Chapter V
CHAPTER - 5

OPTIMIZATION OF COST FUNCTION AND THE WAITING TIME IN HOSPITAL UNDER FUZZY QUEUEING MODEL

In general, waiting time is a facility that depends on the number of customers in the queue, the number of servers and the length of service time provided by the system with too much of service capacity, which involve loss to service providers. Whereas not providing system with enough service capacity results are excessive waiting time involving loss of customers. This warrants the optimal system to accommodate and reduce the waiting time along with staffing and service facilities.

Waiting time has been linked to inefficiencies in health care delivery, which results in prolonged patients suffering and dissatisfaction among the public. Long waiting time for health care is an important health policy issue in many countries and many have introduced some form of national waiting time guarantees. Current national waiting time statistics are of limited use for health care availability among the various countries due to the differences in measurements and data collection. Different methodological issues must be taken into account when making such cross country comparisons.

In this chapter, a queueing model for cost function and waiting list in hospitals is studied in which the arrival rate of the customers is a Fuzzy number and it depends on the number of the customer in the queue. Service rate is
assumed fuzzy decision variable. The objective is to find the optimum value of service rate with minimum total cost of the system. Cost function includes service and delay costs which are naturally a Fuzzy number due to the input Fuzzy data. Waiting time data in hospitals are performance indicators in many countries. Worthenston [155] modeled the hospital system as a state dependent Queue model, focused on the hospitals waiting lists, and analyzed the effect of Queue length on arrival behaviors of customers. The arrival rate depends to the behavior of the customers and the available information about it may be incomplete. In addition, in many practical situations the service time is not deterministic.

Lee [114] worked on as finite server system of M(n)/G/1 with a general service time distribution in which the customers arrival depend on the current number of customers in system (n) with Poisson distribution. More recently Ozaki and Takagi [133] generalized M/M/m/K model such that arrival rates depends on the number of present customers of the system at their times. A typical objective of optimization problems in Queue systems is to find the optimum service rate that balances between service and delay costs. When the cost coefficients or the arrival rate is fuzzy the optimum expected cost per unit time is a fuzzy number.

Chen [41] proposed a mixed integer mathematical programming method to construct the membership function of the fuzzy objective value of the cost based Queueing decision problem when the cost coefficients and the arrival rate are fuzzy numbers.
5.1 Waiting Times in Hospital Management:

Waiting times arise as the result of the demand and supply imbalance. If the demand exceeds the supply, then a queue is formed. Additionally, the waiting time situation can also be difficult to improve long-term if the variation in supply does not adapt to variation in demand. Excess demand during a certain period of time generates queues, whereas temporary excess capacity cannot be saved up for future use. Difference in waiting time for the same procedure can depend on differences in indication or Hospital threshold for when the procedure is performed. Thus, the fact that one care guy has a shorter queue for emergency treatment than others can be due to the fact that its waiting list threshold is higher. The speed at which the patients are taken off the waiting list is affected by the frequency with which surgery is performed, newly published reports point out a negative association between waiting time and the availability of the curative care beds in the hospital and to a lesser extent, between waiting time and public hospital spending per capita. However, supply is not solely the explanation to waiting time.

A Fuzzy cost function is defined for this model to minimize the operating and idle cost of each server, cost of employing each server, queue cost of the customers, and the missed customer cost. The optimum service rate and its membership function are obtained through a mixed integer non-linear programming model.
Finally the effectiveness of the proposed method is illustrated by means of a numerical example.

5.1.1: \( M/\lambda_q/G/m \) queuing model

In this model \( M/\lambda_q/G/m \), \( \lambda_q \) denotes the state-dependent arrival rate and is a decreasing function of queue length ‘q’, service time has a general distribution function by mean.

5.1.2: Performance Measures of Fuzzy Queue

The performance measures of Fuzzy Queue system, for example the number of customers in the system, time that customer spends waiting in the queue or total customer’s time spent in the system.

