6.1 Introduction

In this chapter we extend the dual series equations involving Laguerre Polynomials to the certain space of the generalized functions. In section 6.2, we introduce the dual series equations. In 6.3, following Sneddon [84A] we discuss appearance of the dual series equations in a physical problem. In section 6.4 we mention certain sets of dual series equations which are considered by various authors in the certain space of classical functions. At the end of this chapter in section 6.5 we consider dual series equations involving Laguerre polynomials in the certain space of the generalized functions and then solve it in the distributional sense.

6.2 Dual Series Equations

We know that by a boundary value problem we mean the solution of a differential equation with boundary conditions. When different conditions are imposed on different parts of the boundary we call such boundary value problem a mixed boundary value problem. The mixed boundary value problem can be solved by reducing the
problem to the dual, triple, series and integral equations etc. Therefore we can say that the dual series equations are useful tool for solving the two part mixed boundary value problems which most frequently arise in potential theory and in the theory of elasticity.

In general, mathematically, a pair of equations

\[
\sum_{n=0}^{\infty} A_n G_n P(n, x) = f(x) \quad 0 \leq x < a \\
\sum_{n=0}^{\infty} A_n P(n, x) = g(x) \quad a < x < \infty
\]

where \( G_n \) is a known function of \( n \), \( P(n, x) \) is the given polynomial of order \( n \) and argument \( x \) and sequence \( \{A_n\} \) is to be found are called “Dual series equations” involving a polynomials \( P(n, x) \).

In recent years interest in the study of dual series equations have increased appreciably because of their various applications in mathematical physics and so a large number of papers appear in the literature concerning dual series equations [57] involving various orthogonal polynomials.

6.3 Appearance of Dual Series Equations

Dual series equations arise when boundary conditions of different types are imposed on two parts of one of the boundary of the region concerned, during the solution of mixed boundary value problem.
For instance, if we make use of a series solution of Laplace’s equations we are led to a pair of dual series equations.

Suppose we wish to fixed the asymmetric solution \( V(\rho,z) \) of Laplace’s equations in the semi infinite cylinder \( 0 \leq \rho \leq a, \quad z \geq 0 \) satisfying the boundary conditions.

\[
V(\rho,z) \to 0 \quad \text{as} \quad z \to \infty \tag{6.3.1}
\]

\[
V(a,z) = 0 \quad z = 0 \tag{6.3.2}
\]

\[
V(\rho,0) = f(\rho) \quad 0 \leq \rho \leq 1, \tag{6.3.3}
\]

\[
\left( \frac{\partial V}{\partial z} \right)_{z=0} = 0 \quad 1 < \rho \leq a, \tag{6.3.4}
\]

Then the Harmonic function

\[
v(\rho, z) = \sum_{n=1}^{\infty} \lambda_n^{-1} a_n J_0(\lambda_n \rho) e^{-\lambda_n z}
\]

will satisfy the condition (6.3.2) provided that \( \lambda_i, \quad i=1,2,... \) are the positive zeros of \( J_0(\lambda a) \) the conditions (6.3.3) and (6.3.4) are then equivalent to the pair of equations,

\[
\sum_{n=1}^{\infty} \lambda_n^{-1} a_n J_0(\lambda_n \rho) = f(\rho), \quad 0 \leq \rho \leq 1 \tag{6.3.5a}
\]

\[
\sum_{n=1}^{\infty} a_n J_0(\lambda_n \rho) = 0 \quad 1 < \rho \leq a \tag{6.3.5b}
\]

where \( \{\lambda n\} \) is the sequence of positive zeroes of \( J_0(\lambda a) \).
A pair of equations of this type is called a pair of dual series equations. Thus there are several problems in potential theory [84A] which can be reduced to dual series equations.

6.4 Special Sets of Dual Series Equations

We want to mention here certain sets of dual series equations which appear in the literature.

