4.1 Introduction

Several skewed distributions such as logistic, Weibull, gamma and beta distributions have been used for modelling various random phenomena which arise in engineering, time series modelling, reliability assessment studies etc. Mathai (2005) introduced a real matrix variate probability model which cover almost all real matrix-variate densities used in multivariate statistical analysis. Through the new density introduced, a pathway is created from matrix-variate type I to matrix-variate type II beta to matrix-variate gamma to matrix-variate Gaussian densities. In this chapter we consider the scalar version of the matrix-variate gamma distribution and study the various properties. Recently several $q$-type distributions such as $q$-exponential, $q$-Weibull, and $q$-logistic and various pathway models are

Marshall and Olkin (1997) introduced a method of construction of a family of distributions by adding a new parameter to an existing distribution which resulted in added flexibility of the distribution. Alice and Jose (2003, 2004, 2005a, 2005b), conducted detailed studies of various Marshall-Olkin distributions. Ristic et al. (2007) introduced and studied the various properties of Marshall-Olkin gamma distribution. Jose, Naik and Ristic (2008) introduced Marshall-Olkin $q$-Weibull distribution and also developed time series models having this distribution as marginals. Different first order autoregressive (AR(1)) time series models can be developed using the Marshall-Olkin distribution which are very useful in various fields like time series modelling, reliability studies etc. Here we also extend the $q$-gamma distribution to obtain a more general class called Marshall-Olkin $q$-gamma distribution.

In this chapter we give a detailed study of $q$-gamma distribution and its properties. The estimation of the parameters is discussed and a theorem which is useful in stress-strength analysis is established. The new distribution is found to be a better fit than the standard gamma distribution using a real data set. We introduce the Marshall-Olkin $q$-gamma distribution and develop AR(1) minification model and max-min process. Also the sample path properties are studied.

4.2 The $q$-analogue of the Gamma Distribution

Mathai (2005) introduced a pathway model connecting matrix variate gamma distribution and normal densities. The following is the scalar version of the matrix-variate density introduced by Mathai (2005). When the pathway parameter $q > 1$:

$$f_1(x) = c_1 |x|^\alpha^{-1} \left[ 1 + a(q - 1) |x|^{\delta} \right]^{-\beta/q}; \quad -\infty < x < \infty, a > 0, \beta > 0, \delta > 0, q > 1$$
CHAPTER 4. ON \( q \)-GAMMA DISTRIBUTIONS, MARSHALL-OLKIN \( q \)-GAMMA DISTRIBUTIONS AND MINIFICATION PROCESSES

where

\[
c_1 = \frac{\delta [a(q - 1)]^{\frac{\beta}{q - 1}} \Gamma \left( \frac{\beta}{q - 1} \right)}{2 \Gamma \left( \frac{\alpha}{\delta} \right) \Gamma \left( \frac{\beta}{q - 1} - \frac{\alpha}{\delta} \right)}, \quad \Re \left( \frac{\beta}{q - 1} > \frac{\alpha}{\delta} \right), \Re(\alpha) > 0
\]

where \( \Re(\cdot) \) denotes the real part of \( \cdot \). Since in most statistical problems the parameters are real, we will consider all parameters to be real. Note that for \( q < 1 \), writing \( q - 1 = -(1 - q) \), the density in (4.2.1) reduces to the following form:

\[
f_2(x) = c_2 | x |^{\alpha - 1} \left[ 1 - a(1 - q) \right] | x |^{\delta} \frac{a}{\delta}; \quad q < 1, a > 0, \beta > 0, \delta > 0, 1 - a(1 - q) | x |^{\delta} > 0
\]

(4.2.2)

where the normalizing constant \( c_2 \) is given by

\[
c_2 = \frac{\delta (a(1 - q))^{\frac{\beta}{q - 1}} \Gamma \left( \frac{\beta}{q - 1} + \frac{\alpha}{\delta} + 1 \right)}{2 \Gamma \left( \frac{\alpha}{\delta} \right) \Gamma \left( \frac{\beta}{q - 1} + 1 \right)}, \Re(\alpha) > 0.
\]

As \( q \to 1 \), \( f_1(x) \) and \( f_2(x) \) tend to \( f_3(x) \), which is the generalized gamma density. \( f_3(x) \) is given by

\[
f_3(x) = \delta (a\beta)^{\frac{\alpha}{\delta}} \frac{a}{\delta}, \quad | x |^{\alpha - 1} \exp(-a\beta | x |^{\delta}); \quad -\infty < x < \infty; a, \alpha, \beta, \delta > 0.
\]

(4.2.3)