5.1.3: Pollaczek–Khintchine Mean Value Formula

Let \( w_q \), the time of a customer spends in the queue and \( L_q \), the number of customers in the queue. Using \( W = w_q + w_s \) and \( L_q = L - \rho \)

and obtained, \( W_q = \frac{\lambda E[S^2]}{2(1-\rho)} = \frac{\lambda[(\frac{1}{\mu})^2 + \sigma_s^2]}{2(1-\frac{\lambda}{\mu})} \)

\( L_q = \frac{\lambda^2 E[S^2]}{2(1-\rho)} = \frac{\lambda^2[(\frac{1}{\mu})^2 + \sigma_s^2]}{2(1-\frac{\lambda}{\mu})} \)

Where \( E[S^2] \) is the second moment of service time distribution.
\[ E[s^2] = \sigma_s^2 + E[s]^2, \text{ where} \]

\[ \sigma_s^2 = \text{Variance of service time.} \]

\[ L_q = \text{Number of customers in the queue.} \]

\[ L_s = \text{Number of customers in the system.} \]

5.1.4 Residual time

The residual service time is the time that remains until finishing the service. In other words it is the time that the arriving customer has to wait if there is at least one customer in the process of being served and it is denoted by \( \mathbb{R}. \)

By Pollaczek-Khintchine, the residual time is given by,

\[
E[\mathbb{R}] = w_q - \frac{1}{\mu} L_q = \frac{\lambda E[s^2]}{2(1 - \rho)} - \frac{\lambda^2 E[s^2]}{\mu} = \frac{\lambda E[s^2]}{2(1 - \rho)} (1 - \rho) = \frac{\lambda E[s^2]}{2} \Rightarrow E[\mathbb{R}] = \frac{\lambda E[s^2]}{2}
\]

5.2 Queuing Models for Hospital Waiting List

5.2.1 Model Formulation

Consider the potential arrival rate in this system M/\( \lambda q / G / m, (P_a) \) the actual rate depends on the number of customer in the waiting list. Some of the patients will dispend in the system and the remaining will be queued in the
waiting list. The actual arrival rate of the patients to the system when the queue length = q is,

\[ \lambda_q = P_a(1 - qp) \]  

Where p is the proportion of patients constantly for reneging.

By Worthington [155] showed that the following equality based on some experiments by fitting suitable linear models.

\[(\text{i.e.}) L_q = \frac{1}{p} \left[ 1 - \frac{m}{p_a E(t)} \right] \]  

\[ W_q = \frac{L_q.E(t)}{m} \]  

\[ E(\lambda_q) = \frac{m}{E(t)} \]  

Where E(t) = Mean time, L_q = The mean value of queue length,

w_q = The mean time of the patients are in waiting list, E(\lambda_q) = mean value of arrival rate.

Now the cost function is defined as,

\[ C_F = C_o F[E(t)] + C_E m + C_w E(q) + C_R \left[ L - L_q \right] + C_M \left[ P_a - \lambda_q \right] \]  

Where \( C_F \) = Cost function

\( C_o = \) Operating cost for each server
\[ C_E = \text{Cost of employing for each server} \]

\[ C_w = \text{Cost of wasting time in queue for each customer.} \]

\[ C_R = \text{Cost of wasting time while receiving service} \]

\[ C_M = \text{Cost of missed customer.} \]

From the Little’s formula, then

\[ L - L_q = \lambda_q E(t) = m \] \(………………….…..(6)\)

Also, assumed that

\[ E(t) = G F[E(t)] = \frac{4}{G^2} \] \(…………………..….(7)\)

Using the values (6), (7) in (5) we get

\[ C_F = C_o \frac{4}{G^2} + C_E m + \frac{C_w}{P} \left[ 1 - \frac{m}{P \alpha G} \right] + mC_R + C_M \left[ p_\alpha - \frac{m}{G} \right] \] \(…….(8)\)

By considering ambiguity and vagueness of input data the potential arrival rate assumed as a Trapezoidal Fuzzy Number with four parameters \((x_1, x_2, x_3, x_4)\). Further, the service rate, the output cost are also treated Fuzzy Numbers.

The membership function of the cost function should be extracted and it is given by,

\[ \tilde{C}_F = C_o \frac{4}{G^2} + C_E m + \frac{C_w}{P} \left[ 1 - \frac{m}{\tilde{p} \alpha \tilde{G}} \right] + mC_R + C_M \left[ \tilde{p}_\alpha - \frac{m}{\tilde{G}} \right] \] \(………………..(9)\)
5.2.2 Analysis of Fuzzy Queue Method

Suppose that the arrival rate $\lambda$ is a fuzzy number $\tilde{\lambda}$, Chen [36] uses the concept of $\alpha$-cut to form the cost membership function and $\alpha$-cut is a crisp set as

$$\lambda_\alpha = \{ x \in X / \mu_\tilde{\lambda} \geq \alpha \} \quad \text{..................(10)}$$

The $\alpha$-cut interval is given by,

$$\lambda_\alpha = \{ \min_x \{ x \in X / \mu_\tilde{\lambda} \geq \alpha \}, \max_x \{ x \in X / \mu_\tilde{\lambda} \geq \alpha \} \} \quad \text{........(11)}$$

Where $X$ is the universal set, $\mu_\tilde{\lambda}$ is a membership function and $\alpha$ is a crisp value between 0 and 1. Thus for a fuzzy arrival rate, two crisp numbers as an upper and lower bound of $\alpha$-cut interval can be distinguished.

