH FUZZY TOPOLOGICAL SPACES

The concept of fuzzy semi open sets was introduced by Bin Shahna [15]. In 1975, Bruce Hutton [20] introduced the concept of normality in fuzzy topological spaces. In this chapter, an operation γ on a family of r-fuzzy semi open sets (resp. r-fuzzy G₆-sets) in a smooth fuzzy topological space is introduced. Using the operation γ on r-fuzzy semi open sets (resp. r-fuzzy G₆-sets), the concept of r-fuzzy semi-γ-open set, smooth fuzzy semi-γ-regular space, smooth fuzzy semi-γ-normal space (resp. smooth fuzzy G₆-γ-regular space, smooth fuzzy G₆-γ-normal space) are introduced. Some interesting properties and characterizations of them are investigated. Further, smooth fuzzy semi-γ-T₁ spaces (resp. smooth fuzzy G₆-γ-T₁ spaces) (i = 0, 1, 2, ½) and smooth fuzzy semi-γ-R₀ space are introduced and interrelations among the spaces are discussed with relevant examples.
6.1 r-FUZZY SEMI (resp. Gδ)-γ-OPEN SETS

In this section, the concepts of r-fuzzy semi (resp. Gδ)-γ-open sets are introduced. Some interesting properties are studied.

Notation 6.1.1.

Let (X, T) be a smooth fuzzy topological space. Then

(1) r-SO(X) denotes the family of all r-fuzzy semi open sets of (X, T).
(2) r-Gδ(X) denotes the family of all r-fuzzy Gδ-sets of (X, T).

Definition 6.1.1.

Let (X, T) be a smooth fuzzy topological space. Let γ : r-SO(X) → T_r be an operation such that λ^γ = ∧μ, where λ ≤ μ for T(μ) ≥ r, λ ∈ r-SO(X) and r ∈ I_o.

Definition 6.1.2.

Let (X, T) be a smooth fuzzy topological space. Let γ : r-Gδ(X) → T_r be an operation such that λ^γ = ∧μ, where λ ≤ μ for T(μ) ≥ r, λ ∈ r-Gδ(X) and r ∈ I_o.

Definition 6.1.3.

Let (X, T) be a smooth fuzzy topological space. Let γ be an operation on r-SO(X). δ is called a r-fuzzy semi-γ-open set if for each α with α ≤ δ, there exists a r-fuzzy semi open set λ such that α ≤ λ and λ^γ ≤ δ, α, λ, δ ∈ I^X, r ∈ I_o. The complement of a r-fuzzy semi-γ-open set is a r-fuzzy semi-γ-closed set.

Definition 6.1.4.

Let (X, T) be a smooth fuzzy topological space. Let γ be an
operation on $r$-$\text{G}_\delta(X)$. $\delta$ is called a $r$-fuzzy $\text{G}_\delta$-$\gamma$-open set if for each $\alpha$ with $\alpha \leq \delta$, there exists a $r$-fuzzy $\text{G}_\delta$-set $\lambda$ such that $\alpha \leq \lambda$ and $\lambda^\gamma \leq \delta$, $\alpha, \lambda, \delta \in I^X$, $r \in I_0$. The complement of a $r$-fuzzy $\text{G}_\delta$-$\gamma$-open set is a $r$-fuzzy $\text{F}_0$-$\gamma$-closed set.

**Definition 6.1.5.**

Let $(X, T)$ be a smooth fuzzy topological space. Let $\gamma$ be an operation on $r$-$\text{SO}(X)$. For any $\lambda \in I^X$, $r \in I_0$, the $r$-fuzzy semi-$\gamma$-interior of $\lambda$ (briefly $\gamma$-$\text{SI}_T(\lambda, r)$) is defined by

$$\gamma$-$\text{SI}_T(\lambda, r) = \vee \{ \mu : \mu \leq \lambda, \mu \text{ is a } r\text{-fuzzy semi-$\gamma$-open set} \}.$$  

**Definition 6.1.6.**

Let $(X, T)$ be a smooth fuzzy topological space. Let $\gamma$ be an operation on $r$-$\text{SO}(X)$. For any $\lambda \in I^X$, $r \in I_0$, the $r$-fuzzy semi-$\gamma$-closure of $\lambda$ (briefly $\gamma$-$\text{SC}_T(\lambda, r)$) is defined by

$$\gamma$-$\text{SC}_T(\lambda, r) = \wedge \{ \mu : \mu \geq \lambda, \mu \text{ is a } r\text{-fuzzy semi-$\gamma$-closed set} \}.$$  

**Definition 6.1.7.**

Let $(X, T)$ be a smooth fuzzy topological space. Let $\gamma$ be an operation on $r$-$\text{G}_\delta(X)$. For any $\lambda \in I^X$, $r \in I_0$, the $r$-fuzzy $\text{G}_\delta$-$\gamma$-interior of $\lambda$ (briefly $\gamma$-$\text{I}_{\text{G}_\delta}(\lambda, r)$) is defined by

$$\gamma$-$\text{I}_{\text{G}_\delta}(\lambda, r) = \vee \{ \mu : \mu \leq \lambda, \mu \text{ is a } r\text{-fuzzy } \text{G}_\delta\text{-}$\gamma$-open set}.$$  

**Definition 6.1.8.**

Let $(X, T)$ be a smooth fuzzy topological space. Let $\gamma$ be an
operation on \( r-G_0(X) \). For any \( \lambda \in I^X, r \in I_0 \), the \( r \)-fuzzy \( F_\sigma \)-\( \gamma \)-closure of \( \lambda \) (briefly \( \gamma-C_{T_{(\sigma)}}(\lambda, r) \)) is defined by

\[
\gamma-C_{T_{(\sigma)}}(\lambda, r) = \land \{ \mu : \mu \geq \lambda, \mu \text{ is a } r \text{-fuzzy } F_\sigma \text{-}\gamma \text{-closed set} \}.
\]

**Remark 6.1.1.**

Let \((X, T)\) be a smooth fuzzy topological space. Then

(1) \( \gamma-SI_T(\bar{I} - \lambda, r) = \bar{I} - \gamma-SC_T(\lambda, r) \).

(2) \( \gamma-I_{T_{(\sigma)}}(\bar{I} - \lambda, r) = \bar{I} - \gamma-C_{T_{(\sigma)}}(\lambda, r) \).

**Proposition 6.1.1.**

Let \((X, T)\) be a smooth fuzzy topological space and \( \gamma \) be an operation on \( r-SO(X) \). Then the following properties hold.

(1) For all \( r \)-fuzzy semi-\( \gamma \)-open set \( \lambda, \lambda \mu \) iff \( \lambda \gamma\-SC_T(\mu, r), \mu, \lambda \in I^X, r \in I_0 \).

(2) \( \delta \gamma\-SC_T(\lambda, r) \) iff \( \lambda \mu \) for any \( r \)-fuzzy semi-\( \gamma \)-open set \( \mu \) with \( \delta \leq \mu, \delta, \mu \in I^X, r \in I_0 \).

**Proof**

(1) Let \( \lambda \) be a \( r \)-fuzzy semi-\( \gamma \)-open set such that \( \lambda \mu, \lambda, \mu \in I^X, r \in I_0 \). Since \( \mu \leq \gamma-SO_T(\mu, r) \) and \( \lambda \mu \), it follows that \( \lambda \gamma\-SC_T(\mu, r) \).

Conversely, let \( \lambda \) be a \( r \)-fuzzy semi-\( \gamma \)-open set such that \( \lambda \mu \).

Then, \( \mu \leq \bar{I} - \lambda \).

It follows that, \( \gamma\-SO_T(\mu, r) \leq \gamma\-SO_T(\bar{I} - \lambda, r) = \bar{I} - \lambda \).

Now, \( \gamma\-SO_T(\mu, r) \leq \bar{I} - \lambda \).
Thus, \( \lambda \cdot q \cdot \gamma \cdot SC_T(\mu, r) \). Contradiction. Hence, the result.

(2) Let \( \delta q \cdot \gamma \cdot SC_T(\lambda, r) \), \( \delta, \lambda \in \mathcal{I}^X \), \( r \in I_0 \). Since \( \delta \leq \mu \), \( \mu q \cdot \gamma \cdot SC_T(\lambda, r) \). By (1), \( \mu q \lambda \) for any \( r \)-fuzzy semi-\( \gamma \)-open set \( \mu, \mu \in \mathcal{I}^X \), \( r \in I_0 \).

Conversely, suppose that \( \delta q \cdot \gamma \cdot SC_T(\lambda, r) \). Then, \( \delta \leq \mathcal{I} - \gamma \cdot SC_T(\lambda, r) \).
Let \( \mu = \mathcal{I} - \gamma \cdot SC_T(\lambda, r) \). Then, \( \mu \) is a \( r \)-fuzzy semi-\( \gamma \)-open set. Since \( \lambda \leq \gamma \cdot SC_T(\lambda, r) \) it follows that \( \mu = \mathcal{I} - \gamma \cdot SC_T(\lambda, r) \leq \mathcal{I} - \lambda \). Therefore, \( \lambda \cdot q \mu \).

Contradiction. Hence proved.

**Proposition 6.1.2.**

Let \( (X, T) \) be a smooth fuzzy topological space and \( \gamma \) be an operation on \( r \)-\( G_8(X) \). Then the following properties hold.

1. For all \( r \)-fuzzy \( G_8 \)-\( \gamma \)-open set \( \lambda, \lambda q \mu \) iff \( \lambda q \gamma \cdot C_{T(\mu)}(\mu, r) \), \( \mu, \lambda \in \mathcal{I}^X \), \( r \in I_0 \).
2. \( \delta q \gamma \cdot C_{T(\mu)}(\lambda, r) \) iff \( \lambda q \mu \) for any \( r \)-fuzzy \( G_8 \)-\( \gamma \)-open set \( \mu \) with \( \delta \leq \mu \), \( \delta, \mu \in \mathcal{I}^X \), \( r \in I_0 \).

**Proof**

The proof is similar to Proposition 6.1.1.

### 6.2 Smooth Fuzzy Semi (resp. \( G_8 \))-(\( \gamma \), \( \beta \))-Continuous Functions

In this section, smooth fuzzy semi (resp. \( G_8 \))-(\( \gamma \), \( \beta \))-continuous functions are introduced and their properties are studied.

Throughout this section, let \( (X, T), (Y, S) \) and \( (Z, R) \) be any three smooth fuzzy topological spaces. Let \( \gamma : r \cdot SO(X) \rightarrow T_r \), \( \beta : r \cdot SO(Y) \rightarrow T_r \) and \( \eta : r \cdot SO(Z) \rightarrow T_r \) be operations on \( r \cdot SO(X), r \cdot SO(Y) \) and \( r \cdot SO(Z) \) respectively.
Similarly, let \( \gamma : \mathbb{R}G_{\delta}(X) \to T_r, \beta : \mathbb{R}G_{\delta}(Y) \to T_r \) and \( \eta : \mathbb{R}G_{\delta}(Z) \to T_r \)
be operations on \( \mathbb{R}G_{\delta}(X), \mathbb{R}G_{\delta}(Y) \) and \( \mathbb{R}G_{\delta}(Z) \) respectively.