(i) Dual Series Equations Involving Bessel Functions

The first problem of dual series equations considered was the determination of coefficient $A_n$ in the equations

$$\sum_{n=0}^{\infty} \lambda_n^{-\rho} A_n J_\nu (\lambda_n \rho ) = F (\rho ) \quad 0 \leq \rho \leq 1 \quad (6.4.1a)$$

$$\sum_{n=0}^{\infty} A_n J_\nu (\lambda_n \rho ) = G (\rho ) \quad 1 < \rho \leq a \quad (6.4.1b)$$

where $\rho$ and $\nu$ are real constants satisfying $-1/2 < \rho < 1/2, \nu > 0$ and $\{\lambda_n\}$ are the positive roots of the $J_\nu (a\lambda_n) = 0$. These equations are called Dual Fourier Bessel Series.

The equation (6.4.1) was first considered by Cooke and Tranter [20] for $G(\rho)=0$. They applied the method given by Tranter [96] to solve the dual integral equations. They reduced the problem to a system of algebraic equations which can be easily solved by numerical methods. Later Snedden and Srivastav [86] showed that the same
problem may be reduced to the solution of a single Fredholm integral equations of second kind.

Srivastav[87] introduced the equations

\[ \sum_{n=0}^{\infty} \lambda^2 \rho A_n \psi_n^\nu (\lambda_n \rho) = F (\rho) \quad 0 \leq \rho \leq c \quad (6.4.2a) \]
\[ \sum_{n=0}^{\infty} A_n \psi_n^\nu (\lambda_n \rho) = G (\rho) \quad c < \rho \leq 1 \quad (6.4.2b) \]

where \( \{\lambda_n\} \) is the sequence of positive roots of the transcendental equation

\[ \lambda J_\nu'(\lambda) + H J_\nu(\lambda) = 0 \]

\( H \) and \( \nu \) being the real coefficients with \( \nu \geq -1/2 \) and \( H + \nu \geq 0 \).

Equations (6.3.2) are known as the dual series equations of Dini type.

(ii) Dual Series Equations Involving Jacobi Polynomials

Noble [59] considered the following set of dual series equations:

\[ \sum_{n=0}^{\infty} P_n (\nu, \beta) A_n J_n (\alpha, \beta, x) = f (x) \quad (0 < x \leq a) \quad (6.4.3a) \]
\[ \sum_{n=0}^{\infty} A_n J_n (\alpha, \beta, x) = g (x) \quad (a < x \leq 1) \quad (6.4.3b) \]

where \( J_n (\alpha, \beta, x) \) denotes the Jacobi polynomial and \( P_n (\nu, \beta) \) is a constant defined by

\[ P_n (\nu, \beta) = \frac{\Gamma(n+\nu)\Gamma(1+\alpha+\beta+n)}{\Gamma(1+\alpha+\nu+n)\Gamma(\beta+n)} \]
He solved the above equations by developing multiplying factor method.

Srivastav [88] considered the following set of dual series equations.

\[
\sum_{n=0}^{\infty} \frac{A_n P_n^{(\alpha, \beta)}(\cos \theta)}{\Gamma(\alpha + n + 1) \Gamma(\beta + n + 3/2)} = F(\theta) \quad (0 \leq \theta \leq \phi) \quad (6.4.4a)
\]

\[
\sum_{n=0}^{\infty} \frac{A_n P_n^{(\alpha, \beta)}(\cos \theta)}{\Gamma(\alpha + n + 1/2) \Gamma(\beta + n + 1)} = G(\theta) \quad (\phi \leq \theta \leq \pi) \quad (6.4.4b)
\]

where \( \alpha > -\frac{1}{2}, \beta > -1 \) and \( P_n^{(\alpha, \beta)}(\cos \theta) \) denotes the Jacobi polynomial and obtained closed form solution by using Abel integral equation method.

(iii) Trigonometrical Dual Series Equations

Tranter [93] considered many different kinds of series equations and solve them. After some time Tranter [95] gave an exact solutions of trigonometrical series equations.

Nobel and Whiteman [60,61] also considered other set of trigonometric series equations.

(iv) Dual series equations involving the generalized Laguerre polynomials

Srivastava [89] considered following set of dual Fourier series equations and solved them by applying Sneddon’s [84A] method.
\[ \sum_{n=0}^{\infty} \frac{A_n}{\Gamma(\alpha + n + 1)} L_n^{(\nu)}(x) = f(x) \quad (0 \leq x < y) \quad (6.4.5a) \]

\[ \sum_{n=0}^{\infty} \frac{A_n}{\Gamma(\beta + n + 1)} L_n^{(\nu)}(x) = g(x) \quad (y \leq x < \infty) \quad (6.4.5b) \]

where \( L_n^{(\nu)}(x) \) is the generalized Laguerre Polynomial.

