CHAPTER - V

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The concept of fuzzy $\beta$-open set was introduced and studied in [9]. Further, fuzzy $\beta$-separation axioms have been introduced and investigated in [11] with the help of fuzzy $\beta$-open sets. In this chapter, some generalizations of smooth fuzzy $\beta$-continuous functions are introduced and studied by making use of $r$-fuzzy $\beta$-open sets. The concepts of smooth fuzzy $\beta$-$T_{1/2}$ spaces in the sense of Ramadan [62] is introduced and studied as in [71]. Further generalizations of such spaces are established. Many examples are given in connection with the above functions and spaces. Interesting properties and interrelations among the concepts introduced are established. In this connection, counter examples are also discussed.
5.1 SMOOTH FUZZY ALMOST $\beta$-CONTINUOUS FUNCTIONS

In this section, the concept of smooth fuzzy almost $\beta$-continuous functions is introduced. Characterizations and properties of smooth fuzzy almost $\beta$-continuous functions are discussed.

**Definition 5.1.1.**

Let $(X, T)$ be a smooth fuzzy topological space. For $\lambda \in \mathcal{I}^X$, $r \in I_0$

1. $\lambda$ is called a r-fuzzy $\beta$-open set if $\lambda \leq C_T(I_T(C_T(\lambda, r), r), r)$.

2. The complement of a r-fuzzy $\beta$-open set is a r-fuzzy $\beta$-closed set.

3. The r-fuzzy $\beta$-interior of $\lambda$, denoted by $\beta-I_T(\lambda, r)$, is defined by

$$\beta-I_T(\lambda, r) = \bigvee \{ \mu : \mu \leq \lambda, \mu \text{ is a r-fuzzy } \beta \text{-open set} \}.$$ 

4. The r-fuzzy $\beta$-closure of $\lambda$, denoted by $\beta-C_T(\lambda, r)$, is defined by

$$\beta-C_T(\lambda, r) = \bigwedge \{ \mu : \mu \geq \lambda, \mu \text{ is a r-fuzzy } \beta \text{-closed set} \}.$$ 

**Proposition 5.1.1.**

Let $(X, T)$ be a smooth fuzzy topological space. Then arbitrary union of r-fuzzy $\beta$-open sets is a r-fuzzy $\beta$-open set.

**Proof**

Let $\{ \mu_i \}_{i \in I}$ be r-fuzzy $\beta$-open sets, $\mu_i \in \mathcal{I}^X$, $r \in I_0$.

Now, $\bigvee_{i \in I} \mu_i \leq \bigvee_{i \in I} C_T(I_T(C_T(\mu_i, r), r), r) \leq C_T(I_T(C_T(\bigvee_{i \in I} \mu_i, r), r), r)$. Hence, arbitrary union of r-fuzzy $\beta$-open sets is a r-fuzzy $\beta$-open set.

**Note 5.1.1.**

Finite intersection of r-fuzzy $\beta$-open sets need not be a r-fuzzy $\beta$-open set as shown in the following example.
Example 5.1.1.

Let $X = \{a, b, c\}$. Let $\mu_1, \mu_2, \mu_3, \mu_4, \mu_5 \in I^X$ be defined as follows:

$\mu_1(a) = 0.8, \mu_1(b) = 0.6, \mu_1(c) = 0.5, \mu_2(a) = 0.1, \mu_2(b) = 0.1, \mu_2(c) = 0.5,$

$\mu_3(a) = 0.8, \mu_3(b) = 0.7, \mu_3(c) = 0.5, \mu_4(a) = 0.8, \mu_4(b) = 0.9, \mu_4(c) = 0.5,$

$\mu_5(a) = 0.1, \mu_5(b) = 0.4, \mu_5(c) = 0.5$. Define a smooth fuzzy topology $T : I^X \to I$ as follows.

$$T(\lambda) = \begin{cases} 
1 & \lambda = 0, 1 \\
0.5 & \lambda = \mu_1 \\
0.6 & \lambda = \mu_2 \\
0.5 & \lambda = \mu_3 \\
0.6 & \lambda = \mu_4 \\
0.6 & \lambda = \mu_5 \\
0 & \text{otherwise.}
\end{cases}$$

Let $r = 0.04$. Let $\lambda_1, \lambda_2 \in I^X$ be defined as $\lambda_1(a) = 0.8, \lambda_1(b) = 0.3,$

$\lambda_1(c) = 0.5, \lambda_2(a) = 0.2, \lambda_2(b) = 0.4$ and $\lambda_2(c) = 0.5$. Clearly, $\lambda_1$ and $\lambda_2$ are 0.04-fuzzy $\beta$-open sets.

Further, $\lambda_1 \land \lambda_2 \notin C_T(C_T(\lambda_1 \land \lambda_2, 0.04), 0.04), 0.04)$.

Hence, $\lambda_1 \land \lambda_2$ is not a 0.04-fuzzy $\beta$-open set.

Note 5.1.2.

$r$-fuzzy $\beta$-open sets need not form a topology.

Remark 5.1.1.

Let $(X, T)$ be a smooth fuzzy topological space. Then
\[ \beta - C_T(\lambda - \mu, r) = \lambda - \beta - I(T(\lambda, r), r) \in I^X, r \in I_0. \]

\[ \beta - I_T(\lambda - \mu, r) = \lambda - \beta - C_T(\lambda, r) \in I^X, r \in I_0. \]

**Definition 5.1.2.**

Let \((X, T)\) be any smooth fuzzy topological space.

(1) \(X\) is called a \(r\)-generalized fuzzy \(\beta\)-open set (briefly \(r\)-gf\(\beta\)-open set) \[\iff\] whenever \(\mu\) is a \(r\)-fuzzy \(\beta\)-closed set and \(\mu \leq \lambda, \lambda, \mu \in I^X, r \in I_0.\)

(2) The complement of a \(r\)-gf\(\beta\)-open set is a \(r\)-gf\(\beta\)-closed set.

**Property 5.1.1.**

Let \((X, T)\) be any smooth fuzzy topological space.

(1) Every \(r\)-fuzzy \(\beta\)-closed set is a \(r\)-gf\(\beta\)-closed set.

(2) Every \(r\)-fuzzy \(\beta\)-open set is a \(r\)-gf\(\beta\)-open set.

**Proof**

(1) Let \(\lambda\) be any \(r\)-fuzzy \(\beta\)-closed set, \(\lambda \in I^X, r \in I_0.\) Let \(\lambda \leq \mu, \mu \in I^X, r \in I_0.\) Now, \(\beta - C_T(\lambda, r) = \lambda \leq \mu.\) Hence, \(\lambda\) is a \(r\)-gf\(\beta\)-closed set. Therefore, every \(r\)-fuzzy \(\beta\)-closed set is a \(r\)-gf\(\beta\)-closed set.

(2) Let \(\lambda\) be any \(r\)-fuzzy \(\beta\)-open set, \(\lambda \in I^X, r \in I_0.\) Let \(\lambda \geq \mu, \mu \in I^X, r \in I_0.\) Now, \(\beta - I_T(\lambda, r) = \lambda \geq \mu.\) Hence, \(\lambda\) is a \(r\)-gf\(\beta\)-open set. Therefore, every \(r\)-fuzzy \(\beta\)-open set is a \(r\)-gf\(\beta\)-open set.

**Definition 5.1.3.**

Let \((X, T)\) and \((Y, S)\) be any two smooth fuzzy topological spaces
and let \( f : (X, T) \rightarrow (Y, S) \) be a function.

1. \( f \) is called a smooth fuzzy \( \beta \)-continuous function if \( f^{-1}(\lambda) \) is a \( r \)-fuzzy \( \beta \)-open set for each \( S(\lambda) \geq r, \lambda \in I^Y, r \in I_0 \). Equivalently, if \( f^{-1}(\lambda) \) is a \( r \)-fuzzy \( \beta \)-closed set for each \( S(1 - \lambda) \geq r, \lambda \in I^Y, r \in I_0 \).