For different values of the parameters in (4.2.1) and (4.2.2), we get different distributions like beta type-1, beta type-2, etc. More results are available in Mathai and Haubold (2007) and Mathai and Provost (2006). Here we introduce a \( q \)-analogue of the gamma distribution as a special case of Mathai (2005), by taking \( \delta = 1 \), \( a = 1 \) in \( f_1(x) \). Then for \( x > 0 \) and for \( q > 1 \), \( f_1(x) \) reduces to the following density which we call the \( q \)-gamma distribution:

\[
f_4(x) = k_1 x^{\alpha - 1} \left[ 1 + (q - 1)x \right]^{-\frac{\beta}{\delta - 1}}
\]

(4.2.4)

where the normalizing constant \( k_1 \) is given by
Table 4.1: The q-Gamma distribution characteristics when \( q > 1 \)

<table>
<thead>
<tr>
<th>Characteristics</th>
<th>Functional form</th>
</tr>
</thead>
<tbody>
<tr>
<td>p.d.f ( f_4(x) )</td>
<td>[ f_4(x) = \frac{(q-1)^{\alpha} \Gamma \left( \frac{\beta}{q-1} \right)}{\Gamma \alpha \left( \frac{\beta}{q-1} - \alpha \right)} x^{\alpha-1} \left[ 1 + \left( \frac{q-1}{x} \right) \right]^{-\frac{\beta}{q-1}}; x &gt; 0, \alpha, \beta &gt; 0, \frac{\beta}{q-1} &gt; \alpha. ]</td>
</tr>
<tr>
<td>CDF ( F_4(x) )</td>
<td>[ F_4(x) = \frac{B(x (q-1), \frac{\beta}{q-1} - \alpha)}{B(x, \frac{\beta}{q-1} - \alpha)} ]</td>
</tr>
<tr>
<td>HRF ( h_4(x) )</td>
<td>[ h_4(x) = \frac{(q-1)^{\alpha} x^{\alpha-1} \left[ 1 + \left( \frac{q-1}{x} \right) \right]^{-\frac{\beta}{q-1}}}{B(x, \frac{\beta}{q-1} - \alpha) - B(x (q-1), \frac{\beta}{q-1} - \alpha)} ]</td>
</tr>
<tr>
<td>CHR ( H_4(x) )</td>
<td>[ H_4(x) = \frac{\beta x (q-1), \frac{\beta}{q-1} - \alpha}}{B(x, \frac{\beta}{q-1} - \alpha) - \beta (x (q-1), \frac{\beta}{q-1} - \alpha)} ]</td>
</tr>
<tr>
<td>( s^{th} ) moment ( E(x^s) )</td>
<td>[ E(x^s) = \frac{\Gamma(a + s) \Gamma \left( \frac{\beta}{q-1} - \alpha - s \right)}{\Gamma(a) \Gamma \left( \frac{\beta}{q-1} - \alpha \right)} (q-1)^s. ]</td>
</tr>
<tr>
<td>mean value ( E(x) )</td>
<td>[ E(x) = \frac{\alpha^\beta}{\beta - (\alpha - 1)(q-1)} ]</td>
</tr>
</tbody>
</table>

\[ k_1 = \frac{(q-1)^{\alpha} \Gamma \left( \frac{\beta}{q-1} \right)}{\Gamma \alpha \left( \frac{\beta}{q-1} - \alpha \right)} \] (4.2.5)

For \( x > 0 \) and \( q < 1 \), \( f_2(x) \) reduces to the following density function of the q-gamma distribution:

\[ f_5(x) = k_2 x^{\alpha-1} \left[ 1 - (1 - q) x \right]^{\frac{\beta}{1-q}} \] (4.2.6)

where \( k_2 \) is given by

\[ k_2 = \frac{(1-q)^{\alpha} \Gamma \left( \frac{\beta}{1-q} + \alpha + 1 \right)}{\Gamma \alpha \Gamma \left( \frac{\beta}{1-q} + 1 \right)} \] (4.2.7)

for \( 0 \leq x \leq \frac{1}{(1-q)}, \alpha > 0 \).

The q-gamma distributions exhibits various shapes as can be seen from the figures 4.1 and 4.2. and hence it can be applied for modelling different types of data from various fields.