**Definition 6.2.1.**

Let \((X, T)\) be a smooth fuzzy topological space. \( \lambda \) is called a \( \mathbb{R}\)-fuzzy semi-\( \gamma \)-\( \delta \) closed set if \( \gamma SC_T(\lambda, r) \leq \mu \) whenever \( \lambda \leq \mu \) and \( \mu \) is a \( \mathbb{R}\)-fuzzy semi-\( \gamma \)-open set, \( \lambda, \mu \in I^X, r \in I_0 \). The complement of a \( \mathbb{R}\)-fuzzy semi-\( \gamma \)-\( \delta \) closed set is a \( \mathbb{R}\)-fuzzy semi-\( \gamma \)-\( \delta \) open set.

**Definition 6.2.2.**

Let \((X, T)\) be a smooth fuzzy topological space. \( \lambda \) is called a \( \mathbb{R}\)-fuzzy \( F_\sigma \)-semi-\( \gamma \)-\( \delta \) closed set if \( \gamma C_{T(0)}(\lambda, r) \leq \mu \) whenever \( \lambda \leq \mu \) and \( \mu \) is a \( \mathbb{R}\)-fuzzy \( G_\delta \)-semi-\( \gamma \)-open set, \( \lambda, \mu \in I^X, r \in I_0 \). The complement of a \( \mathbb{R}\)-fuzzy \( F_\sigma \)-semi-\( \gamma \)-\( \delta \) closed set is a \( \mathbb{R}\)-fuzzy \( G_\delta \)-semi-\( \gamma \)-\( \delta \) open set.

**Definition 6.2.3.**

Let \((X, T)\) and \((Y, S)\) be any two smooth fuzzy topological spaces.
Let \( f : (X, T) \to (Y, S) \) be a function. Then

(1) \( f \) is called a smooth fuzzy semi-(\( \gamma, \beta \))-continuous function iff for each \( \mathbb{R}\)-fuzzy semi-\( \beta \)-open set \( \mu \), \( f^1(\mu) \) is a \( \mathbb{R}\)-fuzzy semi-\( \gamma \)-open set, \( \mu \in I^Y, r \in I_0 \). Equivalently, for each \( \mathbb{R}\)-fuzzy semi-\( \beta \)-closed set \( \mu \), \( f^1(\mu) \) is a \( \mathbb{R}\)-fuzzy semi-\( \gamma \)-closed set, \( \mu \in I^Y, r \in I_0 \).

(2) \( f \) is called a smooth fuzzy semi-(\( \gamma, \beta \))-closed function iff for each \( \mathbb{R}\)-fuzzy semi-\( \gamma \)-closed set \( \lambda \), \( f(\lambda) \) is a \( \mathbb{R}\)-fuzzy semi-\( \beta \)-closed set, \( \lambda \in I^X, r \in I_0 \).

(3) \( f \) is called a smooth fuzzy semi-(\( \gamma, \beta \))-\( g \) continuous function iff for
each r-fuzzy semi-β-g closed set μ, \( f^{-1}(μ) \) is a r-fuzzy semi-γ-g closed set, \( μ \in I^Y, r \in I_0 \). Equivalently, for each r-fuzzy semi-β-g open set μ, \( f^{-1}(μ) \) is a r-fuzzy semi-γ-g open set, \( μ \in I^Y, r \in I_0 \).

(4) \( f \) is called a smooth fuzzy semi-(γ, β)-g closed function iff for any r-fuzzy semi-γ-g closed set \( λ \), \( f(λ) \) is a r-fuzzy semi-β-g closed set, \( λ \in I^X, r \in I_0 \).

**Definition 6.2.4.**

Let \((X, T)\) and \((Y, S)\) be any two smooth fuzzy topological spaces. Let \( f: (X, T) \rightarrow (Y, S) \) be a function. Then

(1) \( f \) is called a smooth fuzzy \( G_δ \)-γ, β)-continuous function iff for each r-fuzzy \( G_δ \)-β-open set μ, \( f^{-1}(μ) \) is a r-fuzzy \( G_δ \)-γ-open set, \( μ \in I^Y, r \in I_0 \). Equivalently, for each r-fuzzy \( F_σ \)-β-closed set μ, \( f^{-1}(μ) \) is a r-fuzzy \( F_σ \)-γ-closed set, \( μ \in I^Y, r \in I_0 \).

(2) \( f \) is called a smooth fuzzy \( F_σ \)-(γ, β)-closed function iff for each r-fuzzy \( F_σ \)-γ-closed set \( λ \), \( f(λ) \) is a r-fuzzy \( F_σ \)-β-closed set, \( λ \in I^X, r \in I_0 \).

(3) \( f \) is called a smooth fuzzy \( G_δ \)-(γ, β)-g continuous function iff for each r-fuzzy \( G_δ \)-β-g closed set μ, \( f^{-1}(μ) \) is a r-fuzzy \( G_δ \)-γ-g closed set, \( μ \in I^Y, r \in I_0 \). Equivalently, for each r-fuzzy \( G_δ \)-β-g open set μ, \( f^{-1}(μ) \) is a r-fuzzy \( G_δ \)-γ-g open set, \( μ \in I^Y, r \in I_0 \).

(4) \( f \) is called a smooth fuzzy \( F_σ \)-(γ, β)-g closed function iff for any r-fuzzy \( F_σ \)-γ-g closed set \( λ \), \( f(λ) \) is a r-fuzzy \( F_σ \)-β-g closed set, \( λ \in I^X, r \in I_0 \).
Proposition 6.2.1.

Let \((X, T)\) and \((Y, S)\) be any two smooth fuzzy topological spaces. \(f : (X, T) \rightarrow (Y, S)\) is a smooth fuzzy semi-(\(\gamma, \beta\))-continuous function iff
\[ f(\gamma-SC_T(\lambda, r)) \leq \beta-SC_S(f(\lambda), r) \], for every \(\lambda \in I^X, r \in I_0\).

Proof

Let \(f\) be a smooth fuzzy semi-(\(\gamma, \beta\))-continuous function. Now, \(\beta-SC_S(f(\lambda), r)\) is a \(r\)-fuzzy semi-\(\beta\)-closed set, \(\lambda \in I^X, r \in I_0\).

Then, \(f^{-1}(\beta-SC_S(f(\lambda), r))\) is a \(r\)-fuzzy semi-\(\gamma\)-closed set.

Now, \(\lambda \leq f^{-1}(f(\lambda)) \leq f^{-1}(\beta-SC_S(f(\lambda), r)).\)

Then, \(\gamma-SC_T(\lambda, r) \leq f^{-1}(\beta-SC_S(f(\lambda), r)).\)

That is, \(f(\gamma-SC_T(\lambda, r)) \leq \beta-SC_S(f(\lambda), r)\).

Conversely, suppose that \(\mu \in I^Y, r \in I_0\) is a \(r\)-fuzzy semi-\(\beta\)-closed set. By hypothesis, \(f(\gamma-SC_T(f^{-1}(\mu), r)) \leq \beta-SC_S(f(f^{-1}(\mu)), r) = \mu\). This implies that, \(\gamma-SC_T(f^{-1}(\mu), r) \leq f^{-1}(\mu).\) But \(\gamma-SC_T(f^{-1}(\mu), r) \geq f^{-1}(\mu).\)

Hence, \(\gamma-SC_T(f^{-1}(\mu), r) = f^{-1}(\mu).\)

That is, \(f^{-1}(\mu)\) is a \(r\)-fuzzy semi-\(\gamma\)-closed set. Therefore, \(f\) is a smooth fuzzy semi-(\(\gamma, \beta\))-continuous function.

Proposition 6.2.2.

Let \((X, T)\) and \((Y, S)\) be any two smooth fuzzy topological spaces. \(f : (X, T) \rightarrow (Y, S)\) is a smooth fuzzy \(G_\delta\)-semi-(\(\gamma, \beta\))-continuous function iff
\[ f(\gamma-C_{T(\sigma)}(\lambda, r)) \leq \beta-C_{S(\sigma)}(f(\lambda), r) \], for every \(\lambda \in I^X, r \in I_0\).
Proof

The proof is similar to Proposition 6.2.1.

Proposition 6.2.3.

Let \((X, T)\) and \((Y, S)\) be any two smooth fuzzy topological spaces. \(f : (X, T) \rightarrow (Y, S)\) is a smooth fuzzy semi-(\(\gamma, \beta\))-continuous function iff

\[\gamma-SC_T(f^{-1}(\lambda), r) \leq f^{-1}(\beta-SC_S(\lambda, r)), \text{ for every } \lambda \in I^Y, r \in I_0.\]

Proof

Let \(f\) be a smooth fuzzy semi-(\(\gamma, \beta\))-continuous function. Now, 

\(\beta-SC_S(\lambda, r)\) is a \(r\)-fuzzy semi-\(\beta\)-closed set, \(\lambda \in I^Y, r \in I_0.\)

Then, \(f^{-1}(\beta-SC_S(\lambda, r))\) is a \(r\)-fuzzy semi-\(\gamma\)-closed set.

Then, \(f^{-1}(\lambda) \leq f^{-1}(\beta-SC_S(\lambda, r))\).

Conversely, suppose that \(\mu \in I^Y, r \in I_0\) is a \(r\)-fuzzy semi-\(\beta\)-closed set. By hypothesis, \(\gamma-SC_T(f^{-1}(\mu), r) \leq f^{-1}(\beta-SC_S(\mu, r)) = f^{-1}(\mu)\). This implies that, \(\gamma-SC_T(f^{-1}(\mu), r) \leq f^{-1}(\mu)\). But \(\gamma-SC_T(f^{-1}(\mu), r) \geq f^{-1}(\mu)\).

Hence, \(\gamma-SC_T(f^{-1}(\mu), r) = f^{-1}(\mu)\).

That is, \(f^{-1}(\mu)\) is a \(r\)-fuzzy semi-\(\gamma\)-closed set. Therefore, \(f\) is a smooth fuzzy semi-(\(\gamma, \beta\))-continuous function.

Proposition 6.2.4.

Let \((X, T)\) and \((Y, S)\) be any two smooth fuzzy topological spaces. \(f : (X, T) \rightarrow (Y, S)\) is a smooth fuzzy \(G_\delta\)-semi-(\(\gamma, \beta\))-continuous function iff

\[\gamma-C_{T(\delta)}(f^{-1}(\lambda), r) \leq f^{-1}(\beta-C_{S(\delta)}(\lambda, r)), \text{ for every } \lambda \in I^Y, r \in I_0.\]
Proof

The proof is similar to Proposition 6.2.3.

Proposition 6.2.5.

Let \((X, T), (Y, S)\) and \((Z, R)\) be any three smooth fuzzy topological spaces. If \(f : (X, T) \rightarrow (Y, S)\) is a smooth fuzzy semi-\((\gamma, \beta)\)-continuous function and \(g : (Y, S) \rightarrow (Z, R)\) is a smooth fuzzy semi-\((\beta, \eta)\)-continuous function, then 
\(g \circ f : (X, T) \rightarrow (Z, R)\) is a smooth fuzzy semi-\((\gamma, \eta)\)-continuous function.

Proof

Let \(\lambda \in \mathcal{I}^2, r \in I_0\) be a \(r\)-fuzzy semi-\(\eta\)-closed set. Since \(g\) is a smooth fuzzy semi-\((\beta, \eta)\)-continuous function, 
\(g^{-1}(\lambda)\) is a \(r\)-fuzzy semi-\(\beta\)-closed set. Since \(f\) is a smooth fuzzy semi-\((\gamma, \beta)\)-continuous function, 
\(f^{-1}(g^{-1}(\lambda)) = (g \circ f)^{-1}(\lambda)\) is a \(r\)-fuzzy semi-\(\gamma\)-closed set. Hence, 
\(g \circ f\) is a smooth fuzzy semi-\((\gamma, \eta)\)-continuous function.