2. \( f \) is called a smooth gf\( \beta \)-continuous function if \( f^{-1}(\lambda) \) is a \( r \)-gf\( \beta \) closed set for each \( r \)-fuzzy \( \beta \)-closed set \( \lambda, \lambda \in I^Y, r \in I_0 \). Equivalently, \( f^{-1}(\lambda) \) is a \( r \)-gf\( \beta \) open set for each \( r \)-fuzzy \( \beta \)-open set \( \lambda, \lambda \in I^Y, r \in I_0 \).

3. \( f \) is called a smooth fuzzy gc-\( \beta \)-irresolute function if \( C(\lambda) \) is a \( r \)-gf\( \beta \)-closed set for each \( r \)-gf\( \beta \)-closed set \( \lambda, \lambda \in I^Y, r \in I_0 \). Equivalently, \( C(\lambda) \) is a \( r \)-gf\( \beta \)-open set for each \( r \)-gf\( \beta \)-open set \( \lambda, \lambda \in I^Y, r \in I_0 \).

4. \( f \) is said to be a smooth fuzzy strongly \( \beta \)-continuous function if \( f^{-1}(\lambda) \) is a \( r \)-fuzzy \( \beta \)-open set and a \( r \)-fuzzy \( \beta \)-closed set for each \( \lambda, \lambda \in I^Y, r \in I_0 \).

5. \( f \) is called a smooth M-fuzzy \( \beta \)-continuous function if \( f^{-1}(\lambda) \) is a \( r \)-fuzzy \( \beta \)-open set for each \( r \)-fuzzy \( \beta \)-open set \( \lambda, \lambda \in I^Y, r \in I_0 \). Equivalently, \( f^{-1}(\lambda) \) is a \( r \)-fuzzy \( \beta \)-closed set for each \( r \)-fuzzy \( \beta \)-closed set \( \lambda, \lambda \in I^Y, r \in I_0 \).

**Definition 5.1.4.**

Let \( (X, T) \) be a smooth fuzzy topological space. For \( \lambda \in I^X, r \in I_0 \):

1. \( \lambda \) is called a \( r \)-fuzzy regular \( \beta \)-open set if \( \lambda = \beta \cdot I_T(\beta \cdot C_T(\lambda, r), r) \).

2. \( \lambda \) is called a \( r \)-fuzzy regular \( \beta \)-closed set if \( \lambda = \beta \cdot C_T(\beta \cdot I_T(\lambda, r), r) \).
Property 5.1.2.

(1) \( \lambda \) is a r-fuzzy regular \( \beta \)-open set \( \iff \bar{1} - \lambda \) is a r-fuzzy regular \( \beta \)-closed set, \( \lambda \in I^X, r \in I_0 \).

(2) Every r-fuzzy regular \( \beta \)-open set is a r-fuzzy \( \beta \)-open set.

(3) Every r-fuzzy regular \( \beta \)-closed set is a r-fuzzy \( \beta \)-closed set.

(4) (i) The r-fuzzy \( \beta \)-closure of a r-fuzzy \( \beta \)-open set is a r-fuzzy regular \( \beta \)-closed set.

(ii) The r-fuzzy \( \beta \)-interior of a r-fuzzy \( \beta \)-closed set is a r-fuzzy regular \( \beta \)-open set.

Proof

(1) Proof is simple.

(2) Let \( \lambda \) be any r-fuzzy regular \( \beta \)-open set, \( \lambda \in I^X, r \in I_0 \).

Then, \( \lambda = \beta-I_T(\beta-C_T(\lambda, r), r) \).

Now, \( \beta-I_T(\lambda, r) = \beta-I_T(\beta-I_T(\beta-C_T(\lambda, r), r), r) = \beta-I_T(\beta-C_T(\lambda, r), r) = \lambda \).

Therefore, \( \beta-I_T(\lambda, r) = \lambda \). Hence, \( \lambda \) is a r-fuzzy \( \beta \)-open set.

(3) Let \( \lambda \) be any r-fuzzy regular \( \beta \)-closed set, \( \lambda \in I^X, r \in I_0 \).

Then, \( \lambda = \beta-C_T(\beta-I_T(\lambda, r), r) \).

Now, \( \beta-C_T(\lambda, r) = \beta-C_T(\beta-C_T(\beta-I_T(\lambda, r), r), r) = \beta-C_T(\beta-I_T(\lambda, r), r) = \lambda \).

Therefore, \( \beta-C_T(\lambda, r) = \lambda \). Hence, \( \lambda \) is a r-fuzzy \( \beta \)-closed set.

(4) (i) Let \( \lambda \) be any r-fuzzy \( \beta \)-open set, \( \lambda \in I^X, r \in I_0 \).

Now, \( \beta-I_T(\beta-C_T(\lambda, r), r) \leq \beta-C_T(\lambda, r) \).

Then, \( \beta-C_T(\beta-I_T(\beta-C_T(\lambda, r), r), r) \leq \beta-C_T(\lambda, r) \). (5.1.1)
Now, $\lambda \leq \beta-C_T(\lambda, r)$. Then, $\beta-I_T(\lambda, r) \leq \beta-I_T(\beta-C_T(\lambda, r), r)$. Since $\lambda$ is a r-fuzzy $\beta$-open set, $\lambda \leq \beta-I_T(\beta-C_T(\lambda, r), r)$ and Hence,

$$\beta-C_T(\lambda, r) \leq \beta-C_T(\beta-I_T(\beta-C_T(\lambda, r), r), r).$$  \hspace{1cm} (5.1.2)

From (5.1.1) and (5.1.2) we get, $\beta-C_T(\lambda, r) = \beta-C_T(\beta-I_T(\beta-C_T(\lambda, r), r), r)$. Hence, $\beta-C_T(\lambda, r)$ is a r-fuzzy regular $\beta$-closed set.

4(ii) Let $\lambda$ be any r-fuzzy $\beta$-closed set, $\lambda \in I^X$, $r \in I_0$.

Now, $\beta-C_T(\beta-I_T(\lambda, r), r) \geq \beta-I_T(\lambda, r)$.

Then, $\beta-I_T(\beta-C_T(\beta-I_T(\lambda, r), r), r) \geq \beta-I_T(\lambda, r).$  \hspace{1cm} (5.1.3)

Now, $\lambda \geq \beta-I_T(\lambda, r)$. Then, $\beta-C_T(\lambda, r) \geq \beta-C_T(\beta-I_T(\lambda, r), r)$. Since $\lambda$ is a r-fuzzy $\beta$-closed set, $\lambda \geq \beta-C_T(\beta-I_T(\lambda, r), r)$ and therefore,

$$\beta-I_T(\lambda, r) \geq \beta-I_T(\beta-C_T(\beta-I_T(\lambda, r), r), r).$$  \hspace{1cm} (5.1.4)

From (5.1.3) and (5.1.4) we get, $\beta-I_T(\lambda, r) = \beta-I_T(\beta-C_T(\beta-I_T(\lambda, r), r), r)$. Thus, $\beta-I_T(\lambda, r)$ is a r-fuzzy regular $\beta$-open set.

Remark 5.1.2.

Every r-fuzzy $\beta$-open set need not be a r-fuzzy regular $\beta$-open set as shown in Example 5.1.2.

Example 5.1.2.

Let $X = \{a, b\}$. Let $\lambda_1, \lambda_2 \in I^X$ be defined as follows $\lambda_1(a) = 0.8$, $\lambda_1(b) = 0.9$, $\lambda_2(a) = 0.6$, $\lambda_2(b) = 0.7$. Let $r = 0.04$. Define smooth fuzzy topology $T : I^X \to I$ as follows.
Use the function $T(\lambda) = \begin{cases} 
1 & \lambda = \bar{0}, \bar{1} \\
0.5 & \lambda = \lambda_1 \\
0.6 & \lambda = \lambda_2 \\
0 & \text{otherwise.}
\end{cases}$

Let $\mu \in I_X$ be defined as $\mu(a) = 0.6$, $\mu(b) = 0.8$. Clearly, $\mu$ is a 0.04-fuzzy $\beta$-open set. Further, $\mu$ is not a 0.04-fuzzy $\beta$-closed set.