The graph of the hazard rate function of MO-q-G is flexible and by giving various values for the pathway parameter \( q \) we can get different shapes which makes it applicable in a variety of situations (see figure 4.3).
Table 4.2: The \(q\)-Gamma distribution characteristics when \(q < 1\)

<table>
<thead>
<tr>
<th>Characteristics</th>
<th>Functional form</th>
</tr>
</thead>
<tbody>
<tr>
<td>p.d.f (f_4(x))</td>
<td>(f_4(x) = \frac{(1-q)^\alpha \Gamma(\frac{\beta}{1-q} + \alpha + 1)}{\Gamma(\alpha) \Gamma(\frac{\beta}{1-q} + \alpha)} x^{\alpha - 1} [1 - (1 - q)x]^{-\frac{\beta}{1-q}})</td>
</tr>
<tr>
<td>CDF (F_4(x))</td>
<td>(F_4(x) = \frac{B(x(1-q), \alpha, \frac{\beta}{1-q} - \alpha)}{B(\alpha, \frac{\beta}{1-q} - \alpha)})</td>
</tr>
<tr>
<td>HRF (h_4(x))</td>
<td>(h_4(x) = \frac{(1-q)^{\alpha - 1} [1 - (1-q)x]^{-\frac{\beta}{1-q}}}{B(\alpha, \frac{\beta}{1-q} - \alpha) - B(x(1-q), \alpha, \frac{\beta}{1-q} - \alpha)})</td>
</tr>
<tr>
<td>CHR (H_4(x))</td>
<td>(H_4(x) = \frac{B(x(1-q), \alpha, \frac{\beta}{1-q} - \alpha)}{B(\alpha, \frac{\beta}{1-q} - \alpha) - B(x(1-q), \alpha, \frac{\beta}{1-q} - \alpha)} - \frac{B(x(1-q), \alpha, \frac{\beta}{1-q} - \alpha)}{B(\alpha, \frac{\beta}{1-q} - \alpha) - B(x(1-q), \alpha, \frac{\beta}{1-q} - \alpha)})</td>
</tr>
<tr>
<td>(s^{th}) moment (E(x^s))</td>
<td>(E(x^s) = \frac{\Gamma(\alpha + s) \Gamma(\frac{\beta}{1-q} + \alpha + 1)}{\Gamma(\alpha) \Gamma(\frac{\beta}{1-q} + \alpha + s + 1) (1-q)^{\alpha + s}})</td>
</tr>
<tr>
<td>mean value (E(x))</td>
<td>(E(x) = \frac{\alpha}{\beta + (1-q)(\alpha + 1)})</td>
</tr>
</tbody>
</table>

Figure 4.1: The \(q\)-gamma pdf for \(q > 1\)
CHAPTER 4. ON \textit{q}-\textit{gamma} DISTRIBUTIONS, MARSHALL-OLKIN \textit{q}-\textit{gamma} DISTRIBUTIONS AND MINIFICATION PROCESSES

Figure 4.2: The \textit{q}-gamma pdf for \textit{q} < 1

Figure 4.3: Hazard rate function for various values of \textit{q}
4.3 Estimation

(i) First we consider method of moments. Since the distribution has explicit expressions for the moments we can get the values of the parameters by equating sample and population moments. We kept $\beta = 1$ and estimated the values of the parameters for the data given in Gross and Clark (1975) (Data given in chapter 3). The estimated densities are given and the graph shows that $q$-gamma better fits the data. The estimates were obtained as $\alpha = 4.0127$ and $q = 0.7782$. The estimated densities are given in figure (4.4).

(ii) Now we consider maximum likelihood estimation for the parameters. When $q > 1$ the likelihood function is

$$L = \frac{(q - 1)^{n\alpha}((\Gamma(\beta/(q - 1)))^n}{\Gamma(\alpha)^n(\Gamma(\beta/(q - 1) - \alpha))^n} \prod x_i^{\alpha - 1}[1 + (q - 1)x_i]^{-\beta/(q - 1)}$$

Figure 4.4: Estimated densities of gamma and q-gamma.
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\[ \log L = n\alpha \log(q - 1) + n \log(\Gamma(\beta/(q - 1))) + (\alpha - 1) \sum (\log((x_i) - \beta/(q - 1))) \]
\[ \times \sum (\log((1 + (q - 1)x_i))) - n \log(\Gamma(\alpha)) - n \log(\Gamma(\beta/(q - 1) - \alpha)) \]
\[ \frac{\partial \log L}{\partial \alpha} = n \log(q - 1) + \sum (\log(y)) - n\psi(\alpha) + n\psi(\beta/(q - 1) - \alpha) \]
\[ \frac{\partial \log L}{\partial \beta} = \frac{n\psi(\beta/(q - 1))}{q - 1} - \frac{1}{q - 1} \sum (\log(1 + (q - 1)y)) \]
\[ - \frac{n}{q - 1}\psi(\beta/(q - 1) - \alpha) \]
\[ \frac{\partial \log L}{\partial q} = \frac{n\alpha}{q - 1} - \frac{n}{(q - 1)^2}\psi(\beta/(q - 1)) \]
\[ \frac{1}{q - 1} \sum (\log(1 + (q - 1)y)) \]
\[ - \frac{n}{q - 1}\psi(\beta/(q - 1) - \alpha). \]

The function nlm in R package can be used to solve these normal equations. The case \( q < 1 \) can be considered similarly.