If the cost function is calculated based on $\lambda^L_\alpha$ and $\lambda^U_\alpha$ at each level of $\alpha$-cut will yield lower bound $C^L_{F\alpha}$ and the upper bound $C^U_{F\alpha}$ of fuzzy cost value at $\alpha$-level.

Now $\tilde{x}$ is considered as the fuzzy input number.

Therefore at $\mu_\tilde{x} = \alpha$, the requirement of $\mu_\tilde{x} = \alpha$, is equivalent to $x = \lambda^L_\alpha$ or $x = \lambda^U_\alpha$ of $\alpha$-cut.

(i.e) The pair of following results upper and lower bound of the cost function at each level.

$$\text{(i.e) } C^L_{F\alpha} = \min \{ F_{c_p}(G) + F_{c_e}(G, x) \} \quad \text{.................................(12)}$$
Subject to $x = t\lambda_{\alpha}^L + (1 - t)\lambda_{\alpha}^U, t = 0 \text{ or } 1$

\[ c_{F,\alpha}^U = \max \{ F_{c_{\alpha}} (G^*) + F_{c_{B}} (G^*, x) \} \] .............................................(13)

Subject to $x = \hat{\gamma}\lambda_{\alpha}^L + (1 - \hat{\gamma})\lambda_{\alpha}^U, \hat{\gamma} = 0 \text{ or } 1$

To obtain the analytical form of the membership function, an increasing function $C_{F}^L : \alpha \rightarrow C_{F}^L$ and a decreasing function $C_{F}^U : \alpha \rightarrow C_{F}^U$ are defined. Then the left shape function $L_{s}(C_{F}) = (C_{F}^L)^{-1}$ and the right shape function

$R_{s}(C_{F}) = (C_{F}^U)^{-1}$ can be obtained when both $C_{F,\alpha}^L$ and $C_{F,\alpha}^U$ are invertible with respect to $\alpha$.

\[
(\text{ie}) \quad \mu_{C_{F}} (C_{F}) = \begin{cases} 
L_{s}(C_{F}), & C_{F,\alpha=0}^L \leq C_{F} \leq C_{F,\alpha=1}^L \\
1, & C_{F,\alpha=1}^L \leq C_{F} \leq C_{F,\alpha=1}^U \\
R_{s}(C_{F}), & C_{F,\alpha=1}^U \leq C_{F} \leq C_{F,\alpha=0}^U 
\end{cases} \] .............................................(14)

The fuzzy membership function of the service rate as a variable over than the membership function of cost is a one of merits of Chen [36] proposed model. Suppose the optimal service rate $[E(t)^*]_{\alpha}^L$ and $[E(t)^*]_{\alpha}^U$ are Fuzzy Numbers. Then $\alpha$-cut of the obtained optimal service rate $\{ [E(t)^*]_{\alpha}^L, [E(t)^*]_{\alpha}^U \}$ is the optimal repair rate. Particularly when $\alpha = 1$ (most possible value) and $\alpha = 0$ the range that the optimal service rate could appear. To obtain as exact value for service rate, Chen [37] proposed the expected operator value for Fuzzy Number to defuzzify the resulted Fuzzy service rate.
Therefore the point of estimation of the service rate will be calculated as following

\[ I(\tilde{E}(t)^*) = \int_0^{11} \left[ (\tilde{E}(t)^*)_l^u \right] d\alpha \] (15)

Where \( \tilde{E}(t)^* \) is optimal Fuzzy service rate.

### 5.2.3 Model Solution Procedure:

Now consider the cost function from (9)

\[ \tilde{C}_F = \frac{4}{\tilde{G}^2} + C_E m + \frac{C_w}{\tilde{p}} \left[ 1 - \frac{m}{\tilde{\tilde{p}} / \tilde{G}} \right] + m C_R + C_M \left[ \tilde{\tilde{p}} - \frac{m}{\tilde{G}} \right] \]

Let \( \tilde{X} \) be a Trapezoidal Fuzzy Number with 4 parameters \( (x_1, x_2, x_3, x_4) \).

