Chapter 5

PROBABILISTIC ANALYSIS OF

STORAGE SYSTEMS WITH

RANDOM SALES TIME

DEPENDING ON PRODUCTION

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1The results of the first two models were reported and abstract appeared in the proceedings at national conference held at Nadar Saraswathi college, Theni, and the results of the last model was presented in the international conference held at PSG Tech., Coimbatore. The final enlarged revised and modified version has appeared in Int. J. Contemp. Math. Sciences, Vol. 7, 2012, No. 19, 943 - 951.
5.1 Introduction

A systematic account of the (s,S) inventory policy is provided by Arrow, Karlin and Scarf[4] based on renewal theory. Dvoretsky, Kiefer and Wolfowitz [17] established sufficient conditions on (s,S) policy for the single stage inventory problem to be optimal. Hadley and Whitin [27] worked on several applications of different inventory models. Such systems with random lead times and unit demand were treated by Daniel and Ramanarayanan [15]. Models with bulk demands were analyzed by Ramanarayanan and Jacob [68]. Murthy and Ramanarayanan [53, 54, 55, 56, 57] considered several (s,S) inventory systems. In this chapter we consider four models. In models A and C we consider the operation times have exponential distributions and the production times have general distributions. In models B and D we study the case when the operation times have general distributions and the production times have exponential distributions. After the production, the sales time starts. In models A and B, the sales time varies depending on the number of products produced is within or in excess of the threshold limit. Models C and D study the cases when the sum of the magnitudes of the products produced are within or in excess of the threshold. The joint transforms, the means of production times and sales times, the variances, the covariance of the variables and numerical examples are also presented.
5.2 Model A: Markovian Operation Time and General Single Production

Assumptions

1. The machine operation time T is a random variable with exponential distribution function with parameter $\lambda$.

2. The inter-production times of products are independent identically distributed random variables with Cdf $F(x)$ and pdf $f(x)$. At every production time one product is produced. Let N be the number of products produced during the operation time T.

3. The sale time S has general distribution with Cdf $G_1(y)$ and pdf $g_1(x)$ when the number of products produced $N$ is less than the threshold $U$ and it has Cdf $G_2(y)$ and pdf $g_2(x)$ when $N$ is more than the threshold $U$ to provide change in selling rate. When no unit is produced during operation time sales time $S_1$ is used for other purpose.

4. Threshold U has a general probability function and

$$P(U = n) = p_n, \ n = 1, 2, 3, \ldots.$$  

Let $P(U > n) = P_n, \ \phi(s) = \sum_{n=0}^{\infty} p_n s^n$  

and $\Phi(s) = \sum_{n=0}^{\infty} P_n s^n$
We may derive the joint distribution of \( T \) and \( S \) as follows. The joint probability density function of \( T \) and \( S \) is

\[
\frac{\partial}{\partial x} \left( \frac{\partial}{\partial y} P(T \leq x, S \leq y) \right) = f(x, y)
\]

\[
= \sum_{n} \lambda e^{-\lambda y}[F_n(x) - F_{n+1}(x)][P_ng_1(y) + (1 - P_n)g_2(y)]
\]

(5.2.1)

for \( n = 1, 2, 3 \). The first term of equation (5.2.1) is the part of the pdf that the operation time is \( x \), the sales time is \( y \), the number of productions is \( n \) and the number of products produced is within the threshold. The second term of equation (5.2.1) is the part of the pdf that the operation time is \( x \), the sale time is \( y \), the number of productions is \( n \) and it exceeds the threshold where \( F_k(x) \) is the \( k \) fold cdf convolution and is the probability that the time to \( k \) productions is less than \( x \), for \( k = 1, 2, 3 \). Let us define the joint Laplace transform as follows.