4.4 Application in Stress-Strength Analysis

Stress-Strength analysis is concerned with the probability of failure which is the probability of stress exceeding strength. Let \( X \) represent the stress and \( Y \) represent the strength of the system. Then \( R = P(X < Y) \) is the probability that the system will function. More details on stress-strength analysis can be found in Kotz et al. (2007).

\( H \)-function is defined as follows:

\[ H^{m,n}_{p,q}[z | (a_1, a_1), (a_2, a_2), \ldots, (a_k, a_k)] = \frac{1}{2\pi i} \int_L h(s)z^{-s}ds \]

where \( L \) is a suitable path.

\[ h(s) = \frac{\left\{ \prod_{j=1}^{m} \Gamma(b_j + \beta_j s) \right\} \left\{ \prod_{j=1}^{n} \Gamma(1 - a_j - \alpha_j s) \right\}}{\left\{ \prod_{j=m+1}^{q} \Gamma(1 - b_j - \beta_j s) \right\} \left\{ \prod_{j=n+1}^{p} \Gamma(a_j + \alpha_j s) \right\}} \]

and \( L \) is a suitable path.

Here \( a_1, a_2, \ldots, a_p; b_1, b_2, \ldots, b_q \) are complex numbers and \( \alpha_1, \alpha_2, \ldots, \alpha_p, \beta_1, \beta_2, \ldots, \beta_q \)

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are positive real numbers. The poles of $\Gamma(b_j + \beta_j s)$, $j = 1, 2, \ldots, m$ are at the points $s = -\frac{b_j + \nu}{\beta_j}$ where $j = 1, 2, \ldots, m, \nu = 0, 1, \ldots$ and the poles of $\Gamma(1 - a_j - \alpha_j s)$, $j = 1, 2, \ldots, n$ are at $s = -\frac{1 - a_k + \lambda}{\alpha_k}$ where $k = 1, 2, \ldots, n, \lambda = 0, 1, \ldots$. The details of the existence conditions, various properties and applications of $H$-functions are available in Mathai (1993).

Now we state a theorem which is useful in the context of stress-strength analysis of a multi component system subjected to a number of independent stresses $X_1, X_2, \ldots, X_k$, whose cumulative effect is the actual effective stress against a single strength $Y$. Then $R = P(\sum_{i=1}^{k} X_i < Y)$ is the probability that the system will function and it is called the reliability of the system.

**Theorem 4.4.1.** Let $\{X_i, i = 1, 2, \ldots, k\}$ be i.i.d. random variables having a $q$-Gamma distribution function and $Y$ be distributed as exponential distribution with parameter $\theta$. Then $P(\sum_{i=1}^{k} X_i < Y)$ is an $H$-function.

**Proof:** Let $X_1, X_2, \ldots, X_k$ be continuous i.i.d. $q$-Gamma r.v.s and let $Y$ follow the exponential distribution with parameter $\theta$. Let $g_1(\cdot)$ and $g_2(\cdot)$ be the probability density functions of the $q$-Gamma distribution for $q < 1$ and for $q > 1$. Let $g_k(\cdot)$ and $G_k(\cdot)$ be the probability density function and cumulative distribution function of the random variable $\sum_{i=1}^{k} X_i$. Let $g_1^*(\cdot)$ and $g_2^*(\cdot)$ be the corresponding Laplace transforms of the $q$-Gamma distribution for $q > 1$ and for $q < 1$ respectively. Then

$$P(\sum_{i=1}^{k} X_i < Y) = \theta \int_{0}^{\infty} G_k(x)e^{-\theta x}dx = \begin{cases} \{g_1^*(\theta)\}^k & \text{for } q > 1 \\ \{g_2^*(\theta)\}^k & \text{for } q < 1. \end{cases}$$

Now for $q > 1$ we have to evaluate the integral

$$I_1^*(\alpha, \lambda, q : \theta) = \int_{0}^{\infty} e^{-\theta x} x^{\alpha - 1}[1 + (q - 1)x]^{-\frac{\alpha}{q-1}}dx.$$ 