Then the corresponding formulation is given by,

\[ C_{F\alpha}^L = \min \left\{ \frac{4}{\tilde{G}^2} + C_E m + \frac{C_w}{\tilde{p}} \left[ 1 - \frac{m}{\tilde{\tilde{p}} / \tilde{G}} \right] + m C_R + C_M \left[ \tilde{\tilde{p}} - \frac{m}{\tilde{G}} \right] \right\} \] (16)

Subject that \( \tilde{\tilde{p}} = t[x_1 + (x_2 - x_1)\alpha] + (1 - t)[x_4 - (x_4 - x_3)\alpha] \), \( t = 0, 1 \)

Since \( \tilde{G} \) is absent in the constraint, then the objective function can be treated as unconstraint optimization problem. (i.e) The objective function \( C_{F\alpha}^L \) depends on the value of \( \tilde{G} \). In order to optimize this problem, derivatives of \( C_{F\alpha}^L \) are taken with respect to \( G \).
(i.e) \( \frac{d C_{F\alpha}^L}{dG} = 0 \) gives the minimum or maximum value at the stationary point.

If \( \frac{d^2 C_{F\alpha}^L}{dG^2} < 0 \) at the particular point of \( \tilde{G} = G_1 \) (say) then the cost function \( C_{F\alpha}^L \) attains maximum at that particular point at \( \tilde{G} = G_1 \).

If \( \frac{d^2 C_{F\alpha}^L}{dG^2} > 0 \), at \( \tilde{G} = G_2 \) (say) then cost function \( C_{F\alpha}^L \) attains minimum at that particular point at \( \tilde{G} = G_2 \).

Now, \( \frac{d C_{F\alpha}^L}{dG} = 0 \) ..............................................(17)

\[ \Rightarrow -\frac{8c_o}{\tilde{G}^3} + \frac{C_w}{p} \left( \frac{m}{\tilde{G}^2} \right) + C_M \frac{m}{\tilde{G}} = 0 \]

\[ \Rightarrow p \tilde{p}_a (-8C_o) + MC_w \tilde{G} + p \tilde{p} \alpha m \tilde{G} = 0 \]

\[ \Rightarrow m \tilde{G} (C_w + p \tilde{p}_a) = 8C_0 p \tilde{p}_a \]

\[ \Rightarrow G^* = \frac{8C_0 p \tilde{p}_a}{MC_w + MC_M \tilde{p}_a} \]

\[ \Rightarrow G^* = \frac{8C_0 p \tilde{p}_a}{m[C_w + CM \tilde{p}_a]} \] ..............................................(18)

\[ \frac{d^2 C_{F\alpha}^L}{d\tilde{G}^2} = 24C_o \tilde{G}^4 + \frac{C_w}{p} \left( \frac{-2m}{\tilde{p}_a \tilde{G}^2} \right) \frac{2mC_M}{\tilde{G}^2} = 0 \]
Equation (20) determines a range of $\tilde{G}$ such that the cost function $C_F$ is concave.

From the equation (18) and (20),

$$mC_w + mC_M p \tilde{p}_\alpha > C_w m - mC_M p \tilde{p}_\alpha$$

and $8C_0 p \tilde{p}_\alpha < 12C_0 p \tilde{p}_\alpha$ ...........................................(21)

$$\Rightarrow \frac{12c_0 p \tilde{p}_\alpha}{mC_w - mC_M p \tilde{p}_\alpha} > G^* ...........................................(22)$$

Then the sufficient condition is always true. Consequently $G^*$ is a crisp value that minimizes the lower bound of the cost function $C_F$ at predetermined $\alpha$-cut.

Now, the upper bound of the cost function is obtained by the following procedure.

$$C_{F,a}^U = \max \left\{ C_0 \frac{4}{\tilde{G}^2} + C_E m + \frac{C_w}{\tilde{G}} \left[ 1 - \frac{m}{\tilde{p}_\alpha \tilde{G}} \right] \right\} + mC_R + C_M \left[ \tilde{p}_\alpha - \frac{m}{\tilde{G}} \right] \ldots \ldots (23)$$
Subject to \( \tilde{x} = t[x_1 + (x_2 - x_1)\alpha] + (1 - t)[x_4 - (x_4 - x_3)\alpha] \)

\[
(i.e) \quad G^* = \frac{8c_o p \tilde{p} \alpha}{mC_w + mC_M p \tilde{p} \alpha}, \quad t = 0,1
\]

**Numerical illustration**

The proposed model is illustrated by the numerical example. The following input parameters as,

\( c_0 = 100; \; c_e = 500; \; c_w = 5; \; c_R = 5; \; c_M = 10 \tilde{x} = (10, 11, 13, 14); \)

\( m = 6; \; p = 0.0025 \)

