ABSTRACT

Introduction: The structure of graphs admits a variety of functions that assign real number to their vertices and edges so that certain given conditions are satisfied.

Numbered undirected graphs are becoming an increasingly useful family of mathematical models for a broad range of applications. If a non negative integer \( f(e) \) is assigned to each edge \( e \), then the edges of \( G \) are said to be numbered.

Then each vertex \( v \) is given value \( f.(v) \equiv (\sum f(uv))(\text{mod } (2k-1)) \), where this sum run over all edges through \( v \). Thus every graph can be numbered in infinitely many ways if the additional conditions are absence. Thus the numbered graph models are used after imposing additional conditions.

Rosa [1967] introduced graph labeling later on called as graceful labeling. The graceful labeling problem is to determine which graphs are graceful. Given a graph \( G \) consisting of vertices and edges, a vertex labeling of \( G \) is an assignment \( f \) of labels to the vertices of \( G \) that produces for each edge \( xy \) a label depending on the vertex labels \( f(x) \) and \( f(y) \).

A vertex labeling \( f \) is called a graceful labeling of a graph \( G \) with \( e \) edges if \( f \) is an injection from the vertices of \( G \) to the set \( \{0, 1, \ldots, e\} \) such that when each edge \( xy \) is assigned the label \( |f(x) - f(y)| \), the resulting edge labels are distinct. A graph \( G \) is called graceful if there exists a graceful labeling of \( G \).

Rosa [1967] has identified three reasons why a graph fails to be graceful: (1) \( G \) has “too many vertices” and “not enough edges,” (2) \( G \) “has
too many edges,” and (3) G “has the wrong parity.” An infinite class of graphs that are not graceful for the second reason, is given in [1986]. As an example of the third condition Rosa has shown that if every vertex has even degree and the number of edges is congruent to 1 or 2 (mod 4), then the graph is not graceful. In particular, the cycles $C_{4n+1}$ and $C_{4n+2}$ are not graceful.

Chapter 1 – Preliminaries

In this chapter, various types of graceful labeling and the preliminary concepts are presented. In this chapter the definitions that are needed for the main results are furnished.

Chapter 2: Cycle related graphs

In this chapter, the edge-odd graceful labeling is obtained for the graphs related to cycle with path and star and the results are furnished as follows

1: The connected graph $C_m \Theta S_n$ where $n \geq 2$ is even; all positive integer $m$.
2: The connected graph $C_3 \Theta P_n$ for $n \geq 2$.
3: The connected graph $C_3 \Theta 2P_n$ for $n \geq 2$.
4: The connected graph $C_3 \Theta 3P_n$.
5: The connected graph $C_5 \Theta P_n$ for $n \geq 2$.
6: The connected graph $C_5 \Theta 2P_n$ for $n \geq 2$.
7: The connected graph $C_m \Theta 2P_{n+1}$ for odd $m$ and $n \geq 1$.

Example: The following graphs are given as examples for edge-odd graceful graphs due to various rules in the above graphs.

1. $C_3 \Theta S_n$; 2. $C_3 \Theta P_2$; 3. $C_3 \Theta P_3$;
Chapter 3: Star related graphs

In this chapter, the edge - odd graceful labeling is obtained for the graphs related to star is found.

8: The connected graph $S_{n} \square S_n$, for $n \geq 3$, is edge – odd graceful.
9: The connected graph $S_{2,n}$ is edge – odd graceful for $n > 1$.
10: The connected graph $S_{m,n}$ is edge – odd graceful for all $m, n > 1$.
11: The connected graph $JE_{3,n}$ is edge – odd graceful for $n > 1$.

Example: The following graphs are given as examples for edge-odd graceful graphs due to various rules in the above graphs.