Now, $\beta I_T(\beta C_T(\mu, 0.04), 0.04) \neq \mu$. Hence, $\mu$ is not a 0.04-fuzzy regular $\beta$-open set.

**Definition 5.1.5.**

Let $(X, T)$ and $(Y, S)$ be any two smooth fuzzy topological spaces. $f : (X, T) \rightarrow (Y, S)$ is called a smooth fuzzy almost $\beta$-continuous function if $f^{-1}(\lambda)$ is a $r$-fuzzy $\beta$-open set for each $r$-fuzzy regular $\beta$-open set $\lambda$, $\lambda \in I_Y$, $r \in I_0$.

**Proposition 5.1.2.**

Let $(X, T)$ and $(Y, S)$ be any two smooth fuzzy topological spaces and let $f : (X, T) \rightarrow (Y, S)$ be any function. Then the following statements are equivalent.

1. $f$ is a smooth fuzzy almost $\beta$-continuous function.
2. $f^{-1}(\mu)$ is a $r$-fuzzy $\beta$-closed set for each $r$-fuzzy regular $\beta$-closed set $\mu$, $\mu \in I_Y$, $r \in I_0$.
3. $f^{-1}(\lambda) \leq \beta I_T(f^{-1}(\beta I_T(\beta C_T(\lambda, r), r)), r)$ for each $r$-fuzzy $\beta$-open set $\lambda$, $\lambda \in I_Y$, $r \in I_0$. 

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(4) \( \beta-C_T(f^{-1}(\beta-C_T(\beta-I_1(\mu, r), r)), r) \leq f^{-1}(\mu) \) for each \( r \)-fuzzy \( \beta \)-closed set \( \mu, \mu \in \tilde{I}^Y, r \in I_0. \)

**Proof**

(1) \( \Rightarrow \) (2). Let \( \mu \) be any \( r \)-fuzzy regular \( \beta \)-closed set, \( \mu \in \tilde{I}^Y, r \in I_0. \)

Then, \( \bar{I} - \mu \) is a \( r \)-fuzzy regular \( \beta \)-open set. Therefore by (1), \( f^{-1}(\bar{I} - \mu) \) is a \( r \)-fuzzy \( \beta \)-open set. That is, \( \bar{I} - f^{-1}(\mu) \) is a \( r \)-fuzzy \( \beta \)-open set. This implies that, \( f^{-1}(\mu) \) is a \( r \)-fuzzy \( \beta \)-closed set. Hence, (2) is proved.

Similarly, (2) \( \Rightarrow \) (1) can be shown.

(1) \( \Rightarrow \) (3). Let \( \lambda \) be any \( r \)-fuzzy \( \beta \)-open set, \( \lambda \in \tilde{I}^Y, r \in I_0. \)

Then, \( \lambda \leq \beta-I_T(\beta-C_T(\lambda, r), r). \)

Therefore, \( f^{-1}(\lambda) \leq f^{-1}(\beta-I_T(\beta-C_T(\lambda, r), r)). \)

By Property 4(ii), it follows that \( \beta-I_T(\beta-C_T(\lambda, r), r) \) is a \( r \)-fuzzy regular \( \beta \)-open set. By (1), \( f^{-1}(\beta-I_T(\beta-C_T(\lambda, r), r))) \) is a \( r \)-fuzzy \( \beta \)-open set.

Thus, \( f^{-1}(\lambda) \leq f^{-1}(\beta-I_T(\beta-C_T(\lambda, r), r)) = \beta-I_T(f^{-1}(\beta-I_T(\beta-C_T(\lambda, r), r)), r). \)

Hence, \( f^{-1}(\lambda) \leq \beta-I_T(f^{-1}(\beta-I_T(\beta-C_T(\lambda, r), r)), r). \)

(3) \( \Rightarrow \) (1). Let \( \lambda \) be any \( r \)-fuzzy regular \( \beta \)-open set, \( \lambda \in \tilde{I}^Y, r \in I_0. \)

By (3), it follows that, \( f^{-1}(\lambda) \leq \beta-I_T(f^{-1}(\beta-I_T(\beta-C_T(\lambda, r), r)), r) = \beta-I_T(f^{-1}(\lambda), r). \)

But, \( \beta-I_T(f^{-1}(\lambda), r) \leq f^{-1}(\lambda) \). Hence, \( f^{-1}(\lambda) = \beta-I_T(f^{-1}(\lambda), r) \). That is, \( f^{-1}(\lambda) \) is a \( r \)-fuzzy \( \beta \)-open set. Therefore, \( f \) is a smooth fuzzy almost \( \beta \)-continuous function.

(2) \( \Rightarrow \) (4). Let \( \mu \) be any \( r \)-fuzzy \( \beta \)-closed set, \( \mu \in \tilde{I}^Y, r \in I_0. \)
Then, \( \mu \geq \beta-C_T(\beta-I_T(\mu, r), r) \).

Therefore, \( f^{-1}(\mu) \geq f^{-1}(\beta-C_T(\beta-I_T(\mu, r), r)) \).

By Property 4(i), it follows that \( \beta-C_T(\beta-I_T(\mu, r), r) \) is a \( r \)-fuzzy regular \( \beta \)-closed set. Hence, \( f^{-1}(\beta-C_T(\beta-I_T(\mu, r), r)) \) is a \( r \)-fuzzy \( \beta \)-closed set.

Thus, \( f^{-1}(\mu) \geq f^{-1}(\beta-C_T(\beta-I_T(\mu, r), r)) = \beta-C_T(f^{-1}(\beta-C_T(\beta-I_T(\mu, r), r)), r) \).

Hence, \( f^{-1}(\mu) \geq \beta-C_T(\beta-I_T(\mu, r), r) \).

(4) \( \Rightarrow \) (2). Let \( \mu \) be any \( r \)-fuzzy regular \( \beta \)-closed set, \( \mu \in \mathcal{Y} \), \( r \in I_0 \).

By (4), we have \( \beta-C_T(f^{-1}(\beta-C_T(\beta-I_T(\mu, r), r)), r) \leq f^{-1}(\mu) \).

That is, \( \beta-C_T(f^{-1}(\mu), r) \leq f^{-1}(\mu) \). But, \( f^{-1}(\mu) \leq \beta-C_T(f^{-1}(\mu), r) \).

Therefore, \( f^{-1}(\mu) = \beta-C_T(f^{-1}(\mu), r) \). Hence, \( f^{-1}(\mu) \) is a \( r \)-fuzzy \( \beta \)-closed set.

**Proposition 5.1.3.**

Let \((X, T)\) and \((Y, S)\) be any two smooth fuzzy topological spaces and let \( f : (X, T) \rightarrow (Y, S) \) be a smooth \( M \)-fuzzy \( \beta \)-continuous function. Then \( f \) is a smooth fuzzy almost \( \beta \)-continuous function.

**Proof**

Let \( \lambda \) be any \( r \)-fuzzy regular \( \beta \)-open set, \( \lambda \in \mathcal{Y} \), \( r \in I_0 \). Then, \( \lambda \) is a \( r \)-fuzzy \( \beta \)-open set. Since \( f \) is a smooth \( M \)-fuzzy \( \beta \)-continuous function, \( f^{-1}(\lambda) \) is a \( r \)-fuzzy \( \beta \)-open set. This implies that, \( f \) is a smooth fuzzy almost \( \beta \)-continuous function.
5.2 SMOOTH FUZZY BICONTINUOUS FUNCTIONS

In this section, smooth fuzzy bicontinuous functions, smooth fuzzy $\beta$-bicontinuous functions, smooth M-fuzzy $\beta$-bicontinuous functions and smooth fuzzy gc-$\beta$-biirresolute functions are introduced. Some interesting properties are discussed.