\[
E(e^{-tT}e^{-sS}) = \int_0^\infty \int_0^\infty f(x, y)e^{-tx}e^{-sy}dxdy,
\]

\[
= \int_0^\infty \int_0^\infty \sum_{n=0}^{\infty} \lambda e^{-\lambda y}[F_n(x) - F_{n+1}(x)]P_ng_1(y)e^{-tx}e^{-sy}dxdy
\]

\[
+ \int_0^\infty \int_0^\infty \sum_{n=0}^{\infty} \lambda e^{-\lambda y}[F_n(x) - F_{n+1}(x)](1 - P_n)g_2(y)e^{-tx}e^{-sy}dxdy
\]

and \( t, s \geq 0 \)

On simplification it gives

\[
E(e^{-tT}e^{-sS}) = \left( \frac{\lambda}{\lambda + t} \right)(1 - \phi(f^+g_1(s)))g_1(s)
\]

\[
+ \left( \frac{\lambda}{t + \lambda} \right)\phi(f^+(\lambda + t)g_2(s).
\]

(5.2.2)
Here * indicates Laplace transform

\[ E(e^{-tT}) = \frac{\lambda}{\lambda + t}, \quad E(e^{-sS}) = (1 - \phi(f^*(\lambda)))g_1^*(s) + \phi(f^*(\lambda))g_2^*(s). \]

\[ E(S) = (1 - \phi(f^*(\lambda)))E(S_1) + \phi(f^*(\lambda))E(S_2). \]

Setting \( p = 1 - \phi(f^*(\lambda)) \) and \( q = 1 - p \),

we may note \( E(S) = pE(S_1) + qE(S_2) \)

\[ \text{Var}(S) = p\text{Var}(S_1) + q\text{Var}(S_2) + pq[E(S_1 - S_2)]^2 \quad (5.2.3) \]

Using differentiation of (5.2.2) we get

\[ \text{Cov}(T, S) = \phi'(f^*(\lambda))f^{(\uparrow)}(\lambda)[E(S_1) - E(S_2)]. \quad (5.2.4) \]

When \( f \) is exponential pdf with rate \( a \) and \( U \) has geometric distribution with parameter \( \theta \)

we get

\[ \text{Cov}(T, S) = -[E(S_1) - E(S_2)] \frac{a(1 - \theta)}{[\lambda + a(1 - \theta)]}. \quad (5.2.5) \]

5.3 Model B: General Operation Time and Markovian

Single Production

Assumptions

1. The machine operation time \( T \) is a random variable with cumulative distribution function (cdf) \( F(x) \) and probability density function (pdf) \( f(x) \).
2. The inter-production times of products are independent and identically distributed random variables with exponential distribution with rate \( \lambda \). At every production time one product is produced. Let \( N \) be the number of products produced during the operation time \( T \).

3. The sales time \( S \) has general distribution with cdf \( G_1(y) \) and pdf \( g_1(x) \) when the number of products produced \( N \) is less than the threshold \( U \) and it has cdf \( G_2(y) \) and pdf \( g_2(x) \) when \( N \) is more than the threshold \( U \) to provide change in selling rate. When no unit is produced during operation time sales time \( S_1 \) used for other purpose.

4. Threshold \( U \) has geometric distribution with parameter \( \theta \) and \( P(U = n) = \theta(1 - \theta)^{n-1}, n = 1, 2, 3, \) Noting that the number of products produced during a period has Poisson distribution, we may derive the joint distribution of \( T \) and \( S \) as follows.