Proposition 6.2.6.

Let \((X, T), (Y, S)\) and \((Z, R)\) be any three smooth fuzzy topological spaces. If \(f : (X, T) \rightarrow (Y, S)\) is a smooth fuzzy \(G_\delta\)-\((\gamma, \beta)\)-continuous function and \(g : (Y, S) \rightarrow (Z, R)\) are smooth fuzzy \(G_\delta\)-\((\beta, \eta)\)-continuous function, then 
\(g \circ f : (X, T) \rightarrow (Z, R)\) is a smooth fuzzy \(G_\delta\)-\((\gamma, \eta)\)-continuous function.

Proof

The proof is similar to Proposition 6.2.5.
6.3 FUZZY SEMI (resp. $G_{5}$)-$\gamma$-$T_{1}$ SPACES

In this section, the concepts of fuzzy semi (resp. $G_{5}$)-$\gamma$-$T_{1}$ spaces are introduced and interrelations between them are studied.

**Definition 6.3.1.**

A smooth fuzzy topological space $(X, T)$ is called

1. smooth fuzzy semi-$\gamma$-$T_{0}$ space iff for $\lambda, \mu$ with $\lambda \leq \mu, \lambda, \mu \in I_{X}, r \in I_{0}$, there exists a $r$-fuzzy semi-$\gamma$-open set $\delta$ such that either $\lambda \leq \delta, \mu \leq \delta$ or $\mu \leq \delta, \lambda \leq \delta$, $\delta \in I_{X}, r \in I_{0}$.

2. smooth fuzzy semi-$\gamma$-$T_{1}$ space iff for $\lambda, \mu$ with $\lambda \leq \mu, \lambda, \mu \in I_{X}, r \in I_{0}$, there exist $r$-fuzzy semi-$\gamma$-open sets $\delta, \eta$ such that $\lambda \leq \delta, \mu \leq \eta$ and $\mu \leq \eta, \lambda \leq \eta, \delta, \eta \in I_{X}, r \in I_{0}$.

3. smooth fuzzy semi-$\gamma$-$T_{2}$ space iff for $\lambda, \mu$ with $\lambda \leq \mu, \lambda, \mu \in I_{X}, r \in I_{0}$, there exist $r$-fuzzy semi-$\gamma$-open sets $\delta, \eta$ with $\lambda \leq \delta, \mu \leq \eta$ and $\delta \leq \eta, \delta, \eta \in I_{X}, r \in I_{0}$.

4. smooth fuzzy semi-$\gamma$-$R_{0}$ space iff $\lambda \leq \mu, \lambda \leq \mu \in I_{X}, r \in I_{0}$ implies that $\mu \leq \lambda, \mu \leq \lambda \in I_{X}, r \in I_{0}$.

**Remark 6.3.1.**

From the above definitions we have the following implications.

Smooth fuzzy semi-$\gamma$-$T_{2}$ space $\Rightarrow$ Smooth fuzzy semi-$\gamma$-$T_{1}$ space $\Rightarrow$ Smooth fuzzy semi-$\gamma$-$T_{0}$ space.

The following examples show that the converse statements need not be true.
Example 6.3.1.

Let $X = \{a, b\}$ be a set. Let $\lambda_1, \lambda_2, \lambda_3 \in I^X$ be defined as follows.

$\lambda_1(a) = 0.51$, $\lambda_1(b) = 0.7$, $\lambda_2(a) = 0.57$, $\lambda_2(b) = 0.78$ and $\lambda_3(a) = 0.63$, $\lambda_3(b) = 0.83$. Define a smooth fuzzy topology $T : I^X \to I$ as follows.

$$
T(\beta) = \begin{cases} 
1 & \beta = 0, 1 \\
0.5 & \beta = \lambda_1 \\
0.66 & \beta = \lambda_2 \\
0.5 & \beta = \lambda_3 \\
0 & \text{otherwise.}
\end{cases}
$$

Let $\gamma : r \text{-SO}(X) \to T_r$ be an operation. Let $r = 0.04$. Define $\alpha, \mu \in I^X$ as follows $\alpha(a) = 0.3$, $\alpha(b) = 0.4$ and $\mu(a) = 0.55$, $\mu(b) = 0.75$. Clearly $\mu$ is a 0.04-fuzzy semi open set. Let $\delta, \eta \in I^X$ be defined as follows $\delta(a) = 0.6$, $\delta(b) = 0.8$ and $\eta(a) = 0.65$, $\eta(b) = 0.85$. Now $\alpha \leq \mu$, $\alpha \leq \delta$ and $\alpha \leq \eta$. Further $\mu \gamma \leq \delta$ and $\mu \gamma \leq \eta$. Hence, $\delta$ and $\eta$ are 0.04-fuzzy semi-$\gamma$-open sets. Define $\theta, \lambda \in I^X$ as follows $\theta(a) = 0.3$, $\theta(b) = 0.1$ and $\lambda(a) = 0.2$, $\lambda(b) = 0$, then $\theta \lambda \theta$. Further $\theta \leq \delta$, $\lambda \eta \delta$ and $\lambda \leq \eta$, $\theta \eta$. Hence, $(X, T)$ is a smooth fuzzy semi-$\gamma$-$T_1$ space but not a smooth fuzzy semi-$\gamma$-$T_2$ space.

Example 6.3.2.

Let $X = \{a, b\}$ be a set. Let $\lambda_1, \lambda_2 \in I^X$ be defined as follows.

$\lambda_1(a) = 0.3$, $\lambda_1(b) = 0.2$ and $\lambda_2(a) = 0.45$, $\lambda_2(b) = 0.4$. Define a smooth fuzzy topology $T : I^X \to I$ as follows.
Let $\gamma : r \text{-SO}(X) \to T_r$ be an operation. Let $r = 0.04$. Define $\alpha, \mu \in I_X$ as $\alpha(a) = 0.2$, $\alpha(b) = 0.3$ and $\mu(a) = 0.4$, $\mu(b) = 0.3$. Clearly $\mu$ is a 0.04-fuzzy semi open set. Let $\delta, \eta \in I_X$ be defined as follows $\delta(a) = 0.55$, $\delta(b) = 0.65$ and $\eta(a) = 0.8$, $\eta(b) = 0.7$. Now $\alpha \leq \mu$, $\alpha \leq \delta$ and $\alpha \leq \eta$.

Further $\mu \leq \delta$ and $\mu \leq \eta$. Hence, $\delta$ and $\eta$ are 0.04-fuzzy semi-\(\gamma\)-open sets. Define $\theta, \rho \in I_X$ as $\theta(a) = 0.3$, $\theta(b) = 0.4$ and $\rho(a) = 0.4$, $\rho(b) = 0.2$ such that $\theta \leq \rho$. Now $\theta \leq \delta$ and $\rho \leq \eta$ but $\rho \not\leq \eta$ and $\theta \eta \eta$. Hence, $(X, T)$ is a smooth fuzzy semi-\(\gamma\)-\(T_0\) space but not a smooth fuzzy semi-\(\gamma\)-\(T_1\) space.

**Definition 6.3.2.**

A smooth fuzzy topological space $(X, T)$ is called

(1) smooth fuzzy $G_\delta$-\(\gamma\)-\(T_0\) space iff for $\lambda, \mu$ with $\lambda \leq \delta$, $\mu \in I_X$, $r \in I_0$, there exists a $r$-fuzzy $G_\delta$-\(\gamma\)-open set $\delta$ such that either $\lambda \leq \delta$, $\mu \delta$ or $\mu \leq \delta$, $\lambda \delta$.

(2) smooth fuzzy $G_\delta$-\(\gamma\)-\(T_1\) space iff for $\lambda, \mu$ with $\lambda \leq \delta$, $\mu \in I_X$, $r \in I_0$, there exist $r$-fuzzy $G_\delta$-\(\gamma\)-open sets $\delta, \eta$ such that $\lambda \leq \delta$, $\mu \delta$ and $\mu \leq \eta$, $\lambda \eta$.

(3) smooth fuzzy $G_\delta$-\(\gamma\)$-T_2$ space iff for $\lambda, \mu$ with $\lambda \leq \delta$, $\mu \in I_X$, $r \in I_0$,
there exist \( r \)-fuzzy \( G_8^\gamma \)-open sets \( \delta, \eta \) with \( \lambda \leq \delta, \mu \leq \eta \) and \( \deltaq \eta, \delta, \eta \in I^X, r \in I_0 \).

(4) smooth fuzzy \( G_8^\gamma \)-R \( _0 \) space iff \( \lambda q \gamma - C_{T(\delta)}(\mu, r) \) implies that \( \mu q \gamma - C_{T(\delta)}(\lambda, r) \) for \( \lambda, \mu \in I^X, r \in I_0 \).

**Remark 6.3.2.**

From the above definitions we have the following implications.

**Smooth fuzzy \( G_8^\gamma \)-T \( _2 \) space \( \Rightarrow \) Smooth fuzzy \( G_8^\gamma \)-T \( _1 \) space \( \Rightarrow \) Smooth fuzzy \( G_8^\gamma \)-T \( _0 \) space.**

The following examples show that the converse statements need not be true.

**Example 6.3.3.**

Let \( X = \{a, b\} \) be a set. Let \( \lambda_1, \lambda_2, \lambda_3 \in I^X \) be defined as follows.

\[
\begin{align*}
\lambda_1(a) &= 0.51, \quad \lambda_1(b) = 0.7, \\
\lambda_2(a) &= 0.57, \quad \lambda_2(b) = 0.78 \quad \text{and} \quad \lambda_3(a) = 0.63, \\
\lambda_3(b) &= 0.83.
\end{align*}
\]

Define a smooth fuzzy topology \( T : I^X \rightarrow I \) as follows.

\[
T(\beta) = \begin{cases} 
1 & \beta = \bar{0}, \bar{1} \\
0.5 & \beta = \lambda_1 \\
0.66 & \beta = \lambda_2 \\
0.5 & \beta = \lambda_3 \\
0 & \text{otherwise.}
\end{cases}
\]

Let \( \gamma : r-G_8(X) \rightarrow T_r \) be an operation. Let \( r = 0.04 \). Define \( \alpha \in I^X \) as follows \( \alpha(a) = 0.3, \alpha(b) = 0.4 \). Clearly \( \lambda_1 \) is a 0.04-fuzzy \( G_8 \)-set.
Let $\delta, \eta \in I^X$ be defined as follows $\delta(a) = 0.6$, $\delta(b) = 0.8$ and $\eta(a) = 0.65$, $\eta(b) = 0.85$. Now $\alpha \leq \lambda_1$, $\alpha \leq \delta$ and $\alpha \leq \eta$. Further, $\lambda_1^\gamma \leq \delta$ and $\lambda_1^\gamma \leq \eta$. Hence, $\delta$ and $\eta$ are 0.04-fuzzy $G_{\delta^\gamma}$-open sets. Define $\theta, \lambda \in I^X$ as follows $\theta(a) = 0.3$, $\theta(b) = 0.1$ and $\lambda(a) = 0.2$, $\lambda(b) = 0$, then $\theta q \lambda$. Further $\theta \leq \delta$, $\lambda q \delta$ and $\lambda \leq \eta$, $\theta q \eta$. Hence, $(X, T)$ is a smooth fuzzy $G_{\delta^\gamma}$-$T_1$ space but not a smooth fuzzy $G_{\delta^\gamma}$-$T_2$ space.