The integrand can be taken as a product of two integrable functions. Let $x_1$ and $x_2$ be two
independently distributed real scalar random variables with probability density functions $f_1(x)$ and $f_2(x)$. Consider the transformation $u = \frac{x_1}{x_2}$ and $v = x_2 \Rightarrow dx_1 \wedge dx_2 = v \, du \wedge dv$ where $\wedge$ is the wedge product discussed in (Mathai 1997). Then the joint density of $u$ and $v$ is $g(u, v) = v f_1(uv) f_2(v)$. Now $g_1(u)$ is obtained by integrating out the joint probability density function $g(u, v)$ with respect to $v$. That is $g_1(u) = \int v f_1(uv) f_2(v) \, dv$. Let $f_1(x_1) = c_1 \, e^{-\theta x_1}, x_1 \geq 0$ and $f_2(x_2) = c_2 x_2^{-\alpha-1} \left[1 + (q-1)x_2\right]^{-\beta/q-1}, x_2 \geq 0, \alpha, \lambda > 0$ where $c_1$ and $c_2$ are normalizing constants.

These constants can be obtained by integrating $f_1(x)$ and $f_2(x)$ with respect to $x$. Thus we have

$$g_1(u) = c_1 c_2 \int_0^{\infty} e^{-\theta uv} u^{\alpha-1} \left\{1 + (q-1)v\right\}^{-\beta/q-1} \, dv. \quad (4.4.1)$$

Now $E(x_1^{s-1}) = c_1 \int_0^{\infty} x_1^{s-1} e^{-\theta x_1} \, dx_1 = \theta^{-s} \Gamma(s), \Re(s) > 0$

$$E(x_2^{-s}) = c_2 \int_0^{\infty} x_2^{\alpha-s} \frac{1 + (q-1)x_2}{\left[1 + (q-1)x_2\right]^{-\beta/q-1}} \, dx_2$$

$$= \frac{c_2}{(q-1)^{\alpha-s+1}} \int_0^{\infty} \omega^{\alpha-s} \left(1 + \omega\right)^{-\beta/q-1} \, d\omega$$

$$= \frac{c_2}{(q-1)^{\alpha-s+1}} \frac{\Gamma(\alpha-s+1) \, \Gamma\left(\frac{\beta}{q-1} - \alpha + s - 1\right)}{\Gamma\left(\frac{\beta}{q-1}\right)},$$

for $\Re(1 - s/\alpha) > 0, \Re\left(\frac{1}{q-1} + \frac{s}{\alpha} - 1\right) > 0$.

Then $E(u)^{s-1} = E(x_1)^{s-1} E(x_2)^{1-s}$

$$= c_1 c_2 \frac{\theta^{-s}}{(q-1)^{\alpha-s+1}} \frac{\Gamma(s) \, \Gamma(1 + \alpha - s) \, \Gamma\left(\frac{\beta}{q-1} - \alpha + s - 1\right)}{\Gamma\left(\frac{\beta}{q-1}\right)}, q > 1$$

for $\Re(s) > 0, \Re(\alpha - s) > 0, \Re\left(\frac{\beta}{q-1} - \alpha + s - 1\right) > 0$.

Now the density of $u$ is obtained by the inverse Mellin transform. The detailed existence conditions for Mellin and inverse Mellin transforms are available in Mathai (1993).
Thus
\[
g_1(u) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} u^{-s} E(u)^{s-1} \frac{c_1 c_2}{(q - 1)^{\alpha + 1} \Gamma\left(\frac{\beta}{q-1}\right)} \times \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(s) \Gamma(1 + \alpha - s) \Gamma\left(\frac{\beta}{q-1} - \alpha + s - 1\right) \left(\frac{\theta u}{q-1}\right)^{-s} ds
\]
for \(\Re(s) > 0, \Re(\alpha - s) > 0, \Re\left(\frac{\beta}{q-1} - \alpha - s\right) > 0, q > 1\).

\[
I^*_1(\alpha, \beta, q : \theta) = \frac{1}{\alpha \Gamma\left(\frac{\beta}{q-1} - \alpha\right)} H^{2.1}_{1,2} \left[\frac{\theta}{(q-1)}\right]^{(0,\alpha)}_{\left(0, \frac{\beta-q+1}{q-1}, \alpha\right)}, q > 1.
\]

On substituting this we get,
\[
P\left(\sum_{i=1}^{k} X_i < Y\right) = \left\{\frac{q - 1}{\Gamma\left(\frac{\beta}{q-1} - \alpha\right)} H^{2.1}_{1,2} \left[\frac{\theta}{(q-1)}\right]^{(0,\alpha)}_{\left(0, \frac{\beta-q+1}{q-1}, \alpha\right)}\right\}^k, q > 1.
\]