The special case of these equations were considered by Srivastava [91] and Lowndes [52].

(v) Dual Series Equations Involving The Legendre Polynomials

Collins [19] considered the following set of Dual series equations

\[ \sum_{n=0}^{\infty} (1 + H_n) A_n T_{m-n}^{-m} (\cos \theta) = F(\theta) \quad 0 \leq \theta < \phi \quad (6.4.6a) \]

\[ \sum_{n=0}^{\infty} (2n + 2m + 1) A_n T_{m-n}^{-m} (\cos \theta) = G(\theta) \quad \phi < \theta < \pi \quad (6.4.6b) \]

where \( T_{n}^{-m}(\cos \theta) \) is the Legendre Polynomials. He found the solution of the above equation in the form of Fredholm integral equation of second kind.

(vi) Dual Series Equations Involving The Generalized Bateman K-Functions

Srivastava [90] considered the following set of Dual series equations
\[\sum_{n=0}^{\infty} \frac{A_n}{\Gamma(2\beta + \sigma + n + 1)} K_{2(n+\alpha)}^{2(\alpha+\sigma)}(x) = f(x) \quad (0 \leq x < y) \quad (6.4.7a)\]

\[\sum_{n=0}^{\infty} \frac{A_n}{\Gamma(2\nu + \sigma + n + 1)} K_{2(n+\alpha)}^{2(\alpha+\sigma)}(x) = g(x) \quad (y \leq x < \infty) \quad (6.4.7b)\]

where \(K_v^\alpha(x)\) is the generalized Batman K-Functions. Srivastava [91] and Dwivedi [26] obtained the solution of particular case of the above equations.

(vii) Some Dual series equations involving other special functions

Pathak [71] considered the following equations

\[\sum_{n=0}^{\infty} \frac{A_n T^{-n} \rho^n}{\Gamma(v+1/2+n+\rho)} \rho_{n-\rho}, v(x-t) = f(x,t) \quad (0 < x < y) \quad (6.4.8a)\]

\[\sum_{n=0}^{\infty} \frac{A_n T^{-1} \rho^n}{\Gamma(\mu+1/2+n+\rho)} \rho_{n-\rho\sigma}, (x-t) = g(x,t) \quad (y < x < \infty) \quad (6.4.8b)\]

where \(P_n(x,t)\) is a heat polynomial and solution is obtained in closed form.

Patil [75] studied the Dual series equations

\[\sum_{n=0}^{\infty} \frac{A_n}{\Gamma(\delta + 1+kn)} Z_\delta^n(x,k) = f(x) \quad (0 < x < y) \quad (6.4.9a)\]

\[\sum_{n=0}^{\infty} \frac{A_n}{\Gamma(\delta + \beta + kn)} Z_\beta^n(x,k) = g(x) \quad (y < x < \infty) \quad (6.4.9b)\]

where \(Z_\alpha^n(x,k)\) is the Konhouser biorthogonal polynomial.
In paper [25] Dwivedi & Trivedi have considered some dual series equations involving Laguerre & Jacobi polynomials which appear as generalizations of papers of Lowndes [52] and Noble [59].

6.5 On Dual Laguerre Series of Generalized Functions*

In this section, we consider the problem of determining the sequence \( \{A_n\} \), such that

\[
\sum_{n=0}^{\infty} \frac{A_n}{\Gamma(n+\alpha+1)} L_n^{(\alpha)}(x) = f_1(x), \quad 0 < x < y, \quad (6.5.1a)
\]

\[
\sum_{n=0}^{\infty} \frac{A_n}{\Gamma(n+\alpha+\beta)} L_n^{(\alpha)}(x) = f_2(x), \quad y < x < \infty \quad (6.5.1b)
\]

where \( L_n^{(\alpha)}(x) \) is a Laguerre polynomial, \( f_1(x) \) and \( f_2(x) \) are prescribed generalized functions defined in the indicated intervals. In fact these generalized functions belong to the dual space of a subspace of \( L_2 \)-space defined on \( (0, \infty) \). The problem is solved by using extended forms of Riemann-Liouville and Weyl fractional integral operators.