Example 6.3.4.

Let $X = \{a, b\}$ be a set. Let $\lambda_1, \lambda_2 \in I^X$ be defined as follows. $\lambda_1(a) = 0.3$, $\lambda_1(b) = 0.2$ and $\lambda_2(a) = 0.45$, $\lambda_2(b) = 0.4$. Define a smooth fuzzy topology $T : I^X \rightarrow I$ as follows.

$$T(\beta) = \begin{cases} 1 & \beta = \overline{0}, \overline{1} \\ 0.5 & \beta = \lambda_1 \\ 0.66 & \beta = \lambda_2 \\ 0 & \text{otherwise} \end{cases}$$

Let $\gamma : r$-$SO(X) \rightarrow T_r$ be an operation. Let $r = 0.04$. Define $\alpha \in I^X$ as $\alpha(a) = 0.2$, $\alpha(b) = 0.3$. Clearly $\lambda_1$ is a 0.04-fuzzy $G_\delta$-set. Let $\delta, \eta \in I^X$ be defined as follows $\delta(a) = 0.55$, $\delta(b) = 0.65$ and $\eta(a) = 0.8$, $\eta(b) = 0.7$. Now $\alpha \leq \lambda_1$, $\alpha \leq \delta$ and $\alpha \leq \eta$. Further $\lambda_1^\gamma \leq \delta$ and $\lambda_1^\gamma \leq \eta$. Hence $\delta$ and $\eta$ are 0.04-fuzzy $G_{\delta^\gamma}$-open sets. Define $\theta, \rho \in I^X$ as $\theta(a) = 0.3$, $\theta(b) = 0.4$ and $\rho(a) = 0.4$, $\rho(b) = 0.2$ such that $\theta q \rho$. Now $\theta \leq \delta$ and $\rho q \delta$ but $\rho \leq \eta$ and $\theta q \eta$. Hence, $(X, T)$ is a smooth fuzzy $G_{\delta^\gamma}$-$T_0$ space but not a smooth fuzzy $G_{\delta^\gamma}$-$T_1$ space.
Proposition 6.3.1.

Let \((X, T)\) be a smooth fuzzy topological space. Then the following statements are equivalent.

(1) \((X, T)\) is a smooth fuzzy semi-\(\gamma\)-\(R_0\) space.

(2) If \(\delta \leq \lambda, \lambda = \gamma\text{-SC}_T(\lambda, r)\), there exists a \(r\)-fuzzy semi-\(\gamma\)-open set \(\mu\) such that \(\delta \leq \mu, \delta, \mu, \lambda \in I^X, r \in I_0\).

(3) If \(\delta \leq \lambda, \lambda = \gamma\text{-SC}_T(\lambda, r)\) then \(\gamma\text{-SC}_T(\delta, r) \leq \gamma\text{-SC}_T(\lambda, r)\), \(\delta, \lambda \in I^X, r \in I_0\).

(4) If \(\delta \leq \gamma\text{-SC}_T(\rho, r)\) then \(\gamma\text{-SC}_T(\delta, r) \leq \gamma\text{-SC}_T(\rho, r)\), \(\delta, \rho \in I^X, r \in I_0\).

Proof

(1) \(\Rightarrow\) (2). Let \(\delta \leq \lambda, \lambda = \gamma\text{-SC}_T(\lambda, r)\). Since \(\gamma\text{-SC}_T(\rho, r) \leq \gamma\text{-SC}_T(\lambda, r)\) for each \(\rho \leq \lambda\), it follows that \(\delta \leq \gamma\text{-SC}_T(\rho, r)\). (1) implies that \(\rho \leq \gamma\text{-SC}_T(\delta, r)\).

By Proposition 6.1.1.(2), for each \(\rho \in I^X, r \in I_0\) with \(\rho \leq \gamma\text{-SC}_T(\delta, r)\), there exists a \(r\)-fuzzy semi-\(\gamma\)-open set \(\eta\) such that \(\delta \leq \eta, \eta \in I^X, r \in I_0\).

Let \(\mu = \lor \{ \eta : \delta \leq \eta \}\). Then \(\delta \leq \mu, \lambda \leq \mu\) for all \(r\)-fuzzy semi-\(\gamma\)-open set \(\mu\).

(2) \(\Rightarrow\) (3). Let \(\delta \leq \lambda, \lambda = \gamma\text{-SC}_T(\lambda, r)\). By (2), there exists a \(r\)-fuzzy semi-\(\gamma\)-open set \(\mu\) such that \(\delta \leq \mu, \mu, \lambda \in I^X, r \in I_0\). Since \(\delta \leq \mu\), it follows that \(\delta \leq 1 - \mu\). Therefore,

\[
\gamma\text{-SC}_T(\delta, r) \leq \gamma\text{-SC}_T(1 - \mu, r) = 1 - \mu = 1 - \lambda
\]

Hence, \(\gamma\text{-SC}_T(\delta, r) \leq \lambda\).

(3) \(\Rightarrow\) (4). Let \(\delta \leq \gamma\text{-SC}_T(\rho, r), \delta, \rho \in I^X, r \in I_0\). Now,
\[ \gamma\text{-}SC_T(\gamma\text{-}SC_T(\rho, r), r) = \gamma\text{-}SC_T(\rho, r). \]

By (3), it follows that \( \gamma\text{-}SC_T(\delta, r) \subset \gamma\text{-}SC_T(\rho, r) \).

(4) \implies (1). Let \( \delta \subset \gamma\text{-}SC_T(\rho, r) \), \( \delta, \rho \in I^x, r \in I_0 \). By (4), it follows that \( \gamma\text{-}SC_T(\delta, r) \subset \gamma\text{-}SC_T(\rho, r) \). Now, \( \rho \leq \gamma\text{-}SC_T(\rho, r) \) implies that \( \rho \subset \gamma\text{-}SC_T(\delta, r) \).

Hence, \((X, T)\) is a smooth fuzzy semi-\( \gamma\text{-}R_0 \) space.

**Proposition 6.3.2.**

Let \((X, T)\) be a smooth fuzzy topological space. Then the following statements are equivalent.

1. \((X, T)\) is a smooth fuzzy \(G_\delta\text{-}\gamma\text{-}R_0 \) space.
2. If \( \delta \subset X \), \( X = \gamma\text{-}C_T(\delta)(X, r) \), there exists a \( r\text{-}fuzzy \ G_\delta\text{-}\gamma\text{-}open \ set \mu \) such that \( \delta \subset \mu \), \( \lambda, \mu, \lambda \in I^x, r \in I_0 \).
3. If \( \delta \subset X \), \( X = \gamma\text{-}C_T(\delta)(X, r) \) then \( \gamma\text{-}C_T(\delta)(\delta, r) \subset X, \delta \subset X \).
4. If \( \delta \subset \gamma\text{-}C_T(\delta)(\rho, r) \) then \( \gamma\text{-}C_T(\delta)(\delta, r) \subset \gamma\text{-}C_T(\delta)(\rho, r), \delta, \rho \in I^x, r \in I_0 \).

**Proof**

The proof is similar to Proposition 6.3.1.

**Definition 6.3.3.**

A smooth fuzzy topological space \((X, T)\) is a smooth fuzzy semi-\( \gamma\text{-}T_{1/2} \) space if every \( r\text{-}fuzzy \) semi-\( \gamma\text{-}g \) closed set is a \( r\text{-}fuzzy \) semi-\( \gamma\text{-}c \) closed set. Equivalently, every \( r\text{-}fuzzy \) semi-\( \gamma\text{-}g \) open set is a \( r\text{-}fuzzy \) semi-\( \gamma\text{-}o \) open set.

**Definition 6.3.4.**

A smooth fuzzy topological space \((X, T)\) is a smooth fuzzy \(G_\delta\text{-}\gamma\text{-}T_{1/2} \) space if every \( r\text{-}fuzzy \) \( F_\sigma\text{-}\gamma\text{-}g \) closed set is a \( r\text{-}fuzzy \) \( F_\sigma\text{-}\gamma\text{-}c \) closed...
Equivalently, every r-fuzzy $G_{\delta}^{\gamma}$-g open set is a r-fuzzy $G_{\delta}^{\gamma}$-open set.

**Proposition 6.3.3.**

Let $(X, T)$ and $(Y, S)$ be any two smooth fuzzy topological spaces. Let $f : (X, T) \to (Y, S)$ be a smooth fuzzy semi-$\gamma, \beta$-continuous, smooth fuzzy semi-$\gamma, \beta$-g continuous, smooth fuzzy semi-$\gamma, \beta$-closed and smooth fuzzy semi-$\gamma, \beta$-g closed function.

Then the following conditions hold.

1. If $f$ is injective and $(Y, S)$ is a smooth fuzzy semi-$\beta$-$T_{1/2}$ space, then $(X, T)$ is a smooth fuzzy semi-$\gamma$-$T_{1/2}$ space.

2. If $f$ is surjective and $(X, T)$ is a smooth fuzzy semi-$\gamma$-$T_{1/2}$ space, then $(Y, S)$ is a smooth fuzzy semi-$\beta$-$T_{1/2}$ space.

**Proof**

1. Let $\lambda$ be a r-fuzzy semi-$\gamma$-g closed set, $\lambda \in I_X^r, r \in I_0$. Since $f$ is a smooth fuzzy semi-$\gamma, \beta$-g closed function it follows that $f(\lambda)$ is a r-fuzzy semi-$\beta$-g closed set. Since $(Y, S)$ is a smooth fuzzy semi-$\beta$-$T_{1/2}$ space, $f(\lambda)$ is a r-fuzzy semi-$\beta$-closed set. Since $f$ is a smooth fuzzy semi-$\gamma, \beta$-continuous injective function, it follows that $\lambda = f^{-1}(f(\lambda))$ is a r-fuzzy semi-$\gamma$-closed set. Hence, $(X, T)$ is a smooth fuzzy semi-$\gamma$-$T_{1/2}$ space.

2. Let $\mu$ be a r-fuzzy semi-$\beta$-g closed set, $\mu \in I_Y^r, r \in I_0$. By smooth fuzzy semi-$\gamma, \beta$-g continuity of $f$, it follows that $f^{-1}(\mu)$ is a r-fuzzy semi-$\gamma$-g closed set. Since $(X, T)$ is a smooth fuzzy semi-$\gamma$-$T_{1/2}$ space,
$f^{-1}(\mu)$ is a $r$-fuzzy semi-$\gamma$-closed set. Since $f$ is a smooth fuzzy semi-$(\gamma, \beta)$ closed surjective function, it follows that $\mu = f(f^{-1}(\mu))$ is a $r$-fuzzy semi-$\beta$-closed set. Hence, $(Y, S)$ is a smooth fuzzy semi-$\beta$-$T_{1/2}$ space.

**Proposition 6.3.4.**

Let $(X, T)$ and $(Y, S)$ be any two smooth fuzzy topological spaces and let $f : (X, T) \rightarrow (Y, S)$ be a smooth fuzzy $G_{\delta}$-$(\gamma, \beta)$-continuous, smooth fuzzy $G_{\delta}$-$(\gamma, \beta)$-g continuous, smooth fuzzy $F_{\sigma}$-$(\gamma, \beta)$ closed and smooth fuzzy $F_{\sigma}$-$(\gamma, \beta)$-g closed function. Then the following conditions hold.