The joint probability density function of \( T \) and \( S \) is

\[
\frac{\partial}{\partial x} \left( \frac{\partial}{\partial y} P(T \leq x, S \leq y) \right) = f(x, y) \\
= f(x) \sum_{n=0}^{\infty} \theta^n \left\{ \frac{e^{-\lambda x} (\lambda y)^n}{n!} \right\} g_1(y) \\
+ f(x) \sum_{n=0}^{\infty} (1 - \theta^n) \left\{ \frac{e^{-\lambda x} (\lambda y)^n}{n!} \right\} g_2(y) \tag{5.3.1}
\]

for \( n = 1, 2, 3 \) The first term of equation (5.3.1) is the part of the pdf that the operation time is \( x \), the sale time is \( y \), the number of products is within the threshold limit. The
second term of equation (5.3.1) is the part of the pdf that the operation time is \( x \), the sales time is \( y \), the number of productions is \( n \) and it exceeds the threshold limit. Let us define the joint Laplace transform as follows.

\[
E(e^{-tT}e^{-sS}) = \int_0^\infty \int_0^\infty f(x, y)e^{-tx}e^{-sy}dxdy. 
\]

(5.3.2)

\[
= \int_0^\infty \int_0^\infty f(x)e^{-tx}e^{-sy}e^{-\lambda x(1-\theta)}g_1(y)dxdy \\
+ \int_0^\infty \int_0^\infty f(x)e^{-tx}e^{-sy}(1-e^{-\lambda x(1-\theta)})g_2(y)dxdy,
\]

where

\[
0 \leq \theta \leq 1, \text{ and } t, s \geq 0.
\]

On simplification it gives

\[
E(e^{-tT}e^{-sS}) = f^*(t + \lambda(1-\theta))g_1^*(s) + [f^*(t) - f^*(t + \lambda(1-\theta))]g_2^*(s) 
\]

(5.3.3)

We may note \( E(e^{-tT}) = f^*(t) \);

\[
E(e^{-tS}) = f^*(\lambda(1-\theta))g_1^*(s) + [1 - f^*(\lambda(1-\theta))]g_2^*(s). 
\]

(5.3.4)

Using differentiation and setting \( p = f^*(\lambda(1-\theta)) \) with \( q = 1 - p \), we get

\[
E(S) = pE(S_1) + qE(S_2), E(S^2) = pE(S_1^2) + qE(S_2^2),
\]

\[
Var(S) = pVar(S_1) + qVar(S_2) + pq[E(S_1 - S_2)]^2. 
\]

(5.3.5)

Using differentiation of (5.3.3) we get

\[
E(TS) = E(T)E(S_2) - f^{\prime\prime}(\lambda(1-\theta))[E(S_1) - E(S_2)] \text{ and}
\]

\[
Cov(T, S) = -[E(S_1) - E(S_2)][f^{\prime\prime}(\lambda(1-\theta)) + pE(T)]. 
\]

(5.3.6)
When $T$ is exponential with parameter $a$, we get

$$Cov(T, S) = -[E(S_1) - E(S_2)] \left[ \frac{(\lambda(1 - \theta))}{(a + \lambda(1 - \theta))^2} \right]. \quad (5.3.7)$$

## 5.4 Model C: Markovian Operation Time and General Random Production

### Assumptions

1. The machine operation time $T$ is a random variable with exponential distribution with parameter $\lambda$. Its cumulative distribution function (cdf) is $F(x)$ and probability density function (pdf) is $f(x)$.

2. The inter-production times of products are independent random variables with cumulative distribution function (cdf) $F(x)$ and probability density function (pdf) $f(x)$. Let $N$ be the number of products produced during the operation time $T$.

3. The sizes of the products produced are independent and identically distributed random variable with cdf $H(x)$ and pdf $h(x)$. The sales time $S$ is $S_1$ which has general distribution with cdf $G_1(y)$ and pdf $g_1(x)$ when the total sum of magnitude $Z$ of the products produced is less than the threshold selling capacity $U$ and is $S_2$ which has general distribution cdf $G_2(y)$ and pdf $g_2(x)$ when the total magnitude $Z$ of the products produced is more than the threshold $U$ to provide change selling rate.
4. Threshold $U$ has exponential distribution with parameter $\mu$.