Now for \(q < 1\), we have to evaluate the integral
\[
I^*_2(\alpha, \beta, q : \theta) = \int_0^{\infty} e^{-\theta x} x^{\alpha-1} \left[1 - (1 - q)x\right]^{\frac{\beta}{q-1}} dx.
\]
Let
\[
f_1(x_1) = c_1 e^{-\theta x_1}, x_1 \geq 0
\]
and
\[
f_2(x_2) = \begin{cases} c_2 x_2^{\alpha-1} \left[1 - (1 - q)x\right]^{\frac{\beta}{q-1}} & \text{for } 1 - (1 - q)x > 0 \\ 0 & \text{otherwise} \end{cases}
\]
where \(c_1\) and \(c_2\) are the normalizing constants. Then proceeding as in the case for \(q > 1\), we have
\[
P\left(\sum_{i=1}^{k} X_i < Y\right) = \left\{\frac{1 - q}{\Gamma\left(\frac{\beta}{q-1} + \alpha + 1\right)} H^{2.1}_{1,2} \left[\frac{\theta}{(q-1)}\right]^{(0,\alpha)}_{\left(0, \frac{\beta-q+1}{q-1}, \alpha\right)}\right\}^k, q > 1.
\]
4.5 Marshall-Olkin q-Gamma Distribution

Marshall and Olkin (1997) introduced a new family of survival functions which is constructed by adding a new parameter to an existing distribution. The introduction of a new parameter will result in flexibility in the distribution. Different first order autoregressive (AR(1)) time series models can be developed using the Marshall-Olkin Gamma distribution. Now we shall extend the q-Gamma distributions to obtain a more general class called Marshall-Olkin q-Gamma distribution using (??).

Now consider the case \( q > 1 \). Substituting the q-Gamma distribution function defined in \( F_4(x) \) in (??), we get a new family of distributions called Marshall-Olkin q-Gamma distribution (MO-q-G), whose distribution function and the survival function \( \tilde{G}_1(\cdot) \) are given by

\[
G_1(x; p, q, \alpha, \beta) = \frac{B(x(q-1), \alpha, \frac{\beta}{q-1} - \alpha)}{pB(\alpha, \frac{\beta}{q-1} - \alpha) + (1-p)B(x(q-1), \alpha, \frac{\beta}{q-1} - \alpha)}. \tag{4.5.1}
\]

\[
\tilde{G}_1(x; p, q, \alpha, \beta) = \frac{p [B(\alpha, \frac{\beta}{q-1} - \alpha) - B(x(q-1), \alpha, \frac{\beta}{q-1} - \alpha)]}{pB(\alpha, \frac{\beta}{q-1} - \alpha) + (1-p)B(x(q-1), \alpha, \frac{\beta}{q-1} - \alpha)}. \tag{4.5.2}
\]

Then the p.d.f. corresponding to \( G_1(\cdot) \) is given by

\[
g_1(x; p, q, \alpha, \beta) = \frac{p(q-1)^{\alpha}x^{\alpha-1}[1 + (q-1)x]^{-\frac{\alpha}{q-1}}B(\alpha, \frac{\beta}{q-1} - \alpha)}{pB(\alpha, \frac{\beta}{q-1} - \alpha) + (1-p)B(x(q-1), \alpha, \frac{\beta}{q-1} - \alpha)^2}, \quad x > 0, q > 1.
\]

Similarly when \( q < 1 \), the cdf is given by

\[
G_2(x; p, q, \alpha, \beta) = \frac{B(x(1-q), \alpha, \frac{\beta}{1-q} + 1)}{pB(\alpha, \frac{\beta}{1-q} + 1) + (1-p)B(x(1-q), \alpha, \frac{\beta}{1-q} + 1)}. \tag{4.5.3}
\]
and the p.d.f. corresponding to $G_2(\cdot)$ is given by

$$g_2(x; p, q, \alpha, \beta) = \frac{p(1-q)^{\alpha}x^{\alpha-1}[1-(1-q)x]}{pB(\alpha, \frac{\beta}{1-q} + 1) + (1-p)B(x(1-q), \alpha, \frac{\beta}{1-q} + 1)^2}, \quad 0 < x < (1/1-q).$$