**Definition 5.2.1.**

Let $(X, T)$ and $(Y, S)$ be any two smooth fuzzy topological spaces and let $f : (X, T) \rightarrow (Y, S)$ be any function. Then

1. $f$ is called a smooth fuzzy bicontinuous function if it is onto and if $S(\lambda) \geq r \Leftrightarrow T(f^{-1}(\lambda)) \geq r$, $\lambda \in I^Y$, $r \in I_0$. Equivalently, $S(1 - \lambda) \geq r \Leftrightarrow T(1 - f^{-1}(\lambda)) \geq r$, $\lambda \in I^Y$, $r \in I_0$.

2. $f$ is said to be a smooth fuzzy $\beta$-bicontinuous function if it is onto and if $S(\lambda) \geq r \Leftrightarrow f^{-1}(\lambda)$ is a $r$-fuzzy $\beta$-open set, $\lambda \in I^Y$, $r \in I_0$. Equivalently, $S(1 - \lambda) \geq r \Leftrightarrow f^{-1}(\lambda)$ is a $r$-fuzzy $\beta$-closed set, $\lambda \in I^Y$, $r \in I_0$.

3. $f$ is said to be a smooth M-fuzzy $\beta$-bicontinuous function if it is onto and if $\lambda$ is a $r$-fuzzy $\beta$-open set $\Leftrightarrow f^{-1}(\lambda)$ is a $r$-fuzzy $\beta$-open set, $\lambda \in I^Y$, $r \in I_0$. Equivalently, if $\lambda$ is a $r$-fuzzy $\beta$-closed set $\Leftrightarrow f^{-1}(\lambda)$ is a $r$-fuzzy $\beta$-closed set, $\lambda \in I^Y$, $r \in I_0$.

**Definition 5.2.2.**

Let $(X, T)$ and $(Y, S)$ be any two smooth fuzzy topological spaces. A function $f : (X, T) \rightarrow (Y, S)$ is called a smooth fuzzy $\beta$-open function if $f(\lambda)$ is a $r$-fuzzy $\beta$-open set for each $r$-fuzzy $\beta$-open set $\lambda$, $\lambda \in I^X$, $r \in I_0$. 

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Definition 5.2.3.

Let (X, T) and (Y, S) be any two smooth fuzzy topological spaces. (X, T) and (Y, S) are said to be smooth M-fuzzy \( \beta \)-homeomorphic \( \Leftrightarrow \) there exists \( f : (X, T) \rightarrow (Y, S) \) such that \( f \) is a one-to-one, onto, smooth M-fuzzy \( \beta \)-continuous and smooth fuzzy \( \beta \)-open function. Such an \( f \) is called a smooth fuzzy \( \beta \)-homeomorphism.

Proposition 5.2.1.

Let (X, T) and (Y, S) be any two smooth fuzzy topological spaces and let \( f : (X, T) \rightarrow (Y, S) \) be a smooth M-fuzzy \( \beta \)-bicontinuous and one-to-one function. Then \( f \) is a smooth fuzzy \( \beta \)-homeomorphism.

Proof

Let \( \lambda \) be any \( r \)-fuzzy \( \beta \)-open set, \( \lambda \in I_x, r \in I_0 \). \( f \) being one-to-one, \( \lambda = f^{-1}(f(\lambda)) \). Since \( f \) is a smooth M-fuzzy \( \beta \)-bicontinuous function and \( \lambda \) is a \( r \)-fuzzy \( \beta \)-open set, \( f(\lambda) \) is a \( r \)-fuzzy \( \beta \)-open set. Thus, \( f \) is a smooth fuzzy \( \beta \)-open function. Hence, \( f \) is a smooth fuzzy \( \beta \)-homeomorphism.

Proposition 5.2.2.

Let (X, T), (Y, S) and (Z, R) be any three smooth fuzzy topological spaces. Let \( f : (X, T) \rightarrow (Y, S) \) be a smooth M-fuzzy \( \beta \)-bicontinuous function and \( g : (Y, S) \rightarrow (Z, R) \) be any function. If \( h = g \circ f \) is a smooth M-fuzzy \( \beta \)-continuous function, then \( g \) is a smooth M-fuzzy \( \beta \)-continuous function. If \( h \) is a smooth M-fuzzy \( \beta \)-bicontinuous function, then \( g \) is a smooth M-fuzzy \( \beta \)-bicontinuous function.
Let $\lambda$ be any $r$-fuzzy $\beta$-open set, $\lambda \in I^X$, $r \in I_0$. Since $h$ is a smooth $M$-fuzzy $\beta$-continuous function, $h^{-1}(\lambda)$ is a $r$-fuzzy $\beta$-open set. Now, $h^{-1}(\lambda) = f^{-1}(g^{-1}(\lambda))$ is a $r$-fuzzy $\beta$-open set. Since $f$ is a smooth $M$-fuzzy $\beta$-bicontinuous function, $g^{-1}(\lambda)$ is a $r$-fuzzy $\beta$-open set. Therefore, $g$ is a smooth $M$-fuzzy $\beta$-continuous function.

Next, let us assume that $h$ is a smooth $M$-fuzzy $\beta$-bicontinuous function and that $g^{-1}(\lambda)$ is a $r$-fuzzy $\beta$-open set. Since $f$ is a smooth $M$-fuzzy $\beta$-bicontinuous function, $f^{-1}(g^{-1}(\lambda))$ is a $r$-fuzzy $\beta$-open set. Now, $f^{-1}(g^{-1}(\lambda)) = h^{-1}(\lambda)$ is a $r$-fuzzy $\beta$-open set. Since $h$ is a smooth $M$-fuzzy $\beta$-bicontinuous function, $\lambda$ is a $r$-fuzzy $\beta$-open set. Therefore, $g$ is a smooth $M$-fuzzy $\beta$-bicontinuous function.

**Proposition 5.2.3.**

Let $(X, T)$ and $(Y, S)$ be any two smooth fuzzy topological spaces. If $f : (X, T) \to (Y, S)$ is a smooth $M$-fuzzy $\beta$-continuous, onto and smooth fuzzy $\beta$-open function, then $f$ is a smooth $M$-fuzzy $\beta$-bicontinuous function.

**Proof**

Let $\lambda$ be any $r$-fuzzy $\beta$-open set, $\lambda \in I^X$, $r \in I_0$. Since $f$ is a smooth $M$-fuzzy $\beta$-continuous function, $f^{-1}(\lambda)$ is a $r$-fuzzy $\beta$-open set. Therefore,
f^{-1}(\lambda) is a r-fuzzy \(\beta\)-open set whenever \(\lambda\) is a r-fuzzy \(\beta\)-open set. Conversely, let \(\lambda\) be such that \(f^{-1}(\lambda)\) is a r-fuzzy \(\beta\)-open set. Since \(f\) is a smooth fuzzy \(\beta\)-open function, \(f(f^{-1}(\lambda)) = \lambda\) is a r-fuzzy \(\beta\)-open set. Thus, \(\lambda\) is a r-fuzzy \(\beta\)-open set iff \(f^{-1}(\lambda)\) is a r-fuzzy \(\beta\)-open set. Therefore, \(f\) is a smooth M-fuzzy \(\beta\)-bicontinuous function.

**Definition 5.2.4.**

Let \((X, T)\) and \((Y, S)\) be any two smooth fuzzy topological spaces. \(f : (X, T) \rightarrow (Y, S)\) is called a smooth fuzzy gc-\(\beta\)-biirresolute function if it is onto and if \(\lambda\) is a r-gf\(\beta\)-closed set \(\iff f^{-1}(\lambda)\) is a r-gf\(\beta\)-closed set, \(\lambda \in I^Y, r \in I_0\). Equivalently, if \(\lambda\) is a r-gf\(\beta\)-open set \(\iff f^{-1}(\lambda)\) is a r-gf\(\beta\)-open set, \(\lambda \in I^Y, r \in I_0\).