Noting that the operation time has exponential distribution and inter production times are general, we may derive the joint distribution of $T, N, Z$ and $S$ as follows. The joint probability density function of $T, Z$ and $S$ and probability function of $N$ is

\[
\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} P(T \leq x, N = n, U \leq u, S \leq y)\right) = f(x, n, u, y)
\]

\[
= \lambda e^{-\lambda x}[F_n(x) - F_{n+1}(x)]H_n(u)\mu e^{-\mu y}g_1(y)
\]

\[
+ \lambda e^{-\lambda x}[F_n(x) - F_{n+1}(x)][1 - H_n(u)]\mu e^{-\mu y}g_2(y),
\]

(5.4.1)

for $n = 1, 2, 3$ The first term of equation (5.4.1) is the part of the pdf that the operation time is $x$, the sales time is $y$, the number of productions is $n$ and the size of production $z$ is within $U$, the threshold level. The second term of equation (5.4.1) is the part of the pdf that the operation time is $x$, the sales time is $y$, the number of productions is $n$ and the size of production $z$ exceeds $U$, the threshold level where $F_k(z)$ is the $k$ fold 'cdf convolution’ and is the probability that the size of sum of $k$ productions is less than $z$. Let us define the joint Laplace transform cum generating function as follows.

\[
E(e^{tT}e^{-nS}e^{-uZ}g^N)
\]

\[
= \sum_{n=0}^{\infty} \theta^n \int_0^\infty \int_0^\infty \int_0^\infty f(x, n, z, y)e^{-tx}e^{-ny}e^{-uz} \, dx \, dy \, dz.
\]
\[
= \sum_{n=0}^{\infty} \theta^n \int_0^\infty \int_0^\infty \int_0^\infty \lambda e^{-\lambda \gamma} [F_n(x) - F_{n+1}(x)]
\]
\[
\times H_n(z) \mu e^{-\mu z} e^{-yx} e^{-uz} g_1(y) dx dy dz
\]
\[
+ \sum_{n=0}^{\infty} \theta^n \int_0^\infty \int_0^\infty \int_0^\infty \lambda e^{-\lambda \gamma} [F_n(x) - F_{n+1}(x)]
\]
\[
\times [1 - H_n(z)] \mu e^{-\mu z} e^{-yx} e^{-uz} g_2(y) dx dy dz
\tag{5.4.2}
\]

0 \leq \theta \leq 1, \text{and } t, s, u \geq 0.

On simplification it gives \( E(e^{-tT}e^{-sS}e^{-uZ}\theta^N) \)
\[
= \left( \frac{\mu \lambda}{(\mu + u)(\lambda + t)} \right) \left( \frac{1 - f^s(t + \lambda)}{1 - \theta h^s(\mu + u) f^s(t + \lambda)} \right) (g_1^s(s) - g_2^s(s))
\]
\[
+ \left( \frac{\mu \lambda}{(\mu + u)(\lambda + t)} \right) \left( \frac{1 - f^s(t + \lambda)}{1 - \theta f^s(t + \lambda)} \right) g_2^s(s)
\]

\[
E(e^{-tT}e^{-sS}\theta^N) = \left( \frac{\lambda}{t + \lambda} \right) \left( \frac{1 - f^s(t + \lambda)}{1 - \theta h^s(\mu + u) f^s(t + \lambda)} \right) (g_1^s(s) - g_2^s(s))
\]
\[
+ \left( \frac{\lambda}{t + \lambda} \right) \left( \frac{1 - f^s(t + \lambda)}{1 - \theta f^s(t + \lambda)} \right) g_2^s(s). \tag{5.4.3}
\]