6.5.1 Introduction

Following Zemanian [105, chapter 9], let \( A \) consist of all functions \( \phi(x) \) that possess the following three properties:

---

(i) \( \phi(x) \) is defined, real valued, and smooth on \((0, \infty)\).

(ii) For each \( k \) the quantity:

\[
\alpha_k(\phi) &= \alpha_0(\phi) \left( \int_0^\infty |\eta^k \phi(x)|^2 \, dx \right)^{1/2}
\]

is finite where the operator \( \eta \) is defined by

\[
\eta \phi(x) = x^{-\alpha/2} e^{x/2} \left[ x^{\alpha+1} e^{-x} \frac{d}{dx} \left( x^{-\alpha/2} e^{x/2} \phi(x) \right) \right]
\]

(iii) For each \( n \) and \( k \)

\[
(\eta^k \phi, \psi_n) = (\phi, \eta^k \psi_n)
\]

where \( \psi_n(x) \) are normalized orthonormal Laguerre functions defined by

\[
\psi_n(x) = \left[ \frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)} \right]^{1/2} x^{\alpha/2} e^{-x/2} L_n^{(\alpha)}(x) \quad n=0,1,2\ldots
\]

and \( L_n^{(\alpha)}(x) \) are Laguerre polynomials defined by

\[
L_n^{(\alpha)}(x) = \frac{(1+\alpha)_n}{n!} F_1(-n;1+\alpha;x) = \sum_{k=0}^{n} \frac{(-1)^k (1+\alpha)_n x^k}{k! (n-k)! (1+\alpha)_n}
\]

It is known that the eigen values of the operator \( \eta \) defined above are \( \lambda_n = -n \) for \( n=0,1,2,\ldots \).
Infact in our problem $f_1(x)$ and $f_2(x)$ are assumed to be the members of dual space $A'$ of the space $A$ defined above. In the above book Zemanian has proved that every member of $A'$ can be expanded in the term of $\psi_n(x)$ i.e. in terms of $L_n^{(\alpha)}(x)$.

The problem indicated in the equation(6.5.1) was solved by Srivastava [91] for the special case $\beta=1/2$ and for $f_1$ and $f_2$ to be continuously differentiable function. Srivastava solved the problem by using Abel integral equations and functional integral. Lowndes [52] solved slightly different form of above problem using similar techniques. Several other dual series equations involving various orthogonal polynomials appear in the literature, their reference may be found in above papers.

Following Srivastava [91], we split the problem given above in to two parts:

**Problem (a):** Determine the constants $\{A_n\}$ satisfying the dual series equations

$$\sum_{n=0}^{\infty} \frac{A_n}{\Gamma(n+\alpha+1)} L_n^{(\alpha)}(x) = f_1(x), \quad 0 < x < y,$$

(6.5.6a)

$$\sum_{n=0}^{\infty} \frac{A_n}{\Gamma(n+\alpha+\beta)} L_n^{(\alpha)}(x) = 0, \quad y < x < \infty$$

(6.5.6b)

**Problem (b):** Determine the constants $\{A_n\}$ satisfying the dual series relations
\[
\sum_{n=0}^{\infty} \frac{A_n}{\Gamma(n + \alpha + 1)} L_n^{(\alpha)}(x) = 0, \quad 0 < x < y, \quad (6.5.7a)
\]

\[
\sum_{n=0}^{\infty} \frac{A_n}{\Gamma(n + \alpha + \beta)} L_n^{(\alpha)}(x) = f_2(x), \quad y < x < \infty \quad (6.5.7a)
\]

The solution of the general problem can obviously be obtained merely by adding the solutions of problem (a) and (b) as in Srivastav [91].

**6.5.2 The Results Used in the Analysis**

In the course of analysis we shall use the following results.