From the equation (16)

\[
C_{L_{\alpha}} = \min \left\{ C_o \frac{4}{G^2} + C_em + \frac{C_w}{p} \left[ 1 - \frac{m}{\tilde{p} \alpha} \right] + mC_R + C_M \left[ \frac{\tilde{p}}{\alpha} - \frac{m}{\tilde{G}} \right] \right\} \quad \text{...........(24)}
\]

Replacing \( G^* \) in the objective function, the model (16) is solved, the minimum of the cost function occurs when \( t = 1 \).

put \( t = 1 \) in \([G^*]\) in (18)

\[
G^* = \frac{8c_o p[1\{10 + (11 - 10)\alpha\}]}{m[C_w + C_M p\{1(10 + \alpha)\}]}
\]
\[ G^* = \frac{8C_o p[10 + \alpha]}{m[C_w] + mC_m p[10 + \alpha]} \]

\[ = \frac{8C_o p[10 + \alpha]}{m[C_w + C_m p(10 + \alpha)]} \] (25)

It is clear that the right side of the fuzzy cost number [model (23)] is obtained when \( t = 0 \). Therefore, parametric Fuzzy service rate that construct the RHS of the Fuzzy cost is given by

\[ [G(\alpha)]_{t=0} = \frac{8C_o p[14 - \alpha]}{m[C_w + C_m p(14 - \alpha)]} \] (26)

The parameter relation of the RHS of the fuzzy cost value is given by

\[ \tilde{C_{R\alpha}} = C_o \frac{4}{\tilde{G}^2} + C_E m + \frac{c_w}{p} \left[ 1 - \frac{m}{\tilde{p}} \right] + mC_R + C_M \left[ \tilde{p} - \frac{m}{\tilde{G}} \right] \] (27)

The following table is constructed, which shows clearly \( G^* \) and Expected Cost of the model in different values of \( \alpha \) and value of \( t \).
<table>
<thead>
<tr>
<th>Level of confidence</th>
<th>$[G^*]_{t=0}$</th>
<th>$[C_{F\alpha}]_{t=0}$</th>
<th>$[G^*]_{t=1}$</th>
<th>$[C_{F\alpha}]_{t=1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.86</td>
<td>4644.3</td>
<td>0.63</td>
<td>4137.5</td>
</tr>
<tr>
<td>0.1</td>
<td>0.85</td>
<td>4636.2</td>
<td>0.64</td>
<td>4157.2</td>
</tr>
<tr>
<td>0.2</td>
<td>0.85</td>
<td>4627.6</td>
<td>0.64</td>
<td>4176.3</td>
</tr>
<tr>
<td>0.3</td>
<td>0.84</td>
<td>4619.4</td>
<td>0.65</td>
<td>4195.1</td>
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<tr>
<td>0.4</td>
<td>0.84</td>
<td>4611.1</td>
<td>0.65</td>
<td>4213.3</td>
</tr>
<tr>
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<td>4556.0</td>
<td>0.69</td>
<td>4314.3</td>
</tr>
</tbody>
</table>

Table 5.1: The optimal values of $\tilde{G}^*$ and $C_{F\alpha}$ at different levels of confidence and value of $t$.

**Fuzzy service rate calculation:** The fuzzy service rate is given by

$$\int_0^1 \frac{1}{2} \left\{ [\tilde{G}^L] + [\tilde{G}^U] \right\} d\alpha$$

$$= \frac{16 \cdot \alpha \cdot C_o}{C_w} - \left[ \frac{8 \cdot C_o \cdot C_w \cdot \log(C_M + 10 \cdot C_w \cdot P + \alpha \cdot C_w \cdot P)}{(C_w^2 \cdot m \cdot P)} \right]$$

$$+ \left[ \frac{8 \cdot C_o \cdot C_w \cdot \log(C_M + 14 \cdot C_w \cdot P - \alpha \cdot C_w \cdot P)}{(C_w^2 \cdot m \cdot P)} \right] \quad (28)$$
Substituting the values of $C_o, C_M, C_w, \alpha, P$ in \(\log (C_M + 10 \times C_w \times P + \alpha \times C_w \times P)\) and \(\log (C_M + 14 \times C_w \times P + \alpha \times C_w \times P)\) in (28) we get,

\[
\frac{1}{2} \int_0^1 \{[\tilde{G}]_\alpha^L + [\tilde{G}]_\alpha^U\} d\alpha = 0.7498
\]

Figure: 5.1 Value of $\tilde{G}$ at different levels of $\alpha$
From these observations the Fuzzy cost functions at different level of $\alpha$-values are obtained. It is evident that it minimizes the operating cost, cost of employing for each server, idle cost, queue cost and the cost of missed customers. Thus the researcher can take the optimize decisions.