We may note \( E(e^{-tT}) = \frac{\lambda}{t + \lambda}, \)
\[
E(e^{-sS}) = \left( \frac{1 - f^s(\lambda)}{1 - h^s(\mu)f^s(\lambda)} \right) (g_1^s(s) - g_2^s(s)) + g_2^s(s) \tag{5.4.4}
\]
\[
E(e^{-sS}) = \left( \frac{1 - f^s(\lambda)}{1 - h^s(\mu)f^s(\lambda)} \right) g_1^s(s) + \left( \frac{f^s(\lambda)(1 - h^s(\mu))}{1 - h^s(\mu)f^s(\lambda)} \right) g_2^s(s).
\]
\[
E(S) = \left( \frac{1 - f^s(\lambda)}{1 - h^s(\mu)f^s(\lambda)} \right) E(S_1) + \left( \frac{f^s(\lambda)(1 - h^s(\mu))}{1 - h^s(\mu)f^s(\lambda)} \right) E(S_2).
\]
Setting
\[ p = \left( \frac{1 - f^*(\lambda)}{1 - h^*(\mu)f^*(\lambda)} \right) \text{ and } q = 1 - p, \]

we note
\[ E(S) = pE(S_1) + qE(S_2) \]
\[ Var(S) = p(Var(S_1)) + q(Var(S_2)) + pq[E(S_1 - S_2)]^2 \quad (5.4.5) \]

Using differentiation of (5.4.3) for \( \theta = 1 \) we get
\[ E(TS) = \frac{1}{\lambda} E(S_2) + [E(S_1) - (S_2)] \left\{ \frac{1 - f^*(\lambda)}{\lambda(1 - h^*(\mu)f^*(\lambda))} + \frac{f''(\lambda)(1 - h^*(\mu))}{1 - h^*(\mu)f^*(\lambda)^2} \right\} \]
\[ Cov(T, S) = [E(S_1) - (S_2)] \left\{ \frac{f''(\lambda)(1 - h^*(\mu))}{(1 - h^*(\mu)f^*(\lambda)^2)} \right\} \quad (5.4.6) \]

We may find that
\[ E(\theta^N) = \frac{1 - f^*(\lambda)}{1 - \theta f^*(\lambda)}, \quad E(N) = \frac{f^*(\lambda)}{1 - f^*(\lambda)} \]

and
\[ Var(N) = [\frac{f^*(\lambda)}{1 - f^*(\lambda)}]^2 \quad (5.4.7) \]
\[ E(TN) = \frac{f^*(\lambda)}{\lambda(1 - f^*(\lambda))} - \frac{f''(\lambda)}{[1 - f^*(\lambda)]^2} \]

and
\[ Cov(T, N) = -\frac{f''(\lambda)}{(1 - f^*(\lambda))} \quad (5.4.8) \]

When \( F(x) \) is exponential distribution with parameter \( a \),
\[ Cov(T, N) = \frac{a}{\lambda^2} \quad (5.4.9) \]
5.5 Model D: General Operation and Random

Markovian Production Time

Assumptions

1. The machine operation time $T$ is a random variable with cumulative distribution function (cdf) $F(x)$ and probability density function (pdf) $f(x)$.

2. The inter-production times of products are independent random variables with exponential distribution with rate $\lambda$. Let $N$ be the number of products produced during the operation time $T$.

3. The sizes of the products produced at production epochs are independent and identically distributed random variable with cdf $H(x)$ and pdf $h(x)$. The sales time $S$ is $S_1$ which has general distribution with cdf $G_1(y)$ and pdf $g_1(x)$ when the total magnitude $Z$ of the products produced is less than the threshold selling capacity $U$ and is $S_2$ which has general distribution cdf $G_2(y)$ and pdf $g_2(x)$ when the total magnitude $Z$ of the products produced is more than the threshold $U$ to provide change in selling rate.