(1) If $f$ is injective and $(Y, S)$ is a smooth fuzzy $G_{\delta}$-$\beta$-$T_{1/2}$ space, then $(X, T)$ is a smooth fuzzy $G_{\delta}$-$\gamma$-$T_{1/2}$ space.

(2) If $f$ is surjective and $(X, T)$ is a smooth fuzzy $G_{\delta}$-$\gamma$-$T_{1/2}$ space, then $(Y, S)$ is a smooth fuzzy $G_{\delta}$-$\beta$-$T_{1/2}$ space.

**Proof**

The proof is similar to Proposition 6.3.3.

**Proposition 6.3.5.**

Let $(X, T)$ and $(Y, S)$ be any two smooth fuzzy topological spaces. Let $f : (X, T) \rightarrow (Y, S)$ be a smooth fuzzy semi-$(\gamma, \beta)$-continuous injective function. If $(Y, S)$ is a smooth fuzzy semi-$\beta$-$T_2$ (resp. smooth fuzzy semi-$\beta$-$T_1$) space, then $(X, T)$ is a smooth fuzzy semi-$\gamma$-$T_2$ (resp. smooth fuzzy semi-$\gamma$-$T_1$) space.
Proof

Let \((Y, S)\) be a smooth fuzzy semi-\(\beta\)-\(T_2\) space. Let \(\lambda_1, \lambda_2\) be such that \(\lambda_1 \leq \lambda_2, \lambda_1, \lambda_2 \in I^X, r \in I_0\). This implies that \(f(\lambda_1) \leq f(\lambda_2)\). Then there exist \(r\)-fuzzy semi-\(\beta\)-open sets \(\lambda, \mu\) such that \(f(\lambda_1) \leq \lambda, f(\lambda_2) \leq \mu\) and that \(\lambda \leq \mu, \lambda, \mu \in I^Y, r \in I_0\). Then, \(\lambda \leq 1 - \mu\) implies that \(f^{-1}(\lambda) \leq f^{-1}(\mu)\).

Now, \(\lambda_1 \leq f^{-1}(\lambda)\) and \(\lambda_2 \leq f^{-1}(\mu)\).

Since \(f\) is a smooth fuzzy semi-(\(\gamma\), \(\beta\))-continuous function, \(f^{-1}(\lambda)\) and \(f^{-1}(\mu)\) are \(r\)-fuzzy semi-\(\gamma\)-open sets. Hence, \((X, T)\) is a smooth fuzzy semi-\(\gamma\)-\(T_2\) space.

Similarly we prove the case of smooth fuzzy semi-\(\gamma\)-\(T_1\) space.

Proposition 6.3.6.

Let \((X, T)\) and \((Y, S)\) be any two smooth fuzzy topological spaces. Let \(f : (X, T) \to (Y, S)\) be a smooth fuzzy \(G_\delta\)-(\(\gamma\), \(\beta\))-continuous injective function. If \((Y, S)\) is a smooth fuzzy \(G_\delta\)-\(\beta\)-\(T_2\) (resp. smooth fuzzy \(G_\delta\)-\(\beta\)-\(T_1\)) space, then \((X, T)\) is a smooth fuzzy \(G_\delta\)-\(\gamma\)-\(T_2\) (resp. Smooth fuzzy \(G_\delta\)-\(\gamma\)-\(T_1\)) space.

Proof

The proof is similar to Proposition 6.3.5.

6.4 PROPERTIES OF \(r\)-FUZZY SEMI (resp. \(G_\delta\))-\(\gamma\)-OPEN SETS

In this section, the concepts of \(r\)-fuzzy semi (resp. \(G_\delta\))-\(\gamma\)-border, \(r\)-fuzzy semi (resp. \(G_\delta\))-\(\gamma\)-frontier, \(r\)-fuzzy semi (resp. \(G_\delta\))-\(\gamma\)-exterior are introduced. Some interesting properties are discussed.
Definition 6.4.1.

Let $(X, T)$ be a smooth fuzzy topological space. Let $\gamma : I^X \to T_r$ be an operation. The $r$-fuzzy-$\gamma$-border of $\lambda$, (briefly $\gamma-B_T(\lambda, r)$) is defined by $\gamma-B_T(\lambda, r) = \lambda - \gamma-I_T(\lambda, r), \lambda \in I^X, r \in I_0$.

Definition 6.4.2.

Let $(X, T)$ be a smooth fuzzy topological space. Let $\gamma$ be an operation on $r$-$SO(X)$. The $r$-fuzzy semi-$\gamma$-border of $\lambda$ (briefly $\gamma-SB_T(\lambda, r)$) is defined by $\gamma-SB_T(\lambda, r) = \lambda - \gamma-SI_T(\lambda, r), \lambda \in I^X, r \in I_0$.

Definition 6.4.3.

Let $(X, T)$ be a smooth fuzzy topological space. Let $\gamma$ be an operation on $r$-$SO(X)$. The $r$-fuzzy $G_\delta$-$\gamma$-border of $\lambda$ (briefly $\gamma-B_{T[a]}(\lambda, r)$) is defined by $\gamma-B_{T[a]}(\lambda, r) = \lambda - \gamma-I_{T[a]}(\lambda, r), \lambda \in I^X, r \in I_0$.

Definition 6.4.4.

Let $(X, T)$ be a smooth fuzzy topological space. Let $\gamma : I^X \to T_r$ be an operation. The $r$-fuzzy-$\gamma$-frontier of $\lambda$, (briefly $\gamma-Fr_T(\lambda, r)$) is defined by $\gamma-Fr_T(\lambda, r) = \gamma-C_T(\lambda, r) - \gamma-I_T(\lambda, r), \lambda \in I^X, r \in I_0$.

Definition 6.4.5.

Let $(X, T)$ be a smooth fuzzy topological space. Let $\gamma$ be an operation on $r$-$SO(X)$. The $r$-fuzzy semi-$\gamma$-frontier of $\lambda$ (briefly $\gamma-SFr_T(\lambda, r)$) is defined by $\gamma-SFr_T(\lambda, r) = \gamma-SC_T(\lambda, r) - \gamma-SI_T(\lambda, r), \lambda \in I^X, r \in I_0$. 

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Definition 6.4.6.

Let \((X, T)\) be a smooth fuzzy topological space. Let \(\gamma\) be an operation on \(r-G_6(X)\). The \(r\)-fuzzy \(G_6-\gamma\)-frontier of \(\lambda\) (briefly \(\gamma-\text{Fr}_{T(\sigma)}(\lambda, r)\)) is defined by \(\gamma-\text{Fr}_{T(\sigma)}(\lambda, r) = \gamma-\text{C}_{T(\sigma)}(\lambda, r) - \gamma-\text{I}_{T(\sigma)}(\lambda, r), \lambda \in \mathbb{I}^X, r \in I_0.\)

Definition 6.4.7.

Let \((X, T)\) be a smooth fuzzy topological space. Let \(\gamma : \mathbb{I}^X \rightarrow T_r\) be an operation. The \(r\)-fuzzy-\(\gamma\)-exterior of \(\lambda\), (briefly \(\gamma-\text{Ext}_T(\lambda, r)\)) is defined by \(\gamma-\text{Ext}_T(\lambda, r) = \gamma-\text{I}_T(\mathbb{I} - \lambda, r), \lambda \in \mathbb{I}^X, r \in I_0.\)

Definition 6.4.8.

Let \((X, T)\) be a smooth fuzzy topological space. Let \(\gamma\) be an operation on \(r-\text{SO}(X)\). The \(r\)-fuzzy semi-\(\gamma\)-exterior of \(\lambda\) (briefly \(\gamma-\text{SExt}_T(\lambda, r)\)) is defined by \(\gamma-\text{SExt}_T(\lambda, r) = \gamma-\text{SI}_T(\mathbb{I} - \lambda, r), \lambda \in \mathbb{I}^X, r \in I_0.\)

Definition 6.4.9.

Let \((X, T)\) be a smooth fuzzy topological space. Let \(\gamma\) be an operation on \(r-G_6(X)\). The \(r\)-fuzzy \(G_6-\gamma\)-exterior of \(\lambda\) (briefly \(\gamma-\text{Ext}_{T(\sigma)}(\lambda, r)\)) is defined by \(\gamma-\text{Ext}_{T(\sigma)}(\lambda, r) = \gamma-\text{I}_{T(\sigma)}(\mathbb{I} - \lambda, r), \lambda \in \mathbb{I}^X, r \in I_0.\)

Proposition 6.4.1.

Let \((X, T)\) be a smooth fuzzy topological space. For \(\lambda, \mu \in \mathbb{I}^X, r \in I_0\), the following statements hold.

(1) \(\gamma-\text{SB}_T(\lambda, r) \leq \gamma-\text{SC}_T(\mathbb{I} - \lambda, r).\)

(2) If \(\lambda\) is a \(r\)-fuzzy semi-\(\gamma\)-open set, then \(\gamma-\text{SB}_T(\lambda, r) = \mathbb{0}.\)

(3) \(\gamma-\text{SI}_T(\gamma-\text{SB}_T(\lambda, r), r) \leq \lambda.\)
(4) \( \gamma \text{-SFr}_T(\lambda, r) = \gamma \text{-SFr}_T(\bar{I} - \lambda, r). \)

(5) \( \gamma \text{-SFr}_T(\gamma \text{-SI}_T(\lambda, r), r) \leq \gamma \text{-SFr}_T(\lambda, r). \)

(6) \( \gamma \text{-SFr}_T(\gamma \text{-SC}_T(\lambda, r), r) \leq \gamma \text{-SFr}_T(\lambda, r). \)

(7) \( \lambda - \gamma \text{-SFr}_T(\lambda, r) \leq \gamma \text{-SI}_T(\lambda, r). \)

(8) \( \gamma \text{-SExt}_T(\lambda, r) = \bar{I} - \gamma \text{-SC}_T(\lambda, r). \)

(9) \( \gamma \text{-SExt}_T(\bar{I}, r) = \bar{0}. \)

(10) \( \gamma \text{-SExt}_T(\bar{0}, r) = \bar{I}. \)

(11) \( \gamma \text{-SExt}_T(\gamma \text{-SExt}_T(\lambda, r), r) = \gamma \text{-SI}_T(\gamma \text{-SC}_T(\lambda, r), r). \)

**Proof**

(1) Now, \( \gamma \text{-SB}_T(\lambda, r) = \lambda - \gamma \text{-SI}_T(\lambda, r) \leq \bar{I} - \gamma \text{-SI}_T(\lambda, r) = \gamma \text{-SC}_T(\bar{I} - \lambda, r). \)

Hence, \( \gamma \text{-SB}_T(\lambda, r) \leq \gamma \text{-SC}_T(\bar{I} - \lambda, r). \)

(2) Since \( \lambda \) is a \( r \)-fuzzy semi-\( \gamma \)-open set, \( \gamma \text{-SI}_T(\lambda, r) = \lambda. \)

Now, \( \gamma \text{-SB}_T(\lambda, r) = \lambda - \gamma \text{-SI}_T(\lambda, r) = \bar{0}. \)

Hence, \( \gamma \text{-SB}_T(\lambda, r) = \bar{0}. \)

(3) \( \gamma \text{-SI}_T(\gamma \text{-SB}_T(\lambda, r), r) = \gamma \text{-SI}_T(\lambda - \gamma \text{-SI}_T(\lambda, r), r) \)

\[ \leq \lambda - \gamma \text{-SI}_T(\lambda, r) \]

\[ \leq \lambda \]

Hence, \( \gamma \text{-SI}_T(\gamma \text{-SB}_T(\lambda, r), r) \leq \lambda. \)

(4) \( \gamma \text{-SFr}_T(\lambda, r) = \gamma \text{-SC}_T(\lambda, r) - \gamma \text{-SI}_T(\lambda, r) \)

\[ = \gamma \text{-SC}_T(\lambda, r) - (\bar{I} - \gamma \text{-SC}_T(\bar{I} - \lambda, r)) \]

\[ = \gamma \text{-SC}_T(\bar{I} - \lambda, r) - \gamma \text{-SI}_T(\bar{I} - \lambda, r) \]
\[ \gamma- \text{SI}_T(\lambda, r) = \gamma- \text{SI}_T(1 - \lambda, r) \]

Hence, \( \gamma- \text{SFr}_T(\lambda, r) = \gamma- \text{SFr}_T(\bar{1} - \lambda, r) \).