From Sneddon [84,p.271] we know that Riemann-Liouville operator \( R_\beta \) and the weyl operator \( W_\beta \) are defined as

\[
R_\beta \{ f(u); x \} = \frac{1}{\Gamma(\beta)} \int_0^x f(u)(x-u)^{\beta-1} du, \quad \text{Re} \beta > 0 \quad (6.5.8)
\]

\[
W_\beta \{ f(u); x \} = \frac{1}{\Gamma(\beta)} \int_x^\infty f(u)(u-x)^{\beta-1} du, \quad \text{Re} \beta > 0 \quad (6.5.9)
\]

If \( f \in A' \) then these operators may be generalized as

\[
R_\beta \{ f(u); x \} = \langle H(x-u) f(u), \frac{(x-u)^{\beta-1}}{\Gamma(\beta)} \rangle > (6.5.10)
\]

\[
W_\beta \{ f(u); x \} = \langle H(u-x) f(u), \frac{(u-x)^{\beta-1}}{\Gamma(\beta)} \rangle > (6.5.11)
\]

Following Gel’fand and Shilov [35,p.115] we may write that
\[ R_\beta \{ f(u); x \} = f(x) \ast \frac{x^{\beta-1}}{\Gamma \beta}, \quad (6.5.12) \]

where \( f(x) \ast g(x) \) stands for convolution of generalized functions \( f(x) \) and \( g(x) \). The generalized function \( x^\lambda \) means

\[
x^\lambda_x = \begin{cases} 
  x^\lambda & \text{if } x > 0, \\
  0 & \text{if } x \leq 0.
\end{cases}
\]

Using Gel’fand and Shilov [35,p.117] we have that if

\[ g(x) = R_\beta \{ f(u); x \} = f(x) \ast \frac{x^{\beta-1}}{\Gamma \beta}, \quad (6.5.13) \]

then

\[ f(x) = g(x) \ast \frac{x^{-\beta-1}}{\Gamma \beta} = R_{-\beta} \{ g(u); x \}, \quad (6.5.14) \]

Similarly

\[ g(x) = W_\beta \{ f(u); x \} \quad (6.5.15) \]

then

\[ f(x) = -\frac{d}{dx} W_{1-\beta} \{ g(u); x \} \quad (6.5.16) \]

Further we will be using the following results given in Lowndes[52]

\[ R_\beta \left[ x^\alpha L_n^{(\alpha)}(y); x \right] = \frac{\Gamma(\alpha + n + 1)}{\Gamma(\alpha + \beta + n + 1)} y^{\alpha+\beta} L_n^{(\alpha+\beta)}(y), \quad (6.5.17) \]

\[ W_{1-\beta} \left[ e^{-x} L_n^{(\alpha)}(y); x \right] = e^{-y} L_n^{(\alpha+\beta-1)}(y) \quad (6.5.18) \]
The orthogonality relation for the Laguerre polynomials is

\[
\int_0^\infty x^\alpha e^{-x} L_n^{(\alpha)}(x)L_m^{(\alpha)}(x)dx = \frac{\Gamma(\alpha + n + 1)}{\Gamma(n + 1)} \delta_{mn} \quad \alpha > -1 \tag{6.5.19}
\]

We also have the relations

\[
\frac{d}{dx}\{x^\alpha L_n^{(\alpha)}(x)\} = (n + \alpha)x^{\alpha-1}L_n^{(\alpha-1)}(x), \tag{6.5.20}
\]

\[
S_1(x, y) = \sum_{n=0}^\infty \frac{\Gamma(\alpha + n + 1)}{\Gamma(n + 1)} L_n^{(\alpha)}(x)L_n^{(\alpha+\beta-1)}(y) \tag{6.5.21}
\]

\[
= \frac{e^y x^{-\alpha} (x - y)^{-\beta}}{\Gamma(1 - \beta)} H(y)
\]

\[
S_2(x, y) = \sum_{n=0}^\infty \frac{\Gamma(n + 1)}{\Gamma(\alpha + \beta + n)} L_n^{(\alpha-1)}(x)L_n^{(\alpha+\beta-1)}(y) \tag{6.5.22}
\]

\[
= \frac{e^y (y - x)^{-\beta} y^{1-\alpha-\beta}}{\Gamma \beta} H(y - x)
\]

\[
\int_x^\infty e^{-y} L_n^{(\alpha)}(y)dy = e^{-x} L_n^{(\alpha-1)}(y) \tag{6.5.23}
\]

In the above relations \(H(x)\) denotes the Heaviside unit function.