**Proposition 5.2.4.**

Let \((X, T), (Y, S)\) and \((Z, R)\) be any three smooth fuzzy topological spaces. Let \(f : (X, T) \rightarrow (Y, S)\) be a smooth fuzzy gc-\(\beta\)-biirresolute function and \(g : (Y, S) \rightarrow (Z, R)\) be any function. If \(h = gof\) is a smooth fuzzy gc-\(\beta\)-irresolute function, then \(g\) is a smooth fuzzy gc-\(\beta\)-irresolute function. If \(h\) is a smooth fuzzy gc-\(\beta\)-biirresolute function, then \(g\) is a smooth fuzzy gc-\(\beta\)-biirresolute function.

**Proof**

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
& \searrow h & \downarrow g \\
& Z & \\
\end{array}
\]

Let \(\lambda\) be any r-gf\(\beta\)-open set, \(\lambda \in I^Z, r \in I_0\). Since \(h\) is a smooth
fuzzy gc-β-irresolute function, $h^{-1}(\lambda) = f^{-1}(g^{-1}(\lambda))$ is a r-gfβ-open set. Since $f$ is a smooth fuzzy gc-β-biirresolute function, $g^{-1}(\lambda)$ is a r-gfβ-open set. Therefore, $g$ is a smooth fuzzy gc-β-irresolute function.

Next, let us assume that $g^{-1}(\lambda)$ is a r-gfβ-open set. Since $f$ is a smooth fuzzy gc-β-biirresolute function, it follows that $f^{-1}(g^{-1}(\lambda)) = h^{-1}(\lambda)$ is a r-gfβ-open set. Since $h$ is a smooth fuzzy gc-β-biirresolute function, $\lambda$ is a r-gfβ-open set. Therefore, $g$ is a smooth fuzzy gc-β-biirresolute function.

**Proposition 5.2.5.**

Let $(X, T)$, $(Y, S)$ and $(Z, R)$ be any three smooth fuzzy topological spaces. If $f : (X, T) \rightarrow (Y, S)$ is a smooth M-fuzzy β-continuous function and $g : (Y, S) \rightarrow (Z, R)$ is a smooth fuzzy β-continuous function. Then $gof$ is a smooth fuzzy β-continuous function.

**Proof**

Let $\lambda$ be such that $R(\lambda) \geq r$, $\lambda \in I^2$, $r \in I_0$. Since $g$ is a smooth fuzzy β-continuous function, $g^{-1}(\lambda)$ is a r-fuzzy β-open set. Since $f$ is a smooth M-fuzzy β-continuous function, $f^{-1}(g^{-1}(\lambda)) = (gof)^{-1}(\lambda)$ is a r-fuzzy β-open set. Hence, $gof$ is a smooth fuzzy β-continuous function.

**5.3 SMOOTH GENERALIZED FUZZY β-T1/2 SPACES**

In this section, the concept of smooth generalized fuzzy β-T1/2 spaces is introduced. Some interesting properties are discussed with necessary examples.
Definition 5.3.1.

A smooth fuzzy topological space \((X, T)\) is said to be a smooth generalized fuzzy \(\beta\)-T_{1/2} space if every \(r\)-fuzzy \(\beta\)-closed set \(\lambda\) is such that \(T(\overline{\lambda} - \lambda) \geq r, \lambda \in I^X, r \in I_0\). Equivalently, if every \(r\)-fuzzy \(\beta\)-open set \(\lambda\) is such that \(T(\lambda) \geq r, \lambda \in I^X, r \in I_0\).

Proposition 5.3.1.

Let \((X, T), (Y, S)\) and \((Z, R)\) be any three smooth fuzzy topological spaces. If \(f : (X, T) \rightarrow (Y, S)\) and \(g : (Y, S) \rightarrow (Z, R)\) are smooth fuzzy \(\beta\)-continuous functions and \((Y, S)\) is a smooth generalized fuzzy \(\beta\)-T_{1/2} space, then \(g \circ f\) is a smooth fuzzy \(\beta\)-continuous function.

Proof

Let \(\lambda\) be such that \(R(\lambda) \geq r, \lambda \in I^2, r \in I_0\). Since \(g\) is a smooth fuzzy \(\beta\)-continuous function, \(g^{-1}(\lambda)\) is a \(r\)-fuzzy \(\beta\)-open set. Since \((Y, S)\) is a smooth generalized fuzzy \(\beta\)-T_{1/2} space, \(S(g^{-1}(\lambda)) \geq r\). Since \(f\) is a smooth fuzzy \(\beta\)-continuous function, it follows that \((g \circ f)^{-1}(\lambda) = f^{-1}(g^{-1}(\lambda))\) is a \(r\)-fuzzy \(\beta\)-open set. Therefore, \(g \circ f\) is a smooth fuzzy \(\beta\)-continuous function.

Proposition 5.3.1. is not valid if \((Y, S)\) is not a smooth generalized fuzzy \(\beta\)-T_{1/2} space as shown in the following example.

Example 5.3.1.

Let \(X = \{a, b, c\}\). Define \(\lambda, \mu, \rho \in I^X\) as follows \(\lambda(a) = \lambda(b) = 0, \lambda(c) = 0.75, \mu(a) = 0, \mu(b) = 0, \mu(c) = 1\) and \(\rho(a) = 1, \rho(b) = 0, \rho(c) = 1\). Let
Define a smooth fuzzy topology $T, S, R : I^X \rightarrow I$ as follows

$$T(\delta) = \begin{cases} 1 & \delta = 0, 1 \\ 0.6 & \delta = \lambda \\ 0 & \text{otherwise}, \end{cases}$$

$$S(\delta) = \begin{cases} 1 & \delta = 0, 1 \\ 0.5 & \delta = \mu \\ 0 & \text{otherwise}, \end{cases}$$

$$R(\delta) = \begin{cases} 1 & \delta = 0, 1 \\ 0.6 & \delta = \rho \\ 0 & \text{otherwise}. \end{cases}$$

Define $f : (X, T) \rightarrow (X, S)$ as $f(a) = a, f(b) = a$ and $f(c) = b$. Let $g : (X, S) \rightarrow (X, R)$ be a identity function. Now, $f^{-1}(\mu)(a) = \mu(f(a)) = 0, f^{-1}(\mu)(b) = \mu(f(b)) = 0$ and $f^{-1}(\mu)(c) = \mu(f(c)) = 0$. Therefore, $f^{-1}(\mu) = 0$.

For $S(\mu) > 0.04$, $f^{-1}(\mu)$ is a 0.04-fuzzy $\beta$-open set. Similarly, for $S(\bar{0}) = S(\bar{1}) > 0.04$, $f^{-1}(\bar{0}) = \bar{0}$ and $f^{-1}(\bar{1}) = \bar{1}$ are 0.04-fuzzy $\beta$-open sets. Therefore, $f$ is a smooth fuzzy $\beta$-continuous function. Now, $g$ is a smooth fuzzy $\beta$-continuous function. For $R(\rho) > 0.04$ and $g^{-1}(\rho) = \rho, C_S(\rho, 0.04) = \bar{1}$. Hence, $C_S(I \cap C_S(\rho, 0.04), 0.04) = \bar{1} > \rho$. This shows that $\rho$ is a 0.04-fuzzy $\beta$-open set. $g \circ f$ is not a smooth fuzzy $\beta$-continuous function. For $R(\rho) > 0.04$, $(g \circ f)^{-1}(\rho) = f^{-1}(g^{-1}(\rho)) = f^{-1}(\rho)$

Now, $f^{-1}(\rho)(a) = 1, f^{-1}(\rho)(b) = 1, f^{-1}(\rho)(c) = 0$. Hence, $C_T(f^{-1}(\rho), 0.04) = (1, 1, 0.25)$. Therefore, $C_T(I \cap C_T(f^{-1}(\rho), 0.04), 0.04) = \bar{0} < f^{-1}(\rho)$. This shows that
Proposition 5.3.2.

Let \((X, T)\) be a smooth generalized fuzzy \(\beta\)-\(T_{1/2}\) space and \((Y, S)\) be any smooth fuzzy topological space. If \(f : (X, T) \rightarrow (Y, S)\) is a smooth fuzzy strongly \(\beta\)-continuous function, then \(f\) is a smooth fuzzy continuous function.