4. Threshold $U$ has exponential distribution with parameter $\mu_i$.

Noting that the number of products produced during a period has Poisson distribution, we may derive the joint distribution of $T, N, Z$ and $S$ as follows. The joint probability
The density function of $T$, $Z$ and $S$ and probability function of $N$ is

$$
\frac{\partial}{\partial x}
\left(\frac{\partial}{\partial y}\left(\frac{\partial}{\partial z}P(T \leq x, N = n, U \leq z, S \leq y)\right)\right) = f(x, n, z, y)
$$

$$
= f(x)\{e^{-\lambda x} (\frac{\lambda x}{n!})^n\} H_n(z) \mu e^{-\mu z} g_1(y)
$$

$$
+ f(x)\{e^{-\lambda x} (\frac{\lambda x}{n!})^n\} [1 - H_n(z)] \mu e^{-\mu z} g_2(y)
$$

(5.5.1)

for $n = 0, 1, 2, 3$ The first term of equation (5.5.1) is the part of the pdf that the operation time is $x$, the sale time is $y$, the number of productions is $n$ and the total production magnitude $Z$ is within the threshold $U$. The second term of equation (5.5.1) is the part of the pdf that the operation time is $x$, the sale time is $y$, the number of productions is $n$ and the total production magnitude $Z$ exceeds the threshold $U$. Let us define the joint Laplace transform cum cum generating function as follows.

$$
E(e^{tT} e^{-\lambda S} e^{-\mu U} (\theta)^N)
$$

$$
= \sum_{n=0}^{\infty} \theta^n \int_0^\infty \int_0^\infty \int_0^\infty f(x, n, u, y) e^{-tx} e^{-sy} e^{-uZ} dx dy dz.
$$

(5.5.2)

$$
= \sum_{n=0}^{\infty} \theta^n \int_0^\infty \int_0^\infty \int_0^\infty f(x)\{e^{-\lambda x} (\frac{\lambda x}{n!})^n\}
$$

$$
\times H_n(z) \mu e^{-\mu z} e^{-tx} e^{-sy} e^{-uZ} g_1(y) dx dy dz
$$

$$
+ \sum_{n=0}^{\infty} \theta^n \int_0^\infty \int_0^\infty \int_0^\infty f(x)\{e^{-\lambda x} (\frac{\lambda x}{n!})^n\}
$$

$$
\times [1 - H_n(z)] \mu e^{-\mu z} e^{-tx} e^{-sy} e^{-uZ} g_2(y) dx dy dz.
$$
\[0 \leq \theta \leq 1, \text{ and } t, s, u \geq 0\]

On simplification it gives

\[
E(e^{-tT} e^{-sS} e^{-uU} f^N) = \frac{f^i}{\mu + u} f^i (t + \lambda (1 - \theta h^i (u + \mu))) g_1^i (s)
\]

\[
+ \frac{f^i}{\mu + u} [f^i (t + \lambda (1 - \theta h^i (u))) - f^i (t + \lambda (1 - \theta h^i (u + \mu)))] g_2^i (s).
\]

(5.5.3)

We may note \(E(e^{-tT}) = f^i (t);\)

\[
E(e^{-sS}) = f^i (\lambda (1 - h^i (\mu))) g_1^i (s) + [1 - f^i (\lambda (1 - h^i (\mu)))] g_2^i (s).
\]

(5.5.4)

Using differentiations and setting \(p = f^i (\lambda (1 - h^i (\mu))),\)

with \(q = 1 - p, \) we get

\[
E(S) = pE(S_1) + qE(S_2),
\]

\[
E(S^2) = pE(S_1^2) + qE(S_2^2),
\]

\[
Var(S) = pVar(S_1) + qVar(S_2) + pq[E(S_1 - S_2)]^2.
\]

(5.5.5)

From equation (5.5.3) for \(\theta = 1\) and \(u = 0\) we get

\[
E(e^{-tT} e^{-sS}) = f^i (t + \lambda (1 - h^i (\mu))) g_1^i (s)
\]

\[
+ [f^i (t) - f^i (t + \lambda (1 - h^i (\mu)))] g_2^i (s).
\]