\textbf{6.5.3 Problem (a)}

Let us suppose that for \(0 < x < y\)

\[
\sum_{n=0}^\infty \frac{A_n}{\Gamma(n + \alpha + \beta)} L_n^{(\alpha)}(x) = -e^x \frac{d}{dx} W_\beta\{H(y-u)g_1(u);x\} \tag{6.5.24}
\]

Using the orthogonality property (6.5.19), it can be shown that

\[
A_n = \frac{\Gamma(n+1)\Gamma(n+\alpha+\beta)}{\Gamma(n+\alpha+1)} < -e^x \frac{d}{dx} W_\beta\{H(y-u)g_1(u);x\}, e^{-x}x^\alpha L_n^{(\alpha)}(x) > \tag{6.5.25}
\]
\[
\frac{\Gamma(n+1) \Gamma(n+\alpha+\beta)}{\Gamma(n+\alpha+1)} <W_\beta \{ H(y-u)g_1(u); x \}, \frac{d}{dx} x^\alpha L_n^{(\alpha)}(x) >
\]

\[
\frac{\Gamma(n+1) \Gamma(n+\alpha+\beta)}{\Gamma(n+\alpha+1)} <W_\beta \{ H(y-u)g_1(u); x \}, (n+\alpha)x^{\alpha-1} L_n^{(\alpha-1)}(x) >
\]

(\text{using} \ 6.5.20)

\[
\frac{\Gamma(n+1) \Gamma(n+\alpha+\beta)}{\Gamma(n+\alpha)} <H(y-u)g_1(u), R_\beta \{ x^{\alpha-1} L_n^{(\alpha-1)}(x), u \} >
\]

(\text{Since} \ W_\beta \text{ and } R_\beta \text{ are adjoint operators})

\[
\frac{\Gamma(n+1) \Gamma(n+\alpha+\beta)}{\Gamma(n+\alpha)} <H(y-u)g_1(u), \frac{\Gamma(n+\alpha)}{\Gamma(n+\alpha+\beta)} u^{\alpha+\beta-1} L_n^{(\alpha+\beta-1)}(u) >
\]

(\text{using} \ 6.5.17)

Hence,

\[
A_n = \Gamma(n+1) <H(y-u)g_1(u), u^{\alpha+\beta-1} L_n^{(\alpha+\beta-1)}(u) > \quad (6.5.25)
\]

Substituting the coefficients \( A_n \) from (6.5.25) into (6.5.6a) we get

\[
\sum_{n=0}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)} L_n^{(\alpha)}(x) <H(y-u)g_1(u), u^{\alpha+\beta-1} L_n^{(\alpha+\beta-1)}(u) >= f_1(x),
\]

i.e.

\[
<H(y-u)g_1(u), u^{\alpha+\beta-1} \sum_{n=0}^{\infty} \frac{\Gamma(n+1) L_n^{(\alpha)}(x)L_n^{(\alpha+\beta-1)}(u)}{\Gamma(n+\alpha+1)} >= f_1(x),
\]

i.e.

\[
<H(y-u)g_1(u), u^{\alpha+\beta-1} \frac{e^{x-n} (x-u)^{\beta}}{\Gamma(1-\beta)} H(x-u) >= f_1(x),
\]

(\text{using the relation} \ 6.5.21)

i.e.

\[
<H(y-u)g_1(u), u^{\alpha+\beta-1} \frac{e^{x-n} (x-u)^{\beta}}{\Gamma(1-\beta)} H(x-u) >= x^{\alpha} f_1(x),
\]
\[ <H(y-u)g_1(u), u^{\alpha+\beta-1} \frac{(x-u)^{-\beta}}{\Gamma(1-\beta)} >= x^\alpha f_1(x) \]

Hence we get on using (6.5.10)

\[ R_{1-\beta} \{ g_1(u), u^{\alpha+\beta+1} e^u \}; x \} = x^\alpha f_1(x), \]

i.e.

\[ (g_1(x), x^{\alpha+\beta+1} e^x) \ast \frac{x^{1-\beta-1}}{\Gamma(1-\beta)} = x^\alpha f_1(x), \]

Hence

\[ x^{\alpha+\beta+1} e^x g_1(x) = x^\alpha f_1(x) \ast \frac{x^{\beta-2}}{\Gamma(\beta-1)} \] (6.5.26)

The coefficients \( A_n \) may now be obtained by using the relations (6.5.25) and (6.5.26). For \( \beta=1/2 \) and for \( x^\alpha f_1(x) \) finite and continuously differentiable function the relation (6.5.25) and (6.5.26) reduce to relations of Srivastava [91].