Proof

Let \(\lambda\) be such that \(S(\lambda) \geq r, \lambda \in Y, r \in I_0\). Since \(f\) is a smooth fuzzy strongly \(\beta\)-continuous function, \(f^{-1}(\lambda)\) is a \(r\)-fuzzy \(\beta\)-open and a \(r\)-fuzzy \(\beta\)-closed set. Since \((X, T)\) is a smooth fuzzy \(\beta\)-\(T_{1/2}\) space, \(T(f^{-1}(\lambda)) \geq r\). Therefore, \(f\) is a smooth fuzzy continuous function.

5.4 SMOOTH FUZZY \(\beta\)-\(T_{1/2}\) SPACES

In this section, the concept of smooth fuzzy \(\beta\)-\(T_{1/2}\) spaces is introduced. Some interesting properties are studied by providing necessary examples.

Definition 5.4.1.

A smooth fuzzy topological space \((X, T)\) is said to be a smooth fuzzy \(\beta\)-\(T_{1/2}\) space if every \(r\)-gf\(\beta\)-closed set \(\lambda\) is such that \(T(\overline{I} - \lambda) \geq r, \lambda \in I^X, r \in I_0\). Equivalently, every \(r\)-gf\(\beta\)-open set \(\lambda\) is such that \(T(\lambda) \geq r, \lambda \in I^X, r \in I_0\).
**Proposition 5.4.1.**

Let \((X, T), (Y, S)\) and \((Z, R)\) be any three smooth fuzzy topological spaces. If \(f : (X, T) \rightarrow (Y, S)\) and \(g : (Y, S) \rightarrow (Z, R)\) are smooth \(gf\beta\)-continuous functions and \((Y, S)\) is a smooth fuzzy \(\beta_{T_{1/2}}\) space, then \(g \circ f\) is a smooth \(gf\beta\)-continuous function.

**Proof**

Let \(\lambda\) be any \(r\)-fuzzy \(\beta\)-open set, \(\lambda \in I^Z, r \in I_0\). Since \(g\) is a smooth \(gf\beta\)-continuous function, \(g^{-1}(\lambda)\) is a \(r\)-\(gf\beta\)-open set. Since \((Y, S)\) is a smooth fuzzy \(\beta_{T_{1/2}}\) space, \(S(g^{-1}(\lambda)) \geq r\). Hence, \(g^{-1}(\lambda)\) is a \(r\)-fuzzy \(\beta\)-open set. Since \(f\) is a smooth \(gf\beta\)-continuous function, it follows that \(f^{-1}(g^{-1}(\lambda))\) is a \(r\)-\(gf\beta\)-open set. That is, \((g \circ f)^{-1}(\lambda)\) is a \(r\)-\(gf\beta\)-open set. Therefore, \(g \circ f\) is a smooth \(gf\beta\)-continuous function.

**Proposition 5.4.2.**

Let \((X, T)\) be a smooth fuzzy \(\beta_{T_{1/2}}\) space and \((Y, S)\) be any smooth fuzzy topological space. If \(f : (X, T) \rightarrow (Y, S)\) is a smooth \(gf\beta\)-continuous function, then \(f\) is a smooth fuzzy continuous function.

**Proof**

Let \(\lambda\) be such that \(S(\lambda) \geq r, \lambda \in I^Y, r \in I_0\). Since \(f\) is a smooth \(gf\beta\)-continuous function, \(f^{-1}(\lambda)\) is a \(r\)-\(gf\beta\)-open set. Since \((X, T)\) is a smooth \(gf\beta\)-\(T_{1/2}\) space, \(T(f^{-1}(\lambda)) \geq r\). Therefore, \(f\) is a smooth fuzzy continuous function.

**Proposition 5.4.3.**

Let \((X, T)\) be a smooth fuzzy \(\beta_{T_{1/2}}\) space and \((Y, S)\) be any
smooth fuzzy topological space. If $f : (X, T) \rightarrow (Y, S)$ is a smooth $gf\beta$-continuous function, then $f$ is a smooth $M$-fuzzy $\beta$-continuous function.

**Proof**

Let $\lambda$ be any $r$-fuzzy $\beta$-open set, $\lambda \in I^r$, $r \in I_0$. Since $f$ is a smooth $gf\beta$-continuous function, $f^1(\lambda)$ is a $r$-$gf\beta$-open set. Since $(X, T)$ is a smooth fuzzy $\beta$-$T_{1/2}$ space, $T(f^1(\lambda)) \geq r$ and therefore $f^1(\lambda)$ is a $r$-fuzzy $\beta$-open set. Hence, $f$ is a smooth $M$-fuzzy $\beta$-continuous function.

**5.5 INTERRELATIONS**

In this section, interrelations among the functions and the spaces introduced in sections 5.1, 5.2, 5.3 and 5.4 are discussed providing counter examples wherever necessary.

**Proposition 5.5.1.**

If $(X, T)$ is a smooth fuzzy $\beta$-$T_{1/2}$ space, then $(X, T)$ is a smooth generalized fuzzy $\beta$-$T_{1/2}$ space.

**Proof**

Let $\lambda$ be any $r$-fuzzy $\beta$-closed set, $\lambda \in I^X$, $r \in I_0$. Since every $r$-fuzzy $\beta$-closed set is a $r$-$gf\beta$-closed set, $\lambda$ is a $r$-$gf\beta$-closed set. Since $(X, T)$ is a smooth fuzzy $\beta$-$T_{1/2}$ space, $T(\bar{1} - \lambda) \geq r$ and hence $(X, T)$ is a smooth generalized fuzzy $\beta$-$T_{1/2}$ space.

**Proposition 5.5.2.**

Let $f : (X, T) \rightarrow (Y, S)$ be any function from a smooth fuzzy topological space $(X, T)$ into a smooth generalized fuzzy $\beta$-$T_{1/2}$ space
(Y, S). If f is a smooth fuzzy β-continuous function, then f is a smooth fuzzy almost β-continuous function.

**Proof**

Let λ be any r-fuzzy regular β-open set, λ ∈ I Y, r ∈ I 0. Since every r-fuzzy regular β-open set is a r-fuzzy β-open set, λ is a r-fuzzy β-open set. Since (Y, S) is a smooth generalized fuzzy β-T 1/2 space, S(λ) ≥ r. Since f is a smooth fuzzy β-continuous function, f⁻¹(λ) is a r-fuzzy β-open set. Hence, f is a smooth fuzzy almost β-continuous function.

The Proposition 5.5.2. is not valid if (Y, S) is not a smooth generalized fuzzy β-T 1/2 space as shown in the following example.

**Example 5.5.1.**

Let X = {a, b, c} and Y = {p, q, r}. Define λ ∈ I X as follows λ(a) = 1, λ(b) = λ(c) = 0. Define μ ∈ I Y as follows μ(p) = 1, μ(q) = 1, μ(r) = 0. Define smooth fuzzy topologies T : I X → I and S : I Y → I as follows

\[
T(δ) = \begin{cases} 
1 & δ = \bar{0}, \bar{1} \\
0.6 & δ = λ \\
0 & \text{otherwise},
\end{cases}
\]

\[
S(δ) = \begin{cases} 
1 & δ = \bar{0}, \bar{1} \\
0.5 & δ = μ \\
0 & \text{otherwise}.
\end{cases}
\]

Let r = 0.04. Define f : (X, T) → (Y, S) as f(a) = f(b) = p and f(c) = q. Then, f is a smooth fuzzy β-continuous function. For S(μ) > 0.04,
f^{-1}(p) is a 0.04-fuzzy $\beta$-open set. For any $p \in \beta^T$ define $\rho(p) = 0$, $\rho(q) = 1$ and $\rho(r) = 0$. Since $C_{S}(I_{S}(C_{S}(\rho, 0.04), 0.04)) = \bar{I} > \rho$, $\rho$ is a 0.04-fuzzy $\beta$-open set. Now, $\beta-I_{S}(\beta-C_{S}(\rho, 0.04), 0.04) = \beta-I_{S}(\rho, 0.04) = \rho$.