1. $S_{2} \square S_7$ for $n$ is odd;
2. $S_{2} \square S_6$ for $n \equiv 0 \pmod{6}$;
3. $S_{2} \square S_8$ for $n \equiv 2 \pmod{6}$;
4. $S_{2} \square S_{10}$ for $n \equiv 4 \pmod{6}$;
5. $S_{2,8}$;
6. $S_{2,9}$;
7. $S_{6,5}$, $m \equiv 2 \pmod{4}$ and $n \equiv 1 \pmod{4}$;
8. $S_{6,7}$, $m \equiv 2 \pmod{4}$ and $n \equiv 3 \pmod{4}$;
9. $S_{8,5}$, $m \equiv 0 \pmod{4}$ and $n \equiv 1 \pmod{4}$;
10. $S_{7,5}$, $m \equiv 3 \pmod{4}$ and $n \equiv 1 \pmod{4}$;
11. $S_{7,7}$, $m \equiv 3 \pmod{4}$ and $n \equiv 3 \pmod{4}$;
12. $S_{9,5}$, $m \equiv 1 \pmod{4}$ and $n \equiv 1 \pmod{4}$;
13. $S_{9,5}$, $m \equiv 1 \pmod{4}$ and $n \equiv 3 \pmod{4}$;
14. $S_{6,8}$, $m \equiv 2 \pmod{4}$ and $n \equiv 0 \pmod{4}$;
15. $S_{4,8}$, $m \equiv 0 \pmod{4}$ and $n \equiv 0 \pmod{4}$;
16. $S_{5,6}$, $m \equiv 1 \pmod{4}$ and $n \equiv 2 \pmod{4}$;
17. $S_{6,6}$, $m \equiv 2 \pmod{4}$ and $n \equiv 2 \pmod{4}$;
18. $JE_{3,4}$, $n \equiv 0 \pmod{4}$;
19. $JE_{3,5}$, $n \equiv 1 \pmod{4}$;
20. $JE_{3,6}$, $n \equiv 2 \pmod{4}$;
21. $JE_{3,7}$, $n \equiv 3 \pmod{4}$.

**Chapter 4: Graphs with the combination of Path and Star**

In this chapter, the concept of edge-odd graceful labeling is extended to the graphs with the combination of path and star and the results are furnished below:

12: The connected graph $P_m \Theta S_5$ is edge – odd graceful for $n > 1$.
13: The connected graph $P_4 \circ S_{(2n-1)}$ is edge – odd graceful for $n > 1$.
14: The connected graph $P_6 \circ S_n$ is edge – odd graceful for $n > 1$.
15: The connected graph $P_7 \circ S_n$ is edge – odd graceful for $n > 1$.
16: The connected graph $P_8 \circ S_n$ is edge – odd graceful for odd $n > 5$.

**Example:** The following graphs are given as examples for edge-odd graceful graphs due to various rules in the above graphs.

1. $P_4 \Theta S_5$;
2. $P_4 \circ S_7$;
3. $P_7 \circ S_9$, for $n \equiv 7 \pmod{8}$ and $n \equiv 1 \pmod{8}$
4. $P_8 \circ S_3$, for $n \equiv 0 \pmod{8}$;
5. \( P_8 \circ S_5 \), for \( n \equiv 5 \) (mod 8).

Also the edge-odd graceful labeling is verified for the join of paths and the results are furnished below:

17: The connected graph \( P_2 + P_n \) is edge – odd graceful.
18: The connected graph \( P_3 + P_n \) for \( n = 1, 2, \ldots, (4n + 1) \) is edge – odd graceful.
19: The connected graph \( P_4 + P_n \) for \( n = 1, 2, 3, 4, \ldots, (5n + 2) \) is edge – odd graceful.
20: The connected graph \( P_5 + P_n \) for all \( n \) and \( n \neq 7 \) is edge – odd graceful.
21: The connected graph \( P_6 + P_n \) for all \( n \) and \( n \neq 7 \) and 8 is edge – odd graceful.

Examples of edge-odd graceful graphs are given for join of paths
1. \( P_3 + P_3; P_3 + P_4; P_3 + P_5; \)
2. \( P_4 + P_4; P_4 + P_5; \)
3. \( P_5 + P_5; \)
4. \( P_6 + P_7; P_6 + P_8. \)

Chapter 5: Shell graph and Clock graph (2C_n + nP_2)

In this chapter, the edge-odd graceful labeling for Shell graphs 2C(n, n-3) and 3C(n, n-3) and Clock graph (2C_n + nP_2) is obtained.