(5) \[ \gamma- \text{SFr}_T(\gamma- \text{SI}_T(\lambda, r), r) = \gamma- \text{SC}_T(\gamma- \text{SI}_T(\lambda, r), r) - \gamma- \text{SI}_T(\gamma- \text{SI}_T(\lambda, r), r) \]
\[ \leq \gamma- \text{SC}_T(\lambda, r) - \gamma- \text{SI}_T(\lambda, r) \]
\[ = \gamma- \text{SFr}_T(\lambda, r) \]

Hence, \( \gamma- \text{SFr}_T(\gamma- \text{SI}_T(\lambda, r), r) \leq \gamma- \text{SFr}_T(\lambda, r) \).

(6) \[ \gamma- \text{SFr}_T(\gamma- \text{SC}_T(\lambda, r), r) = \gamma- \text{SC}_T(\gamma- \text{SC}_T(\lambda, r), r) - \gamma- \text{SI}_T(\gamma- \text{SC}_T(\lambda, r), r) \]
\[ \leq \gamma- \text{SC}_T(\lambda, r) - \gamma- \text{SI}_T(\lambda, r) \]
\[ = \gamma- \text{SFr}_T(\lambda, r) \]

Hence, \( \gamma- \text{SFr}_T(\gamma- \text{SC}_T(\lambda, r), r) \leq \gamma- \text{SFr}_T(\lambda, r) \).

(7) \[ \lambda - \gamma- \text{SFr}_T(\lambda, r) \]
\[ = \lambda - (\gamma- \text{SC}_T(\lambda, r) - \gamma- \text{SI}_T(\lambda, r)) \]
\[ \leq \gamma- \text{SC}_T(\lambda, r) - \gamma- \text{SC}_T(\lambda, r) + \gamma- \text{SI}_T(\lambda, r) \]
\[ = \gamma- \text{SI}_T(\lambda, r) \]

Hence, \( \lambda - \gamma- \text{SFr}_T(\lambda, r) \leq \gamma- \text{SI}_T(\lambda, r) \).

(8) The proof follows from definition.

(9) \[ \gamma- \text{SExt}_T(\bar{1}, r) = \gamma- \text{SI}_T(\bar{1} - \bar{1}, r) \]
\[ = \gamma- \text{SI}_T(\bar{0}, r) \]
\[ = \bar{0} \]

Hence, \( \gamma- \text{SExt}_T(\bar{1}, r) = \bar{0} \).

(10) \[ \gamma- \text{SExt}_T(\bar{0}, r) = \gamma- \text{SI}_T(\bar{1} - \bar{0}, r) \]
\[ = \gamma- \text{SI}_T(\bar{1}, r) \]
\[ \gamma \text{-SExt}_r(\overline{0}, r) = \overline{1}. \]

Hence, \( \gamma \text{-SExt}_r(\overline{0}, r) = \overline{1}. \)

(11) \( \gamma \text{-SExt}_r(\gamma \text{-SExt}_r(\lambda, r), r) = \gamma \text{-SI}_r(\overline{1} - \gamma \text{-SExt}_r(\lambda, r), r) \)
\[ = \gamma \text{-SI}_r(\overline{1} - \gamma \text{-SI}_r(\overline{1} - \lambda, r), r) \]
\[ = \gamma \text{-SI}_r(\gamma \text{-SC}_r(\lambda, r), r). \]

Hence, \( \gamma \text{-SExt}_r(\gamma \text{-SExt}_r(\lambda, r), r) = \gamma \text{-SI}_r(\gamma \text{-SC}_r(\lambda, r), r). \)

**Proposition 6.4.2.**

Let \((X, T)\) be a smooth fuzzy topological space. For \(\lambda, \mu \in \mathcal{I}_X\) and \(r \in \mathcal{I}_0\), the following statements hold.

1. \( \gamma \text{-B}_{T(\sigma)}(\lambda, r) \leq \gamma \text{-C}_{T(\sigma)}(\overline{1} - \lambda, r). \)
2. If \(\lambda\) is a \(r\)-fuzzy \(G_{\delta}\)-\(\gamma\)-open set, then \(\gamma \text{-B}_{T(\sigma)}(\lambda, r) = \overline{0}. \)
3. \( \gamma \text{-I}_{T(\sigma)}(\gamma \text{-B}_{T(\sigma)}(\lambda, r), r) \leq \lambda. \)
4. \( \gamma \text{-Fr}_{T(\sigma)}(\lambda, r) = \gamma \text{-Fr}_{T(\sigma)}(\overline{1} - \lambda, r). \)
5. \( \gamma \text{-Fr}_{T(\sigma)}(\gamma \text{-I}_{T(\sigma)}(\lambda, r), r) \leq \gamma \text{-Fr}_{T(\sigma)}(\lambda, r). \)
6. \( \gamma \text{-Fr}_{T(\sigma)}(\gamma \text{-C}_{T(\sigma)}(\lambda, r), r) \leq \gamma \text{-Fr}_{T(\sigma)}(\lambda, r). \)
7. \( \lambda - \gamma \text{-Fr}_{T(\sigma)}(\lambda, r) \leq \gamma \text{-I}_{T(\sigma)}(\lambda, r). \)
8. \( \gamma \text{-Ext}_{T(\sigma)}(\lambda, r) = \overline{1} - \gamma \text{-C}_{T(\sigma)}(\lambda, r). \)
9. \( \gamma \text{-Ext}_{T(\sigma)}(\overline{1}, r) = \overline{0}. \)
10. \( \gamma \text{-Ext}_{T(\sigma)}(\overline{0}, r) = \overline{1}. \)
11. \( \gamma \text{-Ext}_{T(\sigma)}(\gamma \text{-Ext}_{T(\sigma)}(\lambda, r), r) = \gamma \text{-I}_{T(\sigma)}(\gamma \text{-C}_{T(\sigma)}(\lambda, r), r). \)
Proof

The proof is similar to Proposition 6.4.1.

Definition 6.4.10.

A smooth fuzzy topological space \((X, T)\) is said to be a smooth fuzzy semi-\(\gamma\)-T-space if every \(r\)-fuzzy semi-\(\gamma\)-open set is a \(r\)-fuzzy semi open set.

Proposition 6.4.3.

Let \((X, T)\) be a smooth fuzzy semi-\(\gamma\)-T-space. Then for every \(r\)-fuzzy semi-\(\gamma\)-open set \(\lambda, \lambda \in I^X, r \in I_0\) the following statements hold.

1. \(SB_T(\lambda, r) = \overline{0}_T\).
2. \(\gamma-SB_T(\lambda, r) = SB_T(\lambda, r)\).
3. \(\gamma-SB_T(\lambda, r) \leq \gamma-SFr_T(\lambda, r) \leq \gamma-SC_T(\lambda, r)\).

Proof

(1) \(SB_T(\lambda, r) = \lambda - SI_T(\lambda, r)\). Since \(\lambda\) is a \(r\)-fuzzy semi-\(\gamma\)-open set and \((X, T)\) is a fuzzy semi-\(\gamma\)-T-space, it follows that \(\lambda = SI_T(\lambda, r)\). Hence, \(SB_T(\lambda, r) = \overline{0}_T\).

(2) Since \(\lambda\) is a \(r\)-fuzzy semi-\(\gamma\)-open set and \((X, T)\) is a smooth fuzzy semi-\(\gamma\)-T-space, it follows that \(\lambda = SI_T(\lambda, r)\).

\[
\gamma-SB_T(\lambda, r) = \lambda - \gamma-SI_T(\lambda, r)
= \lambda - SI_T(\lambda, r)
= SB_T(\lambda, r).
\]

(3) \(\gamma-SB_T(\lambda, r) = \lambda - \gamma-SI_T(\lambda, r)\)
\[ y - SC_T(\lambda, r) - y - SI_T(\lambda, r) \]
\[ = y - SFr_T(\lambda, r) \]
\[ \leq y - SC_T(\lambda, r). \]

Hence, \( y - SB_T(\lambda, r) \leq y - SFr_T(\lambda, r) \leq y - SC_T(\lambda, r). \)

**Definition 6.4.11.**

A smooth fuzzy topological space \((X, T)\) is said to be a smooth fuzzy \(G_{\delta}-\gamma\)-T-space if every \(r\)-fuzzy \(G_{\delta}-\gamma\)-open set is a \(r\)-fuzzy \(G_{\delta}\)-set.

**Proposition 6.4.4.**

Let \((X, T)\) be a smooth fuzzy \(G_{\delta}-\gamma\)-T-space. Then for every \(r\)-fuzzy \(G_{\delta}-\gamma\)-open set \(X, X. E I x, r E I_0\) the following statements hold.

1. \(B_T(a) (X, r) = 0.\)
2. \(y - B_T(a) (X, r) = B_T(0.) (X, r).\)
3. \(y - a r f o (k, r) \leq y - Fr_T(0) (2, r) \leq y - C_T(a) (X, r).\)

**Proof**

The proof is similar to Proposition 6.4.3.

**Proposition 6.4.5.**

Let \((X, T)\) and \((Y, S)\) be any two smooth fuzzy topological spaces. Let \(f : (X, T) \rightarrow (Y, S)\) be a function and \((X, T)\) be a smooth fuzzy semi-\(\gamma\)-T-space. Then for every \(r\)-fuzzy semi-\(\beta\)-closed set \(\mu, f^{-1}(\mu)\) is a \(r\)-fuzzy semi closed set if \(y - SC_T(f^{-1}(\mu), r) \leq f^{-1}(\beta - SC_\delta(\mu, r)), \mu E I^Y, r E I_0.\)

**Proof**

Let \(\mu\) be a \(r\)-fuzzy semi-\(\beta\)-closed set.

Now, \(y - SC_T(f^{-1}(\mu), r) \leq f^{-1}(\beta - SC_\delta(\mu, r)), \mu E I^Y, r E I_0.\)
Then, $1 - \gamma SC_T(f^{-1}(\mu), r) \geq 1 - f^{-1}(\beta SC_S(\mu, r)) = 1 - f^{-1}(\mu)$.

That is, $\gamma SI_T(1 - f^{-1}(\mu), r) \geq 1 - f^{-1}(\mu)$. But, $\gamma SI_T(1 - f^{-1}(\mu), r) \leq 1 - f^{-1}(\mu)$.