Hence, $\rho$ is a 0.04-fuzzy regular $\beta$-open set. Further, $f^{-1}(p)(a) = 0$, $f^{-1}(p)(b) = 0$ and $f^{-1}(p)(c) = 1$.

Since $C_{T}(I_{T}(C_{T}(f^{-1}(p), 0.04), 0.04)) = 0 < f^{-1}(p)$, $f^{-1}(p)$ is not a 0.04-fuzzy $\beta$-open set. Hence, $f$ is not a smooth fuzzy almost $\beta$-continuous function. $(Y, S)$ is not a smooth generalized fuzzy $\beta$-T$_{1/2}$ space, for $\rho$ is a 0.04-fuzzy $\beta$-open set but $S(\rho) \not\in 0.04$.

**Proposition 5.5.3.**

Let $f : (X, T) \rightarrow (Y, S)$ be any function from a smooth generalized fuzzy $\beta$-T$_{1/2}$ space $(X, T)$ into a smooth generalized fuzzy $\beta$-T$_{1/2}$ space $(Y, S)$. Then $f$ is a smooth M-fuzzy $\beta$-bicontinuous function $\iff f$ is a smooth fuzzy bicontinuous function.

**Proof**

Assume that $f$ is a smooth M-fuzzy $\beta$-bicontinuous function. Let $\lambda$ be such that $S(\lambda) \geq r$, $\lambda \in I^T$, $r \in I_0$. Then, $\lambda$ is a r-fuzzy $\beta$-open set. Since $f$ is a smooth M-fuzzy $\beta$-bicontinuous function, $f^{-1}(\lambda)$ is a r-fuzzy $\beta$-open set. Since $(X, T)$ is a smooth generalized fuzzy $\beta$-T$_{1/2}$ space, it follows that $T(f^{-1}(\lambda)) \geq r$. Let $T(f^{-1}(\lambda)) \geq r$. Then, $f^{-1}(\lambda)$ is a r-fuzzy $\beta$-open set. Since $f$ is a smooth M-fuzzy $\beta$-bicontinuous function, $\lambda$ is a r-fuzzy $\beta$-open set. Since $(Y, S)$ is a smooth generalized fuzzy $\beta$-T$_{1/2}$ space, $S(\lambda) \geq r$. Therefore, $f$ is a smooth fuzzy bicontinuous function.
Conversely, assume that $f$ is a smooth fuzzy bicontinuous function. Let $\lambda$ be any $r$-fuzzy $\beta$-open set, $\lambda \in I^Y$, $r \in I_0$. Since $(Y, S)$ is a smooth generalized fuzzy $\beta$-$T_{1/2}$ space, $S(\lambda) \geq r$. Since $f$ is a smooth fuzzy bicontinuous function, it follows that $T(f^{-1}(\lambda)) \geq r$. Hence, $f^{-1}(\lambda)$ is a $r$-fuzzy $\beta$-open set. Let $f^{-1}(\lambda)$ be any $r$-fuzzy $\beta$-open set for any $\lambda$, $\lambda \in I^Y$, $r \in I_0$. Since $(X, T)$ is a smooth generalized fuzzy $\beta$-$T_{1/2}$ space, $T(f^{-1}(\lambda)) \geq r$. Since $f$ is smooth fuzzy bicontinuous function, $S(\lambda) \geq r$ and hence $\lambda$ is a $r$-fuzzy $\beta$-open set. Therefore, $f$ is a smooth $M$-fuzzy $\beta$-bicontinuous function.

The Proposition 5.5.3. is not valid if $(X, T)$ is not a smooth generalized fuzzy $\beta$-$T_{1/2}$ space as shown in the following example.

Example 5.5.2.

Let $X = \{a, b\}$ and $Y = \{p, q\}$. Define $\lambda_1, \lambda_2 \in I^X$ as $\lambda_1(a) = 0.2, \lambda_1(b) = 0.3, \lambda_2(a) = 0.7, \lambda_2(b) = 0.55$. Let $\mu_1, \mu_2 \in I^Y$ be defined as follows $\mu_1(p) = 0.65, \mu_1(q) = 0.7, \mu_2(p) = 0.15, \mu_2(q) = 0.2$. Let $r = 0.04$. Define smooth fuzzy topologies $T : I^X \to I$ and $S : I^Y \to I$ as follows

$$
T(\delta) = \begin{cases} 
1 & \delta = \overline{0}, I \\
0.5 & \delta = \lambda_1 \\
0.6 & \delta = \lambda_2 \\
0 & \text{otherwise}
\end{cases}
$$
Define \( f : (X, T) \rightarrow (Y, S) \) as \( f(a) = q \) and \( f(b) = p \). Clearly \( f \) is onto and a smooth \( M \)-fuzzy \( \beta \)-continuous function. Now, \( f^{-1}(\mu_1)(a) = 0.7 \), \( f^{-1}(\mu_1)(b) = 0.65 \). For \( S(\mu_1) > 0.04 \), \( T(f^{-1}(\mu_1)) \not\geq 0.04 \). Hence, \( f \) is not a smooth fuzzy bicontinuous function.

Further, \((X, T)\) is not a smooth fuzzy generalized \( \beta \)-T\(_{1/2}\) space. For any \( \rho \in I^X \) define \( \rho(p) = 0.25 \), \( \rho(q) = 0.35 \). Clearly \( \rho \) is a 0.04-fuzzy \( \beta \)-open set but \( T(\rho) \not\geq 0.04 \).

**Proposition 5.5.4.**

Let \( f : (X, T) \rightarrow (Y, S) \) be any function from a smooth generalized fuzzy \( \beta \)-T\(_{1/2}\) space \((X, T)\) into a smooth fuzzy topological space \((Y, S)\). Then \( f \) is a smooth fuzzy \( \beta \)-bicontinuous function \( \iff f \) is a smooth fuzzy bicontinuous function.

**Proof**

Assume that \( f \) is a smooth fuzzy \( \beta \)-bicontinuous function. Let \( \lambda \) be such that \( S(\lambda) \geq r \), \( \lambda \in I^Y \), \( r \in I_0 \). Since \( f \) is a smooth fuzzy \( \beta \)-bicontinuous function, \( f^{-1}(\lambda) \) is a \( r \)-fuzzy \( \beta \)-open set. Since \((X, T)\) is a smooth generalized fuzzy \( \beta \)-T\(_{1/2}\) space, \( T(f^{-1}(\lambda)) \geq r \). Let \( T(f^{-1}(\lambda)) \geq r \). Then, \( f^{-1}(\lambda) \) is a \( r \)-fuzzy \( \beta \)-open set. Since \( f \) is a smooth fuzzy \( \beta \)-bicontinuous function, \( S(\lambda) \geq r \). Therefore, \( f \) is a smooth fuzzy
bicontinuous function.

Conversely, assume that \( f \) is a smooth fuzzy bicontinuous function. Let \( \lambda \) be such that \( S(\lambda) \geq r, \lambda \in I^Y, r \in I_0 \). Since \( f \) is a smooth fuzzy bicontinuous function, \( T(f^{-1}(\lambda)) \geq r \) and hence \( f^{-1}(\lambda) \) is a \( r \)-fuzzy \( \beta \)-open set. Let \( f^{-1}(\lambda) \) be a \( r \)-fuzzy \( \beta \)-open set. Since \( (X, T) \) is a smooth generalized fuzzy \( \beta \)-\( T_1/2 \) space, \( T(f^{-1}(\lambda)) \geq r \). Since \( f \) is a smooth fuzzy bicontinuous function, \( S(\lambda) \geq r \). Therefore, \( f \) is a smooth fuzzy \( \beta \)-bicontinuous function.