6.5.4. Problem (b)

Let us suppose that for \( y < x < \infty \)

\[ \sum_{n=0}^{\infty} \frac{A_n}{\Gamma(n + \alpha + 1)} L_n^{(\alpha)}(x) = x^{-\alpha} R_{1-\beta} \{ H(u-y)g_2(u); x \} \] (6.5.27)

Using the orthogonality relation again we get

\[ A_n = \Gamma(n+1) < x^{-\alpha} R_{1-\beta} \{ H(u-y)g_2(u); x \}, e^{-x} x^\alpha L_n^{(\alpha)}(x) > \]

\[ = \Gamma(n+1) R_{1-\beta} \{ H(u-y)g_2(u); x \}, e^{-x} L_n^{(\alpha)}(x) > \]

\[ = \Gamma(n+1) < H(u-y)g_2(u), W_{1-\beta} \{ e^{-x} L_n^{(\alpha)}(x); u \} > \]
Hence
\[ A_n = \Gamma(n+1) < H(u-y)g_2(u), e^{-u}L_n^{\alpha+\beta-1}(u) > \]  
(6.5.28)  

(using 6.5.18)

Operating the functionals on both sides of (6.5.7b) to the function \( H(u-x)e^{-u} \) we get
\[ \sum_{n=0}^{\infty} \frac{A_n}{\Gamma(n+\alpha+\beta)} L_n^{(\alpha)}(x) = f_2(u), H(u-x)e^{-u} > \]

On using (6.5.23) we obtain
\[ \sum_{n=0}^{\infty} \frac{A_n}{\Gamma(n+\alpha+\beta)} e^{-x} L_n^{(\alpha-1)}(x) = e^x < f_2(u), H(u-x)e^{-u} > \]

i.e.
\[ \sum_{n=0}^{\infty} \frac{A_n}{\Gamma(n+\alpha+\beta)} L_n^{(\alpha-1)}(x) = e^x < f_2(u), H(u-x)e^{-u} > \]

Substituting the values of \( A_n \) from (6.5.28) into the last relation we get
\[ \sum_{n=0}^{\infty} \frac{\Gamma(n+1)L_n^{(\alpha-1)}(x)}{\Gamma(n+\alpha+\beta)} < H(u-y)g_2(u), e^{-u}L_n^{(\alpha+\beta-1)}(u) > = e^x < f_2(u), H(u-x)e^{-u} > \]

i.e.  
\[ < H(u-y)e^{-u}g_2(u), \sum_{n=0}^{\infty} \frac{\Gamma(n+1)L_n^{(\alpha-1)}(x)L_n^{(\alpha+\beta-1)}(u)}{\Gamma(n+\alpha+\beta)} > = e^x < f_2(u), H(u-x)e^{-u} > \]

i.e.  
\[ < H(u-y)e^{-u}g_2(u), \frac{e^x(u-x)^{\beta-1}u^{1-\alpha-\beta}}{\Gamma(\beta)} H(u-x) > = e^x < f_2(u), H(u-x)e^{-u} > \]

(by using 6.5.23)

i.e.  
\[ < H(u-x)e^{-u}u^{1-\alpha-\beta}g_2(u), \frac{(u-x)^{\beta-1}}{\Gamma(\beta)} > = f_2(u), H(u-x)e^{-u} > \]
Hence we get
\[
W_\beta \{g_2(u), e^{-u^{1-\alpha-\beta}}; x\} = <f_2(u), H(u-x)e^{-u}>
\]
which by inversion gives
\[
g_2(x)e^{-x}x^{1-\alpha-\beta} = -\frac{d}{dx} W_{t-\beta} \{<f_2(t), H(t-y)e^{-t}>; x\} \tag{6.5.29}
\]

The coefficients $A_n$ may now be obtained by the relations (6.5.28) and (6.5.29). Again it can be verified that for $\beta=1/2$ and for suitable restricted $f_2$ these relations reduce to the relations (6.5.2) and (6.5.8) of Srivastava [91].

*****