(5.5.6)

\[
E(TS) = E(T) E(S_2) + f^i (\lambda (1 - h^i (\mu))) [E(S_2) - E(S_1)]
\]

and

\[
Cov(T, S) = [E(S_1) - E(S_2)] [-f^i (\lambda (1 - h^i (\mu))) - pE(T)].
\]

(5.5.7)
We may find that

\[ E(\theta^N) = f^*(\lambda(1 - \theta)), \quad E(N) = \lambda E(T) \]

and

\[ Var(N) = \lambda^2 Var(T) + \lambda E(T). \]

From (5.5.2), we get using differentiation

\[ E(TN) = \lambda E(T^2) \]

and

\[ Cov(T, N) = \lambda V_{aT}(T). \quad (5.5.8) \]

When \( T \) is exponential with parameter \( a \) we get

\[ Cov(T, S) = -[E(S_1) - E(S_2)] \frac{[\lambda(1 - h^*(\mu))]}{[a + \lambda(1 - h^*(\mu))]^2} \text{ and } Cov(T, N) = \frac{\lambda}{a^2}. \quad (5.5.9) \]

### 5.6 Numerical Illustration

Models A and B:

We present the usefulness of the results obtained by presenting numerical examples. We consider numerical examples for two models A and B together. We take the operation time is exponential with parameter 2, the inter production time is exponential with parameter 10, Sales times \( S_1 \) and \( S_2 \) are exponential with parameters 1 and 2. We take the
threshold distribution is Geometric with parameter \( \theta \). Assigning different values for \( \theta \) we find the statistical values as given in the table.

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<th>S.No.</th>
<th>( \hat{\theta} )</th>
<th>( p )</th>
<th>( E(S) )</th>
<th>( Var(S) )</th>
<th>( Cov(T, S) )</th>
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</table>

Fig1.1

The increase in the values of \( \hat{\theta} \) (the decrease in expected threshold size) increases expected sales times, and increases its variances. Since the mean of \( S_1 \) is greater than
the mean of $S_2$, the covariance (correlation) is negative. As $\theta$ increases the co-variance increases indicating the variables are more negatively correlated for large values of $\theta$ and $\text{Cov}(T, S)$ approaches to $-1$.

Models C and D:

We now consider models C and D for illustration. To cover the two models C and D, we consider the case of the operation time, the production time and the size of products produced, the sales time $S_1$ and the sales time $S_2$ have exponential distributions with rate parameter values respectively 2, 10, 2, 1 and 2. We vary the threshold rate parameter $\mu$ and give values $\mu = .1, .15, .2$ and .5. Using equations seen we present the statistical values in the following table. Fixing the threshold parameter $\mu$ and varying any other parameter, we may also study the effect of it on the statistical values.

<table>
<thead>
<tr>
<th>S.No.</th>
<th>$\theta$</th>
<th>$p$</th>
<th>$E(S)$</th>
<th>$Var(S)$</th>
<th>$Cov(T, S)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>.1</td>
<td>.80770</td>
<td>.90385</td>
<td>.94268</td>
<td>-.038832</td>
</tr>
<tr>
<td>2</td>
<td>.15</td>
<td>.74138</td>
<td>.87069</td>
<td>.853969</td>
<td>-.047934</td>
</tr>
<tr>
<td>3</td>
<td>.2</td>
<td>.68750</td>
<td>.84735</td>
<td>.819336</td>
<td>-.053711</td>
</tr>
<tr>
<td>4</td>
<td>.5</td>
<td>.5</td>
<td>.75</td>
<td>.6875</td>
<td>-.0625</td>
</tr>
</tbody>
</table>
The increase in the values of $j_1$ (the decrease in expected threshold size) decreases expected sales times, and decreases its variances. Since the mean of $S_1$ is greater than mean of $S_2$ the covariance (correlation) is negative. As $j_1$ increases the co-variance decreases indicating the variables are more negatively correlated for large values of $j_1$ and $Cov(T, S)$ approaches to $-1$. 

Fig 1.2