**The Proposition 5.5.4. is not valid if \((X, T)\) is not a smooth generalized fuzzy \( \beta \)-\( T_1/2 \) space as shown in the following example**

**Example 5.5.3.**

Let \( X = \{a, b, c\} \) and \( Y = \{p, q, r\} \). Define \( \lambda \in I^X \) as follows: \( \lambda(a) = 1, \lambda(b) = 0, \lambda(c) = 1 \). Let \( r = 0.04 \). Define \( \mu \in I^Y \) as follows: \( \mu(p) = 0, \mu(q) = 1, \mu(r) = 1 \). Define smooth fuzzy topologies \( T : I^X \rightarrow I \) and \( S : I^Y \rightarrow I \) as follows

\[
T(\delta) = \begin{cases} 
1 & \delta = 0, 1 \\
0.5 & \delta = \lambda \\
0 & \text{otherwise,}
\end{cases}
\]

\[
S(\delta) = \begin{cases} 
1 & \delta = 0, 1 \\
0.6 & \delta = \mu \\
0 & \text{otherwise.}
\end{cases}
\]

Define \( f : (X, T) \rightarrow (Y, S) \) as \( f(a) = q, f(b) = p \) and \( f(c) = r \). Clearly \( f \) is onto. Now \( S(\mu) > 0.04, T(f^{-1}(\mu)) > 0.04 \) and \( T(\lambda) > 0.04, S(f(\lambda)) > 0.04 \).
Hence, \( f \) is a smooth fuzzy bicontinuous function. \( f \) is not a smooth fuzzy \( \beta \)-bicontinuous function. For, \( \lambda_1 \in I^X \) define \( \lambda_1(a) = 1, \lambda_1(b) = 1, \lambda_1(c) = 0 \). Now, \( C_T(\lambda_1, 0.04) = \tilde{1} \) and

\[
C_T(I_T(C_T(\lambda_1, 0.04), 0.04), 0.04) = \tilde{1} > \lambda_1.
\]

Hence, \( \lambda_1 \) is a 0.04-fuzzy \( \beta \)-open set. But \( f(\lambda_1)(p) = 1, f(\lambda_1)(q) = 1 \) and \( f(\lambda_1)(r) = 0 \). Therefore, \( S(f(\lambda_1)) \geq 0.04 \). Further, \((X, T)\) is not a smooth generalized fuzzy \( \beta \)-\( T_{1/2} \) space. For, \( \lambda_1 \) is a 0.04-fuzzy \( \beta \)-open set and \( T(\lambda_1) \geq 0.04 \).

**Proposition 5.5.5.**

Let \( f : (X, T) \rightarrow (Y, S) \) be any function from a smooth fuzzy \( \beta \)-\( T_{1/2} \) space \((X, T)\) into a smooth fuzzy \( \beta \)-\( T_{1/2} \) space \((Y, S)\). Then \( f \) is a smooth fuzzy gc-\( \beta \)-biirresolute function \( \Leftrightarrow f \) is a smooth fuzzy bicontinuous function.

**Proof**

Assume that \( f \) is a smooth fuzzy gc-\( \beta \)-biirresolute function. Let \( \lambda \) be such that \( S(\tilde{1} - \lambda) \geq r, \lambda \in I^Y, r \in I_0 \). Now, \( \lambda \) is a \( r \)-fuzzy \( \beta \)-closed set and hence \( \lambda \) is a \( r \)-gf\( \beta \)-closed set. Since \( f \) is a smooth fuzzy gc-\( \beta \)-biirresolute function, \( f^{-1}(\lambda) \) is a \( r \)-gf\( \beta \)-closed set. Since \((X, T)\) is a smooth fuzzy \( \beta \)-\( T_{1/2} \) space, \( T(\tilde{1} - f^{-1}(\lambda)) \geq r \). Let \( T(\tilde{1} - f^{-1}(\lambda)) \geq r \). Then, \( f^{-1}(\lambda) \) is a \( r \)-fuzzy \( \beta \)-closed set and hence \( f^{-1}(\lambda) \) is a \( r \)-gf\( \beta \)-closed set. Since \( f \) is a smooth fuzzy gc-\( \beta \)-biirresolute function, \( \lambda \) is a \( r \)-gf\( \beta \)-closed set. Since \((Y, S)\) is a smooth fuzzy \( \beta \)-\( T_{1/2} \) space, \( S(\tilde{1} - \lambda) \geq r \). Therefore, \( f \) is a smooth fuzzy bicontinuous function.
Conversely, assume that \( f \) is a smooth fuzzy bicontinuous function. Let \( \lambda \) be a \( r\)-\( g\beta \)-closed set, \( \lambda \in I^Y, r \in I_0 \). Since \( (Y, S) \) is a smooth fuzzy \( \beta \)-\( T_{1/2} \) space, \( S(\bar{1} - \lambda) \geq r \). Since \( f \) is a smooth fuzzy bicontinuous function, \( T(\bar{1} - f^{-1}(\lambda)) \geq r \). Then, \( f^{-1}(\lambda) \) is a \( r \)-fuzzy \( \beta \)-closed set and hence \( f^{-1}(\lambda) \) is a \( r \)-\( g\beta \)-closed set. Let \( f^{-1}(\lambda) \) be a \( r \)-\( g\beta \)-closed set. Since \( (X, T) \) is a smooth fuzzy \( \beta \)-\( T_{1/2} \) space, \( T(\bar{1} - f^{-1}(\lambda)) \geq r \). Since \( f \) is a smooth fuzzy bicontinuous function, \( S(\bar{1} - \lambda) \geq r \). Then, \( \lambda \) is a \( r \)-fuzzy \( \beta \)-closed set and hence \( \lambda \) is a \( r \)-\( g\beta \)-closed set. Therefore, \( f \) is a smooth fuzzy \( gc \)-\( \beta \)-biirresolute function.

**Proposition 5.5.5.** is not valid if \( (X, T) \) is not a smooth fuzzy \( \beta \)-\( T_{1/2} \) space as shown in the following example.

**Example 5.5.4.**

Let \( X = \{a, b\} \) and \( Y = \{p, q\} \). Define \( \lambda_1, \lambda_2 \in I^X \) as \( \lambda_1(a) = 0.2, \lambda_1(b) = 0.3, \lambda_2(a) = 0.7, \lambda_2(b) = 0.55 \). Let \( \mu_1, \mu_2 \in I^Y \) be defined as follows

\[
\mu_1(p) = 0.65, \mu_1(q) = 0.7, \mu_2(p) = 0.15, \mu_2(q) = 0.2.
\]

Let \( r = 0.04 \). Define smooth fuzzy topologies \( T : I^X \to I \) and \( S : I^Y \to I \) as follows

\[
T(\delta) = \begin{cases} 
1 & \delta = \bar{0}, \bar{1} \\
0.5 & \delta = \lambda_1 \\
0.6 & \delta = \lambda_2 \\
0 & \text{otherwise}
\end{cases}
\]
Define \( f : (X, T) \to (Y, S) \) as \( f(a) = q \) and \( f(b) = p \). Clearly \( f \) is onto and \( f \) is a smooth fuzzy ge-\( \beta \)-biirelolute function, for the inverse image of every 0.04-gf\( \beta \) open set is a 0.04-gf\( \beta \)-open set.

Now, \( f^{-1}(\mu_1)(a) = 0.7 \), \( f^{-1}(\mu_1)(b) = 0.65 \).

Since \( S(\mu_1) > 0.04 \), \( T(f^{-1}(\mu_1)) \not\leq 0.04 \). Hence, \( f \) is not a smooth fuzzy bicontinuous function. Let \( \rho \in I^X \) be defined as \( \rho(p) = 0.25 \), \( \rho(q) = 0.35 \). Now, \( C_T(I_T(C_T(\rho, 0.04), 0.04), 0.04) > \rho \). \( \rho \) is a 0.04-fuzzy \( \beta \)-open set and hence, \( \rho \) is a 0.04-gf\( \beta \)-closed set. Further \( T(\rho) \not\geq 0.04 \).

Therefore \( (X, T) \) is not a smooth fuzzy \( \beta \)-T\( _{1/2} \) space.