The hazard rate functions are given by

$$h_1(x) = \frac{(q-1)^\alpha x^{\alpha-1}[1+(q-1)x]}{pB(\alpha, \frac{\beta}{q-1} - \alpha) + (1-p)U(x)} \left( B(\alpha, \frac{\beta}{q-1} - \alpha) - U(x) \right)$$

for $q > 1$ and

$$h_2(x) = \frac{(1-q)^\alpha x^{\alpha-1}[1-(1-q)x]}{pB(\alpha, \frac{\beta}{1-q} + 1) + (1-p)V(x)} \left( B(\alpha, \frac{\beta}{1-q} + 1) - V(x) \right)$$

where

$$U(x) = B(x(q-1), \alpha, \frac{\beta}{q-1} - \alpha).$$

and

$$V(x) = B(x(1-q), \alpha, \frac{\beta}{1-q} + 1).$$

**Theorem 4.5.1.** Let $\{X_i, i \geq 1\}$ be a sequence of i.i.d. random variables with common distribution function $F(x)$ and let $N$ be a geometric random variable independently distributed of $\{X_i\}$ such that $P[N = n] = \theta(1-\theta)^{n-1}, \quad n = 1, 2, \ldots, 0 < \theta < 1,$ which is for all $i \geq 1$. Let $U_N = \min_{1 \leq i \leq N} X_i$. Then $\{U_N\}$ is distributed as MO-$q$-G if and only if $\{X_i\}$ follows $q$-Gamma distribution.

**Proof:** The survival function of the random variable $U_N$ is

$$\tilde{H}(x) = P(U_N > x) = \theta \sum_{n=1}^{\infty} [F(x)]^n (1-\theta)^{n-1} = \frac{\theta F(x)}{1 - (1-\theta)F(x)}.$$

If $X_i$ has the survival function of the $q$-Gamma distribution given by (6.2), then $U_N$ has the survival function of the MO-$q$-G distribution. The converse easily follows.
4.6 AR(1)models with $q$-Gamma Marginal Distribution

Tavares (1977, 1980) introduced two stationary Markov processes with similar structural forms which had applications in hydrological applications. Lewis and McKenzie (1991) discuss various aspects of first order autoregressive minification processes. In this section we develop autoregressive minification processes of order one and order $k$ with minification structures where MO-$q$-G distribution is the stationary marginal distribution. We call the process as MO-$q$-G AR(1) process. Now we have the following theorem.

**Theorem 4.6.1.** Consider an AR(1) structure given by

$$X_n = \begin{cases} 
\epsilon_n, & \text{w.p. } p_1 \\
\min(X_{n-1}, \epsilon_n), & \text{w.p. } 1 - p_1
\end{cases}$$

$0 < p_1 < 1$ and $\{\epsilon_n\}$ is a sequence of i.i.d. random variables independently distributed of $X_n$. Then $\{X_n\}$ is a stationary Markovian AR(1) process with MO-$q$-G marginal if and only if $\{\epsilon_n\}$ is distributed as $q$-Gamma distribution.

**Proof:** From the given structure it follows that

$$\bar{F}_{X_n}(x) = p_1 \bar{F}_{\epsilon_n}(x) + (1 - p_1) \bar{F}_{X_{n-1}}(x) \bar{F}_{\epsilon_n}(x).$$

Under stationary equilibrium, it reduces to

$$\bar{F}_X(x) = \frac{p_1 \bar{F}_{\epsilon}(x)}{1 - (1 - p_1) \bar{F}_{\epsilon}(x)}. \quad (4.6.1)$$

On substituting the survival function $\bar{F}_{\epsilon}(x)$ of $\epsilon$, for $q > 1$, we get

$$\bar{F}_{1X}(x) = \frac{p_1 [\beta(\alpha, \frac{\beta}{q-1} - \alpha) - \beta(x(q-1), \alpha, \frac{\beta}{q-1} - \alpha)]}{p_1 \beta(\alpha, \frac{\beta}{q-1} - \alpha) + (1 - p_1) \beta(x(q-1), \alpha, \frac{\beta}{q-1} - \alpha)}$$

which resembles the survival function $\bar{G}_{1}(\cdot)$ of the MO-$q$-G distribution. Conversely, if
we take the survival function of the above form, we get the survival function of \( \epsilon_n \) as the \( q \)-Gamma distribution under stationary equilibrium. Similar is the case when \( q < 1 \).

The following theorem generalizes the results to a \( k \)th order autoregressive structure.

**Theorem 4.6.2.** Consider an AR\((k)\) structure given by

\[
X_n = \begin{cases} 
\epsilon_n, & \text{w.p. } p_0 \\
\min(X_{n-1}, \epsilon_n), & \text{w.p. } p_1 \\
\min(X_{n-2}, \epsilon_n), & \text{w.p. } p_2 \\
\vdots & \vdots \\
\min(X_{n-k}, \epsilon_n), & \text{w.p. } p_k
\end{cases}
\]

where \( \{\epsilon_n\} \) is a sequence of i.i.d. random variables independently distributed of \( X_n \), \( 0 < p_i < 1 \), \( p_1 + p_2 + \cdots + p_k = 1 - p_0 \). Then the stationary marginal distribution of \( \{X_n\} \) is MO-\( q \)-G if and only if \( \{\epsilon_n\} \) is distributed as \( q \)-Gamma distribution.