Hence, $\gamma SI_T(1 - f^{-1}(\mu), r) = 1 - f^{-1}(\mu)$. This implies that, $1 - f^{-1}(\mu)$ is a \( r \)-fuzzy semi-\( \gamma \)-open set. Since \((X, T)\) is a smooth fuzzy semi-\( \gamma \)-\( T \) space, it follows that $1 - f^{-1}(\mu)$ is a \( r \)-fuzzy semi open set. Therefore, $f^{-1}(\mu)$ is a \( r \)-fuzzy semi closed set.

**Proposition 6.4.6.**

Let \((X, T)\) and \((Y, S)\) be any two smooth fuzzy topological spaces. Let $f : (X, T) \rightarrow (Y, S)$ be a function and \((X, T)\) be a smooth fuzzy $G_\delta$-\( \gamma \)-\( T \)-space. Then for every \( r \)-fuzzy $F_\sigma$-\( \beta \)-closed set $\mu$, $f^{-1}(\mu)$ is a \( r \)-fuzzy $F_\sigma$-set if $\gamma C_T(f^{-1}(\mu), r) \leq f^{-1}(\beta C_S(\mu, r))$, $\mu \in I^Y$, $r \in I_0$.

**Proof**

The proof is similar to Proposition 6.4.5.

**Proposition 6.4.7.**

Let \((X, T)\) and \((Y, S)\) be any two smooth fuzzy topological spaces. Let $f : (X, T) \rightarrow (Y, S)$ be a function and \((Y, S)\) be a smooth fuzzy semi-\( \beta \)-\( T \)-space. Then for every \( r \)-fuzzy semi-\( \gamma \)-open set $\lambda$, $f(\lambda)$ is a \( r \)-fuzzy semi open set if $\beta SB_S(f(\lambda), r) \leq f(\gamma SB_T(\lambda, r))$, $\lambda \in I^X$, $r \in I_0$.

**Proof**

Let $\lambda$ be a \( r \)-fuzzy semi-\( \gamma \)-open set.

Now, $\beta SB_S(f(\lambda), r) \leq f(\gamma SB_T(\lambda, r))$, $\lambda \in I^X$, $r \in I_0$.

Then, $f(\lambda) - \beta SI_S(f(\lambda), r) \leq f(\lambda) - f(\gamma SI_T(\lambda, r))$. 

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That is, $\beta-SI_{S}(f(\lambda), r) \geq f(\gamma-SI_{T}(\lambda, r)) = f(\lambda)$. But, $\beta-SI_{S}(f(\lambda), r) \leq f(\lambda)$. Hence, $\beta-SI_{S}(f(\lambda), r) = f(\lambda)$. This implies that, $f(\lambda)$ is a r-fuzzy semi-$\beta$-open set. Since $(Y, S)$ is a smooth fuzzy semi-$\beta$-T-space, it follows that $f(\lambda)$ is a r-fuzzy semi open set.

**Proposition 6.4.8.**

Let $(X, T)$ and $(Y, S)$ be any two smooth fuzzy topological spaces. Let $f : (X, T) \rightarrow (Y, S)$ be a function and $(Y, S)$ be a smooth fuzzy $G_{S}$-$\beta$-T-space. Then for every r-fuzzy $G_{S}$-$\gamma$-open set $\lambda$, $f(\lambda)$ is a r-fuzzy $G_{S}$-set if $\beta-B_{S(o)}(f(\lambda), r) \leq f(\gamma-B_{T(o)}(\lambda, r)), \lambda \in I^{X}, r \in I_{0}$.

**Proof**

The proof is similar to Proposition 6.4.7.

**6.5 SMOOTH FUZZY SEMI (resp. $G_{S}$)-$\gamma$-REGULAR, SMOOTH FUZZY SEMI (resp. $G_{S}$)-$\gamma$-NORMAL AND SMOOTH FUZZY SEMI (resp. $G_{S}$)-$\gamma$-EXTREMALLY DISCONNECTED SPACES**

In this section, smooth fuzzy semi (resp. $G_{S}$)-$\gamma$-regular, smooth fuzzy semi (resp. $G_{S}$)-$\gamma$-normal and smooth fuzzy semi (resp. $G_{S}$)-$\gamma$-extremally disconnected spaces are introduced and its characterizations are studied.

**Definition 6.5.1.**

A smooth fuzzy topological space $(X, T)$ is said to be a smooth fuzzy semi-$\gamma$-regular space if for every r-fuzzy semi-$\gamma$-closed set $\lambda$ and each $\alpha$ with $\alpha \leq \lambda$, $\alpha, \lambda \in I^{X}, r \in I_{0}$, there exist r-fuzzy semi-$\gamma$-open sets $\mu, \delta$ with $\delta \sqcup \mu$ such that $\alpha \leq \delta, \lambda \leq \mu, \delta, \mu \in I^{X}, r \in I_{0}$.
Definition 6.5.2.

A smooth fuzzy topological space $(X, T)$ is said to be a smooth fuzzy $G_{\delta}-\gamma$-regular space if for every $r$-fuzzy $F_{\alpha}-\gamma$-closed set $\lambda$ and each $\alpha$ with $\alpha \not\leq \lambda$, $\alpha, \lambda \in I^X$, $r \in I_0$, there exist $r$-fuzzy $G_{\delta}-\gamma$-open sets $\mu, \delta$ with $\delta \not\leq \mu$ such that $\alpha \leq \delta$, $\lambda \leq \mu$, $\delta, \mu \in I^X$, $r \in I_0$.

Proposition 6.5.1.

Let $(X, T)$ be a smooth fuzzy topological space. Then the following statements are equivalent.

(1) $(X, T)$ is a smooth fuzzy semi-$\gamma$-regular space.

(2) For each $\alpha$ and a $r$-fuzzy semi-$\gamma$-open set $\lambda$ with $\alpha \not\leq \lambda$, $\alpha, \lambda \in I^X$, $r \in I_0$, there exists a $r$-fuzzy semi-$\gamma$-open set $\delta$ with $\alpha \leq \delta$ such that $\gamma-SC_T(\delta, r) \leq \lambda$, $\delta \in I^X$, $r \in I_0$.

Proof

$(1) \Rightarrow (2)$. Let $\lambda$ be a $r$-fuzzy semi-$\gamma$-open set with $\alpha \not\leq \lambda$, $\alpha, \lambda \in I^X$, $r \in I_0$. By (1), there exist $r$-fuzzy semi-$\gamma$-open set $\mu, \delta$ with $\delta \not\leq \mu$ such that $\alpha \leq \delta$ and $\bar{1} - \lambda \leq \mu$, $\mu, \delta \in I^X$, $r \in I_0$. Since $\delta \leq \bar{1} - \mu$ it follows that $\gamma-SC_T(\delta, r) \leq \gamma-SC_T(\bar{1} - \mu, r) = \bar{1} - \mu \leq \lambda$. Hence, $\gamma-SC_T(\delta, r) \leq \lambda$.

$(2) \Rightarrow (1)$. Let $\lambda$ be a $r$-fuzzy semi-$\gamma$-closed set with $\alpha \not\leq \lambda$, $\alpha, \lambda \in I^X$, $r \in I_0$. Now, $\bar{1} - \lambda$ is a $r$-fuzzy semi-$\gamma$-open set. By (2), there exists a $r$-fuzzy semi-$\gamma$-open set $\delta$ with $\alpha \leq \delta$ such that $\gamma-SC_T(\delta, r) \leq \bar{1} - \lambda$, $\delta \in I^X$, $r \in I_0$. This implies that $\lambda \leq \bar{1} - \gamma-SC_T(\delta, r)$. Now, $\delta \leq \bar{1} - (\bar{1} - \gamma-SC_T(\delta, r))$ such that $\alpha \leq \delta$ and $\lambda \leq \bar{1} - \gamma-SC_T(\delta, r)$. Hence $(X, T)$ is a smooth fuzzy.
semi-γ-regular space.

**Proposition 6.5.2.**

Let \((X, T)\) be a smooth fuzzy topological space. Then the following statements are equivalent.

1. \((X, T)\) is a smooth fuzzy \(G_δ\)-γ-regular space.

2. For each \(α\) and a \(r\)-fuzzy \(G_δ\)-γ-open set \(λ\) with \(αqλ, α, λ ∈ I^X, r ∈ I_0\), there exists a \(r\)-fuzzy \(G_δ\)-γ-open set \(δ\) with \(α ≤ δ\) such that \(γ-C_{T(α)}(δ, r) ≤ λ, δ ∈ I^X, r ∈ I_0\).

**Proof**

The proof is similar to Proposition 6.5.1.

**Proposition 6.5.3.**

Let \((X, T)\) and \((Y, S)\) be any two smooth fuzzy topological spaces. Let \(f : (X, T) → (Y, S)\) be a smooth fuzzy semi-(γ, β)-continuous and smooth fuzzy semi-(γ, β)-open and bijective function. If \((X, T)\) is a smooth fuzzy semi-γ-regular space then \((Y, S)\) is a smooth fuzzy semi-β-regular space.

**Proof**

Let \(λ\) be a \(r\)-fuzzy semi-β-closed set and \(η\) be such that \(η ≤ λ, η, λ ∈ I^Y, r ∈ I_0\). Since \(f\) is a smooth fuzzy semi-(γ, β)-continuous function, \(f^{-1}(λ)\) is a \(r\)-fuzzy semi-γ-closed set. Let \(f(ν) = η, ν ∈ I^X, r ∈ I_0\). Since \(f\) is bijective \(ν = f^{-1}(η)\). Since \((X, T)\) is a smooth fuzzy semi-γ-regular space and \(ν ≤ f^{-1}(λ)\), there exist \(r\)-fuzzy semi-γ-open sets \(μ, δ\) such that \(δqμ\) such that \(ν ≤ δ, f^{-1}(λ) ≤ μ\). Since \(f\) is a bijective and a
smooth fuzzy semi-(\(\gamma\), \(\beta\))-open function, \(f(\delta)\) and \(f(\mu)\) are r-fuzzy semi-\(\beta\)-open sets such that \(\eta \leq f(\delta)\) and \(\lambda \leq f(\mu)\) with \(f(\delta) \sqsubseteq f(\mu)\). Therefore, \((Y, S)\) is a smooth fuzzy semi-\(\beta\)-regular space.

**Proposition 6.5.4.**

Let \((X, T)\) and \((Y, S)\) be any two smooth fuzzy topological spaces. Let \(f : (X, T) \rightarrow (Y, S)\) be a bijective, smooth fuzzy \(G^\delta_\gamma - (\gamma, \beta)\)-continuous and smooth fuzzy \(G^\delta_\gamma - (\gamma, \beta)\)-open function. If \((X, T)\) is a smooth fuzzy \(G^\delta_\gamma\)-\(\gamma\)-regular space then \((Y, S)\) is a smooth fuzzy \(G^\delta_\gamma\)-\(\beta\)-regular space.

**Proof**

The proof is similar to Proposition 6.5.3.

**Proposition 6.5.5.**

Let \((X, T)\) and \((Y, S)\) be any two smooth fuzzy topological spaces. Let \(f : (X, T) \rightarrow (Y, S)\) be a bijective, smooth fuzzy semi-(\(\gamma\), \(\beta\))-continuous and smooth fuzzy semi-(\(\gamma\), \(\beta\))-closed function. If \((Y, S)\) is a smooth fuzzy semi-\(\beta\)-regular space then \((X, T)\) is a smooth fuzzy semi-\(\gamma\)-regular space.