22: One edge union of shell graph 2C(n, n-3) is edge odd graceful.
23: One edge union of shell graph 3C(n, n-3) is edge odd graceful.
24: The connected graph 2C_n + nP_2 is edge – odd graceful.
**Example:** The following graphs are given as examples for edge-odd graceful graphs due to various rules in the above graphs.

1. $2C(7, 4)$;
2. $3C(7, 4)$.
3. $2C_7 + 7P_2, n \equiv 2 \pmod{3}$;
4. $2C_8 + 8P_2, n \equiv 1 \pmod{3}$;
5. $2C_6 + 6P_2, n \equiv 0 \pmod{3}$.

**Chapter 6: Edge–vertex–even Gracefulness of the Graphs**

In this chapter, Graceful labeling is verified for the following graph.

25: The spanning tree of the graph of Cartesian product of $S_m$ and $S_n$.

Also, in this chapter, a new type of labeling called edge–vertex–even graceful labeling is introduced and is verified for some graphs.

**Various analogues to graceful labeling:** After the introduction of graceful labeling for the graphs by Rosa in [1967], several authors invented various labeling by imposing different conditions to the edges as well as to the vertices.

Lo [1985] introduced similar kind of labeling called edge-graceful. Gnanajothi [1991] has developed a concept similar to edge-graceful called line-graceful. Also in 1991, Gnanajothi defined another labeling called odd-graceful. Lee, Pan, and Tsai [2005] found another graph labeling called vertex-graceful. Vertex-graceful graphs can be viewed as the dual of edge-graceful graphs. B. Gayathri, M. Duraisamy, and M. Tamilselvi obtained the concept of even- edge graceful in [2007]. A.Solairaju and K.Chitra in 2008 introduced the concept of edge-odd graceful labeling. These different
labelings motivate us to study another kind of labeling called edge-vertex-even graceful.

Edge-vertex-even gracefulfulness of the graph is defined as follows:

A (p, q) connected graph is edge-vertex-even graceful graph if there exists an injective map \( f: E(G) \rightarrow \{2, 4, \ldots, 2q\} \) so that induced map \( f_+: V(G) \rightarrow \{0, 2, 4, \ldots, (2k-2)\} \) defined by \( f_+(x) = (\Sigma f(x, y))(\text{mod } 2k) \), where the vertex \( x \) is incident with other vertex \( y \) and \( k = \max \{p, q\} \), makes the resulting labels distinct and even. When the graph admits the labeling of edge-vertex-even graceful, the graph is called edge-vertex-even graceful graph.

The edge-vertex-even graceful labeling is obtained for some graphs and the results are furnished below:

26: The path \( P_n \) is edge–vertex–even graceful for odd \( n \).
27: The circuit graph \( C_n \) is an edge–vertex–even graceful for odd \( n \).
28: The star graph \( S_n \) (or \( K_{1, n-1} \)) is an edge–vertex–even graceful for odd \( n \).
29: The \( n \circ C_3 \) is edge–vertex–even graceful where \( n \) is even.
30: The graph \( P_2 \square C_n \) is edge–vertex–even graceful.
31: A spanning tree of the Cartesian product graph \( S_3 \square S_n \) is edge–vertex–even graceful.
32: The connected graph \( P_2 \square P_n \) is edge–vertex–even graceful.
33: The connected graph \( 2C_n \square nP_2 \) is edge–vertex–even graceful.

**Example:** The following graphs are given as examples for edge–vertex–even graceful graphs due to various rules in the above graphs.

A spanning tree of the Cartesian product of \( S_2 \) and \( S_6 \);

1. A spanning tree of the Cartesian product graph \( S_3 \square S_3 \);
2. The path $P_9$;
3. The circuit $C_9$;
4. The star $S_7$ (or $K_{1,6}$);
5. $6 \circ C_3$;
6. $P_2 \boxtimes C_4$;
7. The connected graph $P_2 + P_{10}$ for $n= 4(\mod 6)$;
8. The connected graph $P_2 + P_6$ for $n= 2(\mod 6)$;
9. The connected graph $P_2 + P_5$ for $n= 1, 3, 5(\mod 6)$;
10. The connected graph $P_2 + P_6$ for $n= 0(\mod 6)$;
11. The connected graph $2C_7 + 7P_2$ and $2C_8 + 8P_2$. 