**Proof:** From the given structure the survival function is given as follows:

\[
\bar{F}_{X_n}(x) = p_0 \bar{F}_{\epsilon_n}(x) + p_1 \bar{F}_{X_{n-1}}(x) \bar{F}_{\epsilon_n}(x) + \cdots + p_k \bar{F}_{X_{n-k}}(x) \bar{F}_{\epsilon_n}(x).
\]

Under stationary equilibrium, this yields

\[
\bar{F}_X(x) = p_0 \bar{F}_\epsilon(x) + p_1 \bar{F}_X(x) \bar{F}_\epsilon(x) + \cdots + p_k \bar{F}_X(x) \bar{F}_\epsilon(x).
\]

This reduces to

\[
\bar{F}_X(x) = \frac{p_0 \bar{F}_\epsilon(x)}{1 - (1 - p_0) \bar{F}_\epsilon(x)}.
\]

Then the theorem easily follows by similar arguments as in Theorem (4.6.1). For \( q > 1 \) and for \( q < 1 \), if we substitute the survival function of \( q \)-Gamma distribution we get the survival function of the MO-\( q \)-G distribution. The sample path of the minification process is given in figure 4.5.
4.7 The Max-min AR(1) Processes

Next we introduce the model called the max-min process given by Jose et al. (2008b). The structure is given as follows.

**Theorem 4.7.1.** Consider an AR(1) structure given by

\[
X_n = \begin{cases} 
\epsilon_n, & \text{w.p. } p_1 \\
\max(X_{n-1}, \epsilon_n), & \text{w.p. } p_2 \\
\min(X_{n-1}, \epsilon_n), & \text{w.p. } p_3 
\end{cases}
\]

subject to the conditions \(0 < p_1, p_2 < 1, p_2 < p_1 \) and \( p_1 + p_2 + p_3 = 1 \), where \( \{\epsilon_n\} \) is a sequence of i.i.d. random variables independently distributed of \( X_n \). Then \( \{X_n\} \) is a stationary Markovian AR(1) max-min process with \( q \)-gamma stationary marginal distribution if and only if \( \{\epsilon_n\} \) follows Marshall-Olkin \( q \)-gamma distribution.
**Proof:** From the given structure it follows that

\[ P(X_n > x) = p_1 P(\epsilon_n > x) + p_2 P(\max(X_{n-1}, \epsilon_n) > x) + p_3 P(\min(X_{n-1}, \epsilon_n) > x) \]

On simplification we get,

\[ P(X_n > x) = p_1 P(\epsilon_n > x) + p_2 \left[ 1 - P(\max(X_{n-1}, \epsilon_n) \leq x) \right] + p_3 F_{X_{n-1}}(x) F_{\epsilon_n}(x) \]

\[ = p_1 \bar{F}_{\epsilon_n}(x) + p_2 \left[ 1 - (1 - F_{X_{n-1}}(x))(1 - \bar{F}_{\epsilon_n}(x)) \right] + p_3 F_{X_{n-1}}(x) \bar{F}_{\epsilon_n}(x). \]

Under stationary equilibrium,

\[ \bar{F}_{\epsilon}(x) = \frac{\frac{p_1 + p_3}{p_1 + p_2} \bar{F}_{X}(x)}{1 - (1 - \frac{p_1 + p_3}{p_1 + p_2}) \bar{F}_{X}(x)} \]

\[ = \frac{p' \bar{F}_{X}(x)}{1 - (1 - p') \bar{F}_{X}(x)} \]

This has the same functional form of Marshall-Okin survival function. Substituting for \( \bar{F}_{X}(x) \) we can obtain the survival function of \( \epsilon_n \).

\[ \bar{F}_{\epsilon}(x) = \frac{p' \left[ \beta(x(q - 1), \alpha, \frac{\beta}{q-1} - \alpha) \right]}{\beta(\alpha, \frac{\beta}{q-1} - \alpha) + (1 - p') \beta(x(q - 1), \alpha, \frac{\beta}{q-1} - \alpha)} \]

which is the required MO-q-G function. The converse can be proved using the same procedure as in theorem (4.6.1).
4.8 Conclusions

In this chapter we introduced the $q$-analogue of gamma distribution and studied its properties. Estimation problems are addressed. Its application in stress-strength analysis is discussed. The goodness of fit of the new distribution also tested. It is generalized using Marshall-Olkin procedure and minification and max-min model developed.

References


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