**Proof**

Let \(\lambda\) be a r-fuzzy semi-\(\gamma\)-closed set and \(\alpha\) be such that \(\alpha \not\subseteq \lambda\), \(\alpha, \lambda \in I^X\), \(r \in I_0\). Since \(f\) is a smooth fuzzy semi-(\(\gamma\), \(\beta\))-closed function, \(f(\lambda)\) is a r-fuzzy semi-\(\beta\)-closed set with \(f(\alpha) \not\subseteq f(\lambda)\). Since \((Y, S)\) is a smooth fuzzy semi-\(\gamma\)-regular space, there exist r-fuzzy semi-\(\beta\)-open sets \(\mu, \delta\) such that \(\delta \sqsubseteq \mu\) such that \(f(\alpha) \subseteq \mu\) and \(f(\lambda) \subseteq \delta\). Since \(f\) is a smooth fuzzy semi-(\(\gamma\), \(\beta\))-continuous function, \(f^{-1}(\delta)\) and \(f^{-1}(\mu)\) are
r-fuzzy semi-γ-open sets such that \( \lambda \leq f^{-1}(\delta) \) and \( \alpha \leq f^{-1}(\mu) \) with \( f^{-1}(\delta) \cap f^{-1}(\mu) \). Therefore, \((X, T)\) is a smooth fuzzy semi-β-regular space.

**Proposition 6.5.6.**

Let \((X, T)\) and \((Y, S)\) be any two smooth fuzzy topological spaces. Let \( f : (X, T) \to (Y, S) \) be a bijective, smooth fuzzy \( G_\delta(\gamma, \beta) \)-continuous and smooth fuzzy \( F_\sigma(\gamma, \beta) \)-closed function. If \((Y, S)\) is a smooth fuzzy \( G_\delta \)-β-regular space then \((X, T)\) is a smooth fuzzy \( G_\delta \)-γ-regular space.

**Proof**

The proof is similar to Proposition 6.5.5.

**Definition 6.5.3.**

A smooth fuzzy topological space \((X, T)\) is said to be a smooth fuzzy semi-γ-normal space if for every r-fuzzy semi-γ-closed set \( \lambda \) and for every r-fuzzy semi-γ-open set \( \alpha \) with \( \lambda \leq \alpha, \lambda \in I^X, r \in I_0 \), there exists a \( \delta \) such that \( \lambda \leq \gamma-\text{SL}_r(\delta, r) \leq \lambda \leq \gamma-\text{SC}_r(\delta, r) \leq \alpha, \delta \in I^X, r \in I_0 \).

**Definition 6.5.4.**

A smooth fuzzy topological space \((X, T)\) is said to be a smooth fuzzy \( G_\delta \)-γ-normal space if for every r-fuzzy \( F_\sigma \)-γ-closed set \( \lambda \) and for every r-fuzzy \( G_\delta \)-γ-open set \( \alpha \) with \( \lambda \leq \alpha, \lambda \in I^X, r \in I_0 \), there exists a \( \delta \) such that \( \lambda \leq \gamma-I_{\text{TI}(\delta, r)}(\delta, r) \leq \gamma-C_{\text{TI}(\delta, r)}(\delta, r) \leq \alpha, \delta \in I^X, r \in I_0 \).

**Proposition 6.5.7.**

Let \((X, T)\) be a smooth fuzzy topological space. Then the following statements are equivalent.

1. \((X, T)\) is a smooth fuzzy semi-γ-normal space.
(2) For every r-fuzzy semi-$\gamma$-closed set $\lambda$ and for every r-fuzzy semi-$\gamma$-open set $\alpha$ with $\lambda \leq \alpha$, $\alpha, \lambda \in I^X$, $r \in I_0$, there exists a r-fuzzy semi-$\gamma$-open set $\delta$ such that $\gamma-SC_T(\lambda, r) \leq \delta \leq \gamma-SC_T(\delta, r) \leq \alpha$, $\delta \in I^X$, $r \in I_0$.

(3) For every r-fuzzy semi-$\gamma$-g closed set $\lambda$ and for every r-fuzzy semi-$\gamma$-open set $\alpha$ with $\lambda \leq \alpha$, $\alpha, \lambda \in I^X$, $r \in I_0$, there exists a r-fuzzy semi-$\gamma$-open set $\delta$ such that $\gamma-SC_T(\lambda, r) \leq \delta \leq \gamma-SC_T(\delta, r) \leq \alpha$, $\delta \in I^X$, $r \in I_0$.

**Proof**

(1) $\Rightarrow$ (2). The proof is trivial.

(2) $\Rightarrow$ (3). Let $\lambda$ be a r-fuzzy semi-$\gamma$-g closed set and $\alpha$ be a r-fuzzy semi-$\gamma$-open set with $\lambda \leq \alpha$, $\alpha, \lambda \in I^X$, $r \in I_0$. Since $\lambda$ is a r-fuzzy semi-$\gamma$-g closed set $\lambda$, $\gamma-SC_T(\lambda, r) \leq \alpha$. Now, $\gamma-SC_T(\lambda, r)$ is a r-fuzzy semi-$\gamma$-closed set and $\alpha$ is a r-fuzzy semi-$\gamma$-open set. By (2), there exists a r-fuzzy semi-$\gamma$-open set $\delta$ such that

$$\gamma-SC_T(\lambda, r) \leq \delta \leq \gamma-SC_T(\delta, r) \leq \alpha, \delta \in I^X, r \in I_0.$$ 

(3) $\Rightarrow$ (1). The proof is trivial.

**Proposition 6.5.8.**

Let $(X, T)$ be a smooth fuzzy topological space. Then the following statements are equivalent.

(1) $(X, T)$ is a smooth fuzzy $G_\delta-\gamma$-normal space.

(2) For every r-fuzzy $F_\sigma-\gamma$-closed set $\lambda$ and for every r-fuzzy
G₇-γ-open set α with λ ≤ α, α, λ ∈ I^X, r ∈ I₀, there exists a r-fuzzy G₇-γ-open set δ such that γ-C_{T₀}(λ, r) ≤ δ ≤ γ-C_{T₀}(δ, r) ≤ α, δ ∈ I^X, r ∈ I₀.

(3) For every r-fuzzy F₀-γ-g closed set λ and for every r-fuzzy G₇-γ-open set α with λ ≤ α, α, λ ∈ I^X, r ∈ I₀, there exists a r-fuzzy G₇-γ-open set δ such that γ-C_{T₀}(λ, r) ≤ δ ≤ γ-C_{T₀}(δ, r) ≤ α, δ ∈ I^X, r ∈ I₀.

Proof

The proof is similar to Proposition 6.5.7.

**Definition 6.5.5.**

A smooth fuzzy topological space (X, T) is said to be a smooth fuzzy semi-γ-extremally disconnected space if the r-fuzzy semi-γ-closure of each r-fuzzy semi-γ-open set is a r-fuzzy semi-γ-open set.

**Proposition 6.5.9.**

Let (X, T) be a smooth fuzzy topological space. For λ, μ ∈ I^X, r ∈ I₀ the following statements are equivalent:

(1) (X, T) is a smooth fuzzy semi-γ-extremally disconnected space.

(2) Whenever λ is a r-fuzzy semi-γ-closed set, γ-SIₜ(λ, r) is a r-fuzzy semi-γ-closed set.

(3) Whenever λ is a r-fuzzy semi-γ-open set, we have

γ-SCₜ(₁ - γ-SCₜ(λ, r), r) = ₁ - γ-SCₜ(λ, r).

(4) For every pair of r-fuzzy semi-γ-open sets λ and μ with γ-SCₜ(λ, r) + μ = ₁, we have γ-SCₜ(λ, r) + γ-SCₜ(μ, r) = ₁.
Proof

(1) $\Rightarrow$ (2). Let $\lambda$ be a r-fuzzy semi-$\gamma$-closed set. Then $\tilde{1} - \lambda$ is a r-fuzzy semi-$\gamma$-open set.

From (1), $\gamma$-$SC_T(\tilde{1} - \lambda, r)$ is a r-fuzzy semi-$\gamma$-open set.

Now, $\tilde{1} - \gamma$-$SC_T(\tilde{1} - \lambda, r)$ is a r-fuzzy semi-$\gamma$-closed set.

But $\tilde{1} - \gamma$-$SC_T(\tilde{1} - \lambda, r) = \gamma$-$SI_T(\lambda, r)$ implies that $\gamma$-$SI_T(\lambda, r)$ is a r-fuzzy semi-$\gamma$-closed set.

(2) $\Rightarrow$ (3). Let $\lambda$ be a r-fuzzy semi-$\gamma$-open set. Then $\tilde{1} - \lambda$ is a r-fuzzy semi-$\gamma$-closed set. Now, $\gamma$-$SI_T(\tilde{1} - \lambda, r)$ is a r-fuzzy semi-$\gamma$-closed set.

Therefore, $\tilde{1} - \gamma$-$SC_T(\lambda, r)$ is a r-fuzzy semi-$\gamma$-closed set. Hence,

$$\gamma$-$SC_T(\tilde{1} - \gamma$-$SC_T(\lambda, r), r) = \tilde{1} - \gamma$-$SC_T(\lambda, r).$$

(3) $\Rightarrow$ (4). Let $\lambda$ and $\mu$ be any two r-fuzzy semi-$\gamma$-open sets, such that $\gamma$-$SC_T(\lambda, r) + \mu = \tilde{1}$. By (3), we have

$$\gamma$-$SC_T(\tilde{1} - \gamma$-$SC_T(\lambda, r), r) = \tilde{1} - \gamma$-$SC_T(\lambda, r).$$

Then,

$$\gamma$-$SC_T(\mu, r) = \tilde{1} - \gamma$-$SC_T(\lambda, r).$$

Therefore, $\gamma$-$SC_T(\mu, r) + \gamma$-$SC_T(\lambda, r) = \tilde{1}$.

(4) $\Rightarrow$ (1). Let $\lambda$ and $\mu$ be a pair of r-fuzzy semi-$\gamma$-open set with $\gamma$-$SC_T(\lambda, r) + \mu = \tilde{1}$. Then, $\mu = \tilde{1} - \gamma$-$SC_T(\lambda, r)$.

By (4), $\gamma$-$SC_T(\lambda, r) + \gamma$-$SC_T(\mu, r) = \tilde{1}$.

This implies that $\gamma$-$SC_T(\lambda, r)$ is a r-fuzzy semi-$\gamma$-open set.

Definition 6.5.6.

A smooth fuzzy topological space $(X, T)$ is said to be a smooth
fuzzy $G_δ-γ$-extremally disconnected space if the $r$-fuzzy $F_σ-γ$-closure of each $r$-fuzzy $G_δ-γ$-open set is a $r$-fuzzy $G_δ-γ$-open set.

**Proposition 6.5.10.**

Let $(X, T)$ be a smooth fuzzy topological space. For $λ, μ ∈ I^X$, $r ∈ I_0$ the following statements are equivalent:

1. $(X, T)$ is a smooth fuzzy $G_δ-γ$-extremally disconnected space.
2. Whenever $λ$ is a $r$-fuzzy $F_σ-γ$-closed set, $γ-I_{T(σ)}(λ, r)$ is a $r$-fuzzy $F_σ-γ$-closed set.
3. Whenever $λ$ is a $r$-fuzzy $G_δ-γ$-open set, we have
   
   $γ-C_{T(σ)}(λ, r) + μ = I, \text{ we have } γ-C_{T(σ)}(λ, r) + γ-C_{T(σ)}(μ, r) = I.

**Proof**

The proof is similar to Proposition 